Analysis & Topology

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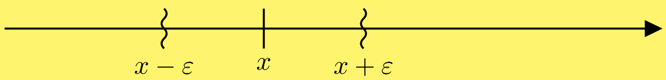
1 Uniform Convergence and Uniform Continuity

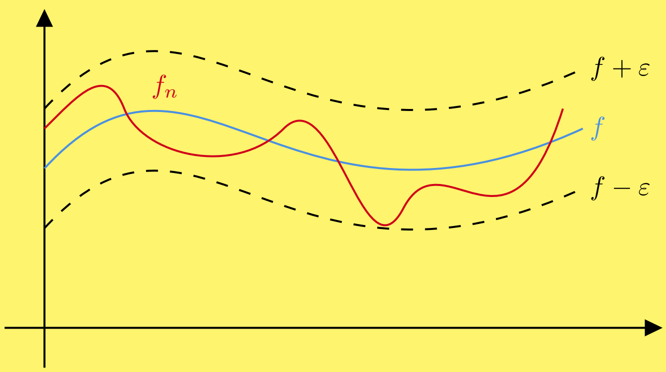
Note. From IA: $x_n \to x$ as $n \to \infty$ (in $\mathbb{R}$ or $\mathbb{C}$) if

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} \forall n \geq N \ |x_n - x| < \varepsilon$$



$$\forall \varepsilon > 0 \exists N \in \mathbb{N} \ x_n \in (x - \varepsilon, x + \varepsilon)$$

i.e. x_n is ε-close to x. We say $(x - \varepsilon, x + \varepsilon)$ is the ε-neighbourhood of x.
We aim to define "$f_n \to f$" for functions



Definition. Let S be a set, $f_n : S \to \mathbb{R}, n \in \mathbb{N}, f : S \to \mathbb{R}$ be functions. Say (f_n) converges to f **unformly** on S if

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} \forall n \geq N \forall x \in S \ |f_n(x) - f(x)| < \varepsilon$$

Notes.

(i) N depends only on ε, not on any $x \in S$ (hence "uniform")

(ii) Can replace $\mathbb{R}$ with $\mathbb{C}$

(iii) Equivalently:

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} \forall n \geq N \ \sup_{x \in S} |f_n(x) - f(x)| < \varepsilon$$

or

$$\sup_{x \in S} |f_n(x) - f(X)| \to 0 \text{ as } n \to \infty$$

(iv) For each $x \in S$, $(f_n(x))_{n=1}^{\infty}$ converges to $f(x)$. So f is unique (i.e. if $f_n \to f$ and $f_n \to g$ uniformly on S, then $f = g$). We call f the uniform limit of f_n) on S

Definition. $S, (f_n), f$ as before. Say (f_n) **converges pointwise** to f on S if $(f_n(x))_{n=1}^{\infty}$ converges to $f(x)$ for every $x \in S$ ie

$$\forall x \in S \forall \varepsilon > 0 \exists N \in \mathbb{N} \forall n \geq N \ |f_n(x) - f(x)| < \varepsilon$$

Notes.
(i) N depends on ε and x
(ii) Again f is unique - call it the pointwise limit of (f_n) on S

Remark. Uniform converge $\implies$ pointwise convergence

Example. $f_n(x) = x^2 e^{-nx}, x \in [0, \infty), n \in \mathbb{N}$. Does (f_n) converge unformly on $[0, \infty)$?
Fix $x \geq 0$. Then $x^2 e^{-nx} \to 0$ as $n \to \infty$ so $f_n \to 0$ pointwise on $[0, \infty)$. Does (f_n) converge to 0 (the zero function) unfiromly on $[0, \infty)$, i.e.

$$\sup_{x \in [0,\infty)} |f_n(x) - 0| = \sup_{x \in [0,\infty)} f_n(x) \to 0 \text{ as } n \to \infty?$$

We could differenitable but a much better way to find an upper bound on $|f_n(x) - f(x)|$ that does not depend on x. In our case:

$$0 \leq x^2 e^{-nx} = \frac{x^2}{1 + nx + \frac{n^2x^2}{2} + \dots} \leq \frac{2}{n^2} \forall x \geq 0$$

So $\sup f_n(x) \leq \frac{2}{n^2} \to 0$ as $n \to \infty$. So $f_n \to 0$ uniformly on $[0, \infty)$

Example. $f_n(x) = x^n, x \in [0, 1], n \in \mathbb{N}$. Does (f_n) converge unfiromly on $[0, 1]$?

$$x^n \to \begin{cases} 0 & 0 \leq x < 1 \\ 1 & x = 1 \end{cases}$$

So $f_n \to f$ pointwise on $[0, 1]$ where

$$f(x) = \begin{cases} 0 & 0 \leq x < 1 \\ 1 & x = 1 \end{cases}$$

$$\sup_{x \in [0,1]} |f_n(x) - f(x)| = \sup_{x \in [0,1)} f_n(x) = 1$$

as $f_n(x) \to 1$ as $x \to q$ for each n)
So $f_n \not\to f$ uniformly on $[0, 1]$ and hence (f_n) does not converge unfiromly on $[0, 1]$ or

$$\sup_{x \in [0,1)]} \geq f_n \left(\left(\frac{1}{2} \right)^{1/n} \right) = \frac{1}{2}$$

Remark. "$f_n \not\to f$ uniformly on S" means:

$$\exists \varepsilon > 0 \forall N \in \mathbb{N} \exists n \geq N \exists x \in S \ |f_n(x) - f(x)| \geq \varepsilon$$

Theorem 1.1 (The uniform limit of continuous functions is continuous)**.** Let S be a subset of $\mathbb{R}$ or $\mathbb{C}$. We're given functions $f_n : S \to \mathbb{R}$ (or $\mathbb{C}$), $n \in \mathbb{N}$ and $f : S \to \mathbb{R}$ ($\mathbb{C}$). Assume f_n is continuous for every $n \in N$ and $f_n \to f$ unfiromly on S. Then f is continuous.

Proof.

Idea: Fix $a \in S$. Want $x \simeq a \implies f(X) \simeq f(a)$. Choose n s.t. $f_n \simeq f$ everywhere. Then as f_n is continuous, $x \simeq a \implies f_n(x) \simeq f_n(a)$ so

$$f(x) \simeq f_n(x) \simeq f_n(a) \simeq f(a)$$

Fix $a \in S, \varepsilon > 0$. We seek $\delta > 0$ s.t. $\forall x \in S\ |x - a| < \delta \implies |f(x) - f(a) < \varepsilon$.
Choose $n \in \mathbb{N}$ s.t. $\forall x \in S |f_n(x) - f(x)| < \varepsilon$.
Fix such an n. Since f_n is continuous, there exists $\delta > 0$ s.t. $\forall x \in S$

$$|x - a| < \delta \implies |f)n(x) - f_n(a)| < \varepsilon$$

So $\forall x \in S$ if $|x - a| < \delta$ then

$$|f(x) - f(a)| \leq |f(x) - f_n(a)| + |f_n(x) - f_n(a)| + |f_n(a) - f(a)| < 3\varepsilon$$

Remarks.

(i) This is called a 3ε-proof.

(ii) Not true for pointwise convergence e.g. $f_n(x) = x^n$ for $x \in [0,1]$, $n \in \mathbb{N}$ and $f(x) = \begin{cases} 0 & 0 \leq x \leq 1 \\ 1 & x = 1 \end{cases}$
$f_n \to f$ pointwise on $[0,1]$, f_n continuous $\forall n$ but f is not continuous on $[0,1]$

(iii) Not true for differentiability (see example sheet)

(iv)

$$\begin{aligned} \lim_{x\to a} \lim_{n\to\infty} f_n(x) &= \lim_{x\to a} f(x) = f(a) \\ &= \lim_{n\to\infty} f_n(a) = \lim_{n\to\infty} \lim_{x\to a} f_n(a) \end{aligned}$$

(swapped the limits)

Lemma 1.2 (The uniform limit of bounded functions is bounded)**.** Assume $f_n \to f$ uniformly on some set S. If f_n is bounded for every n, then so is f.

Proof. Fix $n \in \mathbb{N}$ s.t. $\forall x \in S\ |f_n(x) - f(x)| < 1$. Since f_n is bounded, there is $m \in \mathbb{R}$ s.t. $\forall x \in S |f_n(x)| \leq M$. So $\forall x \in S$

$$|f(x)| \leq |f(X) - f_n(x)| + |f_n(x)| \leq 1 + M$$

From IA: Let $f : [a, b] \to \mathbb{R}$ be a bounded function. For a dissection $\mathcal{D} : a = x_0 < x_1 < \cdots < x_n = b$ of $[a, b]$ we define the upper and lower sums of f w.r.t $\mathcal{D}$ by

$$U_{\mathcal{D}}(f) = \sum_{k=1}^{\infty} (x_k - x_{k-1}) \cdot \sup_{[x_{k-1}, x_k]} f$$

$$L_{\mathcal{D}}(f) = \sum_{k=1}^{\infty} (x_k - x_{k-1}) \cdot \inf_{[x_{k-1}, x_k]} f$$

Reimann's criterion: f is intergrable iff $\forall \varepsilon > 0 \exists \mathcal{D}$ s.t. $U_{\mathcal{D}} - L_{\mathcal{D}} < \varepsilon$.
Easy exercise: for any $I \subset [a, b]$

$$\sup_I f - \inf_I f = \sup_{x,y \in I} (f(x) - f(y)) = \sup_{x,y \in I} |f(x) - f(y)|$$

This is called the oscillation of f on I

Theorem 1.3 (We can swap limit and integral for uniform convergence)**.** Let $f_n : [a,b] \to \mathbb{R}$ be integrable for all $n \in \mathbb{N}$. If $f_n \to f$ uniformly on $[a,b]$ then f is integrable and moreover

$$\int_a^b f_n \to \int_a^b f \text{ as } n \to \infty$$

Proof. We prove that f is is bounded and satisfies Riemann's criterion.
f bounded: by definition each f_n is bounded, so by the lemma, f is bounded.
Now fix $\varepsilon > 0$. Fix $n \in \mathbb{N}$ s.t.

$$\forall x \in [a,b] \ |f_n(x) - f(x)| < \varepsilon$$

Since f_n is integrable, $\exists$ dissection $\mathcal{D} = x_n < x_1 < \cdots < x_N = b$ of $[a,b]$ s.t. $U_{\mathcal{D}}(f_n) - L_{\mathcal{D}}(f_n) < \varepsilon$.
Fix $k \in \{1,\dots,N\}$. For $x, y \in [x_{k-1}, x_k]$

$$\begin{aligned}|f(x) - f(y)| &\le |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)| \\ &\le |f_n(x) - f_n(y)| + 2\varepsilon\end{aligned}$$

hence

$$\sup_{x,y \in [x_{k-1},x_k]} |f(x) - f(y)| \le \sup_{[x_{k-1},x_k]} |f_n(x) - f_n(y)| + 2\varepsilon$$

Multiply by $(x_k - x_{k-1})$ and take $\sum_{k=1}^{N}$

$$U_{\mathcal{D}}(f) - L_{\mathcal{D}}(f) \le U_{\mathcal{D}}(f_n) - L_{\mathcal{D}}(f_n) + 2\varepsilon(b-a) \le \varepsilon(2(b-a)+1)$$

So f is integrable.

$$\left| \int_a^b f_n - \int_a^b f \right| \le \int_a^b |f_n - f| \le (b-a) \sup_{[a,b]} |f_n - f| \to 0 \text{ as } n \to \infty$$

Note.

$$\int_a^b \lim_{n\to\infty} f_n(x)\,\mathrm{d}x = \lim_{n\to\infty} \int_a^b f_n(x)\,\mathrm{d}x$$

Corollary 1.4 (We can swap infinite sum and integral for uniform convergence)**.** Let $f_n : [a, b] \to \mathbb{R}$ be integrable for every n.
If $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on $[a, b]$, then $x \to \sum_{n=1}^{\infty} f_n(x)$ is integrable and

$$\int_a^b \sum_{n=1}^{\infty} f_n(x)\,\mathrm{d}x = \sum_{n=1}^{\infty} \int_a^b f_n(x)\,\mathrm{d}x$$

Proof. Let

$$F_n(x) = \sum_{k=1}^{n} f_k(x)\ x \in [a, b],\ n \in \mathbb{N}$$

$$F(x) = \sum_{k=1}^{\infty} f_k(x)\ x \in [a, b]$$

By assumption, $F_n \to F$ uniformly on $[a, b]$. From IA: F_n is integrable and

$$\int_a^b F_n = \sum_{k=1}^{n} \int_a^b f_k$$

which follows from the previous theorem

Theorem 1.5 (Can differentiate term by term if derivative sum converges uniformly)**.** Let $f_n : [a,b] \to \mathbb{R}$ be continuously differentiable for every n. Assume:

(i) $\sum_{k=1}^{\infty} f_k'(x)$ converges uniformly on $[a,b]$

(ii) There exists $c \in [a,b]$ s.t. $\sum_{n=1}^{\infty} f_n(c)$ converges

Then $\sum_{k=1}^{\infty} f_k(x)$ converges uniformly on $[a,b]$ to a continuously differentiable function f and moreover

$$\left(\sum_{k=1}^{\infty} f_k\right)'(x) = f'(x) = \sum_{k=1}^{\infty} f_k'(x)$$

Proof. Let

$$g(x) = \sum_{k=1}^{\infty} f_k'(x) \ x \in [a,b]$$

Solve $f' = g$ with initial condition $f(c) = \sum_{n=1}^{\infty} f_n(c)$
Let $\lambda = \sum_{n=1}^{\infty} f_n(c)$ and define $f : [a,b] \to \mathbb{R}$ by

$$f(x) = \lambda + \int_c^x g(t)\,\mathrm{d}t$$

Since $\sum_{k=1}^{\infty} f_k'(x)$ converges uniformly to g on $[a,b]$, g is continuous and hence integrable. By Fundamental Theorem of Calculus (FTC) $f' = g$ on $[a,b]$ (so f' is continuous) and $f(c) = \lambda$. Also by FTC:

$$f_k(c) = f_k(x) + \int_c^x f_k'(t)\,\mathrm{d}t \ k \in \mathbb{N} \ x \in [a,b]$$

Fix $\varepsilon > 0$. By assumption, $\exists N \in \mathbb{N}$ s.t.

$$\left|\lambda - \sum_{k=1}^{n} f_k(c)\right| < \varepsilon \ \forall n \geq N$$

$$\left|g(t) - \sum_{k=1}^{n} f_k'(t)\right| < \varepsilon \ \forall n \geq N \ \forall t \in [a,b]$$

Now for $x \in [a,b], n \geq N$, we have

$$\begin{aligned}\left|f(x) - \sum_{k=1}^{n} f_k(x)\right| &= \left|\lambda + \int_c^x g(t)\,\mathrm{d}t - \sum_{k=1}^{n}\left(f_k(c) + \int_c^x f_k'(t)\,\mathrm{d}t\right)\right| \\ &\leq \left|\lambda - \sum_{k=1}^{n} f_k(c)\right| + \left|\int_c^x \left(f(t) - \sum_{k=1}^{n} f_k'(t)\right)\,\mathrm{d}t\right| \\ &\leq \varepsilon + |x-c|\varepsilon \leq (b-a+1)\varepsilon\end{aligned}$$

From IA: a scalar sequence (x_n) is Cauchy if $\forall \varepsilon > 0 \exists N \in \mathbb{N} \forall m,n \geq N |x_m - x_n| < \varepsilon$
General Principle of Convergence (GPC): every Cauchy sequence converges

Definition. A sequence (f_n) of scalar functions on a set S is **uniform Cauchy** if

$$\forall \varepsilon > 0 \ \exists N \in \mathbb{N} \ \forall m,n \geq N \ \forall x \in S \ |f_m(x) - f_n(x)| < \varepsilon$$

Theorem 1.6 (General Principle of Uniform Convergence, GPUC)**.** If (f_n) is a uniformly Cauchy sequence of functions on a set S, then it converges uniformly on S to some function

Proof. Fix $x \in S$. Wel'll show $(f(x))_{n=1}^{\infty}$ is convergent.
Given $\varepsilon > 0$, we have $N \in \mathbb{N}$ s.t.

$$\forall m, n \geq N \; \forall t \in S \; |f_m(t) - f_n(t)| < \varepsilon$$

In particular,

$$\forall m, n \geq N \; |f_m(x) - f_n(x)| < \varepsilon$$

So $(f_n(x))_{n=1}^{\infty}$ is Cauchy and hence convergent by GPC.
Let $f(x) = \lim_{n=\infty} f_n(x)$. Doing this for every $x \in S$, we obtain $f : S \to$ scalars s.t. $f_n \to f$ pointwise on S.
Claim: $f_n \to f$ uniformly on S
Fix $\varepsilon > 0$. There's $n \in \mathbb{N}$ s.t.

$$\forall m, n \geq N \; \forall x \in D \; |f_m(x) - f_n(x)| < \varepsilon$$

We now show that $\forall n \geq N \; \forall x \in S \; |f_n(x) - f(x)| < 2\varepsilon$. Then done.
Fix $x \in S$, fix $n \geq N$. Since$f_m(x) \to f(x)$ as $m \to \infty$, we can choose $m \in \mathbb{N}$ s.t.

$$|f_m(x) - f(x)| < \varepsilon \text{ and } m \geq N$$

(m depends on x). Now

$$|f_n(x) - f(x)| < |f_n(x) - f_m(x)| + |f_m(x) - f(x)| < \varepsilon + \varepsilon = 2\varepsilon$$

Note. Alternative end of proof:
Fix $x \in S, \; n \geq N$. Then

$$|f_n(x) - f_m(x)| < \varepsilon \; \forall m \geq N$$

Let $m \to \infty$:

$$|f_n(x) - f(x)| \leq \varepsilon$$

Theorem 1.7 (Weierstass M-test)**.** Let (f_n) be a sequence of scalar functions on a set S. Assume that for every $n \in \mathbb{N}$ there is an $M_n \in \mathbb{R}^+$ s.t.

$$|f_n(x)| \leq M_n \text{ for all } x \in S$$

If $\sum_{n=1}^{\infty} M_n < \infty$ then $\sum_{n=1}^{\infty} f_n(x)$ is uniformly convergent on S

Proof. Let $F_n(x) = \sum_{k=1}^{n} f_k(x)$ $x \in S, n \in \mathbb{N}$.
For $x \in S,\ n \geq m$ in $\mathbb{N}$, we have

$$|F_n(x) - F_m(x)| \leq \sum_{k=m+1}^{n} |f_k(x)| \leq \sum_{k=m+1}^{n} M_k$$

Given $\varepsilon > 0$ choose $N \in \mathbb{N}$ s.t. $\sum_{k=N+1}^{\infty} M_k < \varepsilon$.
Then $\forall x \in S\ \forall n \geq m \geq N$ we have

$$|F_n(x) - F_m(x)| \leq \sum_{k=m+1}^{n} M_k < \varepsilon$$

So (F_n) is uniformly Cauchy on S and hence uniformly convergent on S by previous theorem

Consider the power series $\sum_{n=1}^{\infty} C_n(z-a)^n$
Here $C_n \in \mathbb{C}$ $(n \in \mathbb{N})$, $a \in \mathbb{C}$ fixed and $z \in \mathbb{C}$ variable
Let $R \in [0, \infty]$ be the r.o.c. (radius of convergence) of this power series. Recall

$$|z-a| < R \implies \sum_{n=0}^{R} C_n(x-a)^n \text{ converges absolutely}$$

$$|z-a| > R \implies \sum_{n=0}^{R} C_n(x-a)^n \text{ diverges}$$

Let $D(a,R) = \{z \in \mathbb{C} \| z-a| < R\}$ (the open disc centre a, radius R). Define

$$f : D(a,R) \to \mathbb{C} \text{ with } f(z) = \sum_{n=0}^{\infty} C_n(z-a)^n$$

f is the pointwise limit on $D(a,R)$ of the power series. We ask: is the convergence uniform? In general, it is not.

Examples.

(i)

$$\sum_{n=1}^{\infty} \frac{z^n}{n^2} \text{ has } R = 1$$

Let $f_n : D(0,1) \to \mathbb{C}$ be $f_n(z) = z^n/n^2$

$$\forall z \in D(0,1) \ |f_n(z)| \leq \frac{1}{n^2}$$

Since $\sum_{n=1}^{\infty} 1/n^2$ is convergent, by the M-test, the power series converges uniformly on $D(0,1)$

(ii)

$$\sum_{n=0}^{\infty} \text{ has } R = 1$$

and we know

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$$

$$|\sum_{n=0}^{N} z^n| \leq N + 1 \forall z \in D(0,1)$$

By lemma 2, the series does NOT converge uniformly on $D(0,1)$ ($1/(1-z)$ is not bounded on $D(0,1)$)

OR

$$\sup_{|z|<1} \left| \frac{1}{1-z} - \sum_{k=0}^{n} z^k \right| = \sup_{|z|<1} \left| \frac{z^{n+1}}{1-z} \right| = \infty$$

Theorem 1.8 (Power series converges uniformly on disk smaller than r.o.c.)**.** Assume the power series $\sum_{n=0}^{\infty} C_n(z-a)^n$ has r.o.c R. Then for any r with $0 < r < R$ the power series converges uniformly on $D(a,r)$

Proof. Fix $w \in \mathbb{C}$ s.t. $r < |w-a| < R$ e.g. $w = a + \frac{r+R}{2}$. Set $\rho = \frac{r}{|w-a|}$ so $\rho \in (0,1)$.

Since $\sum_{n=0}^{\infty} C_n(w-a)^n$ converges, we have

$$C_n(w-a)^n \to 0 \text{ as } n \to \infty$$

$$\therefore \exists M \in \mathbb{R}^+ |C_n(w-a)^n| \leq M \text{ for all } n \in \mathbb{N}$$

("convergent $\implies$ bounded")

For $z \in D(a,r), n \in \mathbb{N}$ we have

$$|C_n(z-a)^n| = |C_n(w-a)^n| \cdot \left(\frac{|z-a|}{|w-a|}\right)^n \leq M \left(\frac{r}{|w-a|}\right)^n = M\rho^n$$

Since $\sum_{n=0}^{\infty} M\rho^n$ is convergent, by M-test:

$$\sum_{n=0}^{\infty} C_n(z-a)^n \text{ converges uniformly on } D(a,r)$$

Remarks.

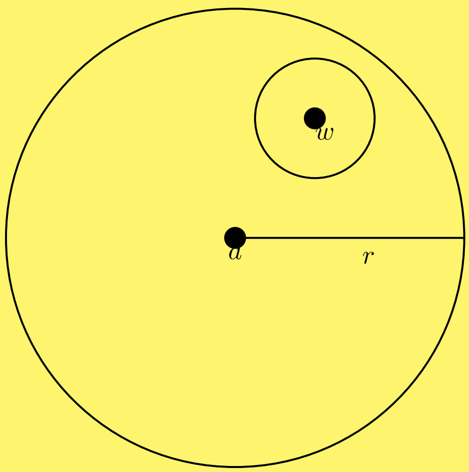
(i)
$$f : Da, R \to \mathbb{C},\ f(z) \sum_{n=0}^{\infty} C_n (z-a)^n$$

is, by previous theorem, the uniform limit on $F(a, r)$ of polynomials for any r with $0 < r < R$, and hence f is continuous on $D(a, r)$ by theorem 1.1, since $D(a, R) = \cup_{0<r<R} D(a, r)$, it follows that f is continuous on $F(a, R)$

(ii) $\sum_{n=1}^{\infty} C_n \cdot n \cdot (z-a)^{n-1}$ has r.o.c. R, i.e. same as the original series (from IA) so converges uniformly on $D(a, r)$ if $0 < r < R$.

By a result analagous to theorem 1.5, we have that $\sum C_n (z-a)^n$ is complex differentiable on $F(a, R)$ with derivarive $\sum C_n \cdot n(z-a)^{n-1}$ (see Complex Analysis)

(iii) Fix $w \in D(a, R)$. Fix r s.t. $|w-a| < r < R$, fix $\delta > 0$ s.t. $w-a| + \delta < r$



If $|z-w| < \delta$ then
$$|z-a| < |z-w| + |w-a| < \delta + |w-a|$$

So $D(w, \delta) \subset D(a, r)$. Hence $\sum_{n=0}^{\infty} C_n (z-a)^n$ converges uniformly on $D(w, \delta)$

Definition. A subset U of $\mathbb{C}$ is **open** if

$$\forall w \in U \exists \delta > 0\ D(w, \delta) \subset U$$

Definition. Let U be an open subset of $\mathbb{C}$ and (f_n) a sequence of scalar functions on U. Say (f_n) **converges locally uniformly** on U if $\forall w \in U \exists \delta > 0$ s.t. (f_n) converges uniformly on $D(w, \delta) \subset U$

Remarks.

(i) The third remark alone shows that a power series converges locally uniformly inside the r.o.c. (i.e. on $D(a, R)$)

(ii) We'll return to this when discussing compactness

1.1 Uniform Continuity

Definition. Let U be a subset of $\mathbb{R}$ or $\mathbb{C}$. Let f be a scalar function on U. For $x \in E$, f is **continuous** at x if:

$$\forall \varepsilon > 0 \exists \delta > 0 \forall y \in U |y - x| < \delta \implies |f(y) - f(x)| < \varepsilon$$

f is **continuous** on U if f is continuous at x for every $x \in U$

$$\forall x \in U \forall \varepsilon > 0 \exists \delta > 0 \forall y \in U |y - x| < \delta \implies |f(y) - f(x)| < \varepsilon$$

Note. δ depends on ε and x

Definition. Let U, f be as before. Say f is **uniformly continuous** on U if

$$\forall \varepsilon > 0 \exists \delta > 0 \forall x, y \in U |x - y| < \delta \implies |f(x) - f(y)| < \varepsilon$$

Note. δ depends on ε only. We have that uniform continuity implies continuity

Examples. (i) $f : \mathbb{R} \to \mathbb{R}, f(x) = 2x + 17$ is uniformly continuous. Given $\varepsilon > 0$, let $\delta = \varepsilon/2$. Then $\forall x, y \in \mathbb{R}$ if $|x - y| < \delta$ then $|f(x) - f(y)| = 2|x - y| < 2\delta = \varepsilon$

(ii) $f : \mathbb{R} \to \mathbb{R},\ f(x) = x^2$, is continuous but not uniformly continuous. Let $\varepsilon = 1$. Given $\delta > 0$ let $x > 0$ and $y = x + \delta/2$. Then $|y - x| < \delta$ and $|f(x) - f(y)| = (x + \delta/2)^2 - x^2 = \delta x + \delta^2/4$ so for $x = 1/\delta$ and $y = x + \delta/2$ we have $|x - y| < \delta$ but $|f(x) - f(y)| = 1 + \delta^2/4 > 1 = \varepsilon$. So f is not uniformly continuous.

Note. For U, f as in definition above, f is NOT uniformly continuous on U means:

$$\exists \varepsilon > 0 \forall \delta > 0 \exists x, y \in U |x - y| < \delta \text{ and } |f(x) - f(y)| \geq \varepsilon$$

Theorem 1.9. Let f be a scalar function on a closed, bounded interval $[a, b]$. If f is continuous on $[a, b]$, then f is uniformly continuous on $[a, b]$

One idea: fix $\varepsilon > 0$. For all $x \in [a, b]$ $\exists \delta_x > 0$ s.t. $\forall y \in [a, b]$ if $|y - x| < \delta_x$ then $|f(y) - f(x)| < \varepsilon$. Let

$$\delta = \inf_{x \in [a,b]} \delta_x$$

but we have the problem that $\delta = 0$ is possible.

Proof. We argue by contradiction. Assume theres an $\varepsilon > 0$ s.t. $\forall \delta > 0 \exists x, y \in [a, b]$ s.t .$|x - y| < \delta$ and $|f(x) - f(y)| \geq \varepsilon$. In particular, $\forall n \in \mathbb{N} \exists x_n, y_n \in [a, b]$ s.t. $|x_n - y_n| < 1/n$ and $|f(X_n) - f(y_n)| > \varepsilon$. By Bolzano Weierstass $\exists$ subsequence (x_{k_n}) of (x_n) that converges. $(k_1 < k_2 < k_3 < \dots$ and so $k_n \geq n \forall n)$. Let $x = \lim_{n \to \infty} x_{k_n}$. Then $x \in [a, b]$. Then

$$|y_{k_n} - x| \leq |y_{k_n} - x_{k_n}| + |x_{k_n} - x| < \frac{1}{n} + |x_{k_n} - x| \to 0$$

So $y_{k_n} \to x$. Since f is continuous, we have $f(x_{k_n}) \to f(x)$ and $f(y_{k_n}) \to f(x)$. Now

$$\varepsilon \leq |f(x_{k_n}) - f(y_{k_n})| \to |f(x) - f(x)| = 0 \text{ ※}$$

Corollary 1.10. A continuous function $f : [a, b] \to \mathbb{R}$ is integrable

Proof. Since continuous function on closed bounded interval is bounded, we have f is bounded. Fix $\varepsilon > 0$. By theorem 1.9, f is uniformly continuous so $\exists \delta > 0$ s.t. $\forall x, y \in [a, b]$ if $|x - y| < \delta$ then $|f(x) - f(y)| < \varepsilon$. Choose dissection $\mathcal{D}$ of $[a, b]$ s.t. all intervals in δ have length $< \delta$ (e.g. choose $n \in \mathbb{N}$ s.t. $\frac{b-a}{n} < \delta$ and let $\mathcal{D}$ consist of $a + k \cdot \frac{b-a}{n},\ k = 0, 1, \dots, n)$ If I is one interval of $\mathcal{D}$ then $\forall x, y \in I$, we have $|x - y| < \delta$, and so $|f(x) - f(y)| < \varepsilon$

$$\therefore \sup_{x,y \in I} |f(x) - f(y)| \leq \varepsilon$$

multiply by the length of I and sum over all I to get

$$U_{\mathcal{D}}(f) - L_{\mathcal{D}}(f) \leq (b - a)\varepsilon$$

So f satisfies Riemann's critereon

2 Metric Spaces

Remark. In $\mathbb{R}$ and $\mathbb{C}$ we measured "closeness" of a point x, y by the expression $|x - y|$. The most important property of this "distance" was the Δ-inequality.

Definition. Let M be a set. A metric on M is a function $d : M \times M \to \mathbb{R}$ s.t.

(i)
$$\forall x, y \in M \ d(x, y) \geq 0 \text{ and moreover } d(x, y) = 0 \iff x = y \tag{Positivity}$$

(ii)
$$\forall x, y \in M \ d(x, y) = d(y, x) \tag{Symmetry}$$

(iii)
$$\forall x, y, z \in M \ d(x, z) \leq d(x, y) + d(y, z) \tag{triangle-inequality}$$

A **metric space** is a pair (M, d) where M is a set and d is a metric on M.

Examples. (i) $M = \mathbb{R}$ or $\mathbb{C}$ and $d(x,y) = |x-y|$. This is the standard metric on M

(ii) $M = \mathbb{R}^n$ or $\mathbb{C}^n$. We define the Euclidean norm (Euclidean length) of $x \in M$ by

$$\|x\| = \|x\|_2 = \left(\sum_{k=1}^{n} |x_k|^2\right)^{1/2} \quad (x = (x_k)_{k=1}^n)$$

This satisfies

$$\|x+y\| \leq \|x\| + \|y\| \ \forall x, y \in M$$

It follows that

$$d(x,y) = d_2(x,y) = \|x-y\|_2 = \left(\sum_{k=1}^{n} |x_k - y_k|^2\right)^{1/2}$$

defines a metric on M, called the **Euclidean Metric**.
E.g. $\forall x, y, z \in M$

$$d(x,z) = \|x-z\| = \|(x-y) + \|(y-z)\| \leq \|x-y\| + \|y-z\| = d(x,y) + d(y,z)$$

This will be the standard metric on $M = \mathbb{R}^n$ or $\mathbb{C}^n$. The metric space (M,d) is called: n-dimensional real or complex Euclidean space. We sometimes denote this by l_2^n, the euclidean norm is also called the l_2-norm and the Euclidean metrix is also called the l_2 metric.

(iii) $M = \mathbb{R}^n$ or $\mathbb{C}^n$, the l_1 norm of $x \in M$ is

$$\|x\| = \sum_{k=1}^{n} |x_k|$$

which defines the l_1-metric

$$d_1(x,y) = \sum_{k=1}^{n} |x_k - y_k| = \|x-y\|$$

(M, d_1) is denoted by l_1^n.
In fact, you can do this for $p \in \mathbb{R}, 1 \leq p < \infty$. In this course we will only work with $p = 1, 2$ and $(p = \infty)$

(iv) $M = \mathbb{R}^n$ or $\mathbb{C}^n$. We define the l_∞-norm of $x \in M$ by

$$\|x\|_\infty = \max_{1 \leq k \leq n} |x_k|$$

This defines the l_∞ metric

$$d_\infty(x,y) = \|x-y\|_\infty = \max_{1 \leq k \leq n} |x_k - y_k|$$

We denote (M, d_∞) by l_∞^n

(v) Let S be a set. Let $l_\infty(S)$ be the set of all bounded scalar functions on S. We define the l_∞-norm of $f \in l_\infty(S)$ by

$$\|f\| = \|f\|_\infty = \sup_{x \in S} |f(x)|$$

(also called sup norm or uniform norm)

Note. For $f, g \in l_\infty(S), x \in S$, we have

$$|f(x) + g(x)| \leq |f(x)| + |g(x)| \leq \|f\| + \|g\|$$

So $\|f+g\| \leq \|f\|+\|g\|$. It follows that $d(f,g) = \|f-g\|$ defines a metric on $l_\infty(S)$, called the uniform metric on $l_\infty(S)$

Examples. $l_\infty(\{1,2,\ldots,n\})$ in l_∞^n. If$S = \mathbb{N}$, then we will write l_∞ for $l_\infty(\mathbb{N})$. This is the space of scalar sequences with uniform metric.

Examples. (vi) $(C[a,b])$ is the set of all continuous functions on the closed, bounded interval $[a,b]$. For $p = 1, 2$ we define the **L_p-norm** of $f \in C[a,b]$ by

$$\|f\|_p = \left(\int_a^b |f(x)|^p \,\mathrm{d}x\right)^{1/p}$$

This defines the **L_p-metric**

$$d_p(f,g) = \|f-g\|_p$$

e.g.

$$\|f+g\|_2^2 = \int_a^b |f+g|^2 \leq \int_a^b |f|^2 + |g|^2 + 2|f|\cdot|g| \leq \|f\|_2^2 + \|g\|_2^2 + 2\|f\|_2\|g\|_2 = (\|f\|_2 + \|g\|_2)^2$$

So

$$\|f+g\|_2 \leq \|f\|_2 + \|g\|_2$$

This easily implies the triangle inequality for d_2

(vii) Let M beany set. Then

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

defines a metric called the **discrete metric** and (M,d) is called a **discrete metric space**

(viii) Let G be a group generated by $S \subseteq G$. Then

$$d(x,y) = \min\{n \geq 0 : \exists s_1, s_2, \ldots, s_n \in S \text{ s.t. } yxs_1s_x\ldots s_n\}$$

with $x \in S \implies x^{-1} \in S$ defines a metric called the **word metric** (Geometric group theory)

(ix) Fix a prime $p \in \mathbb{Z}$. Then

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ p^{-n} & \text{if } x \neq y \text{ where } x - y = p^n \cdot m,\ n \geq 0, m \in \mathbb{Z}, p \neq\mid m \end{cases}$$

defines a metric on $\mathbb{Z}$ called the p-adic metric (Number Theory)

2.1 Subspaces

Definition. Let (M,d) be a metric space and $N \subset M$ then $d|_{N\times N}$ is a metric on N. N with this metric is a **subspace of M**. We usually use d to denote the metric on N

Examples. (i) $\mathbb{Q}$ with the metric $d(x,y) = |x-y|$ is a subspace of $\mathbb{R}$

(ii) Since every continuous function on a closed, bounded interval is bounded, it follows that $C[a,b]$ is a subset of $l_\infty([a,b])$. So $X[a,b]$ with the uniform metric is a subspace of $l_\infty(a,b])$.

2.2 Product Spaces

Let (M,d) and $M',d')$ be metric spaces. Then any of the following defines a metric on $M \times M'$:

$$d_1((x,x'),(y,y')) = d(x,y) + d'(x',y')$$

$$d_2((x,x'),(y,y')) = (d(x,y)^2 + d'(x',y')^2)^{1/2}$$

$$d_\infty((x,x'),(y,y')) = \max\{d(x,y), d'(x',y')\}$$

Notation. We denote the metric space $(M \times M', d_p)$ by $M \oplus_p M'$ $(p = 1,2,\infty)$

Note.

$$d_\infty \le d_2 \le d_1 \le 2d_\infty$$

Can generalise: for $n \in \mathbb{N}$ and metric spaces $M_k, \rho_k)$ $k = 1,2,\ldots,n$ we define

$$\left(\bigoplus_{k=1}^{n} M_k\right)_p = M_1 \oplus_p M_2 \oplus_p \cdots \oplus_p M_n$$

to be the metric space $(M_1 \times M_2 \times \cdots \times M_n, d_p)$ e.g.

$$d_2((x_1,\ldots,x_n),(y_1,\ldots,y_n)) = \left(\sum_{k=1}^{n} \rho_k(x_k,y_k)^2\right)^{1/2}$$

Example.

$$\mathbb{R} \oplus_1 \mathbb{R} = l_1^2, \mathbb{R} \oplus_2 \mathbb{R} \oplus_R = l_2^3$$

$$\underbrace{\mathbb{R} \oplus_\infty \mathbb{R} \oplus_\infty \cdots \oplus_\infty \mathbb{R}}_{n} = l_\infty^n$$

Note. $\mathbb{R} \oplus_1 \mathbb{R} \oplus_2 \mathbb{R}$ makes no sense since $(R \oplus_1 \mathbb{R}) \oplus_2 \mathbb{R}$, $\mathbb{R} \oplus_1 (\mathbb{R} \oplus_2 \mathbb{R})$ are different metric spaces

2.3 Convergence

Definition. Let M be a metric space and (x_n) a sequence in M. Given $x \in M$, say (x_n) **converges** to x in M (write $x_n \to x$ as $n \to \infty$) if

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} \forall n \geq N d(x_n, x) < \varepsilon$$

Say (x_n) is **convergent in M** if $\exists x \in M$ s.t. $x_n \to x$ as $n \to \infty$, otherwise we say (x_n) is **divergent**

Note. $x_n \to x$ in $M \iff d(x_n, x) \to 0$ in $\mathbb{R}$

Lemma 2.1. Assume $x_n \to x$ and $x_n \to y$ in a metric space M. Then $x = y$.

Proof. Assume $x \neq y$.Let $\varepsilon = d(x, y)/3$. Then $\varepsilon > 0$, so since $x_n \to x$ and $x_n \to y$,

$$\exists N_1 \in \mathbb{N} \ \forall n \geq N_1 \ d(x_n, x) < \varepsilon$$

$$\exists N_2 \in \mathbb{N} \ \forall n \geq N_2 \ d(x_n, y) < \varepsilon$$

Dix $n \in \mathbb{N}$ s.t. $n \geq N_1$ and $n \geq N_2$ then

$$d(x, y) \leq d(x, x_n) + d(x_n, y) < 2\varepsilon = \frac{2}{3} d(x, y) \text{ ※}$$

Definition. Given a convergent subsequence in a metric space M, the **limit of (x_n)** is the unique $x \in N$ s.t. $x_n \to x$ as $n \to \infty$, denoted $\lim_{n \to \infty} x_n$

Examples. (i) This has the usual meaning in $\mathbb{R}$ or $\mathbb{C}$

(ii) Constant sequences converge. More generally, let (x_n) be an eventually constant sequence in a metric space M
$$\exists x \in M \exists N \in \mathbb{N} \forall n \geq N \; x_n = x$$
Then $x_n \to x$ as $n \to \infty$. The converse is false (consider $1/n$ in $\mathbb{R}$).
However, assume $x_n \to x$ is a discrete metric space:
$$\exists N \in \mathbb{N} \forall n \geq N d(x_n, x) < 1$$
so $\forall n \geq N \; x_n = x$

(iii) In the 4-adic metric $3^n \to 0$ as $n \to \infty$ since $d(3^n, 0) = 3^{-n} \to 0$ as $n \to \infty$

(iv) Let S be a set. Then $f_n \to f$ in $l_\infty(S)$ in the uniform metric iff
$$d(f_n, f) = \|f_n - f\|_\infty = \sup_S |f_n - f| \to 0 \text{ as } n \to \infty$$
iff $f_n \to f$ uniformly on S.

> **Note.** For $f_n(x) = x + 1/n, \; x \in \mathbb{R}, \; n \in \mathbb{N}$ with $f(x) = x, \; x \in \mathbb{R}$. Then $f_n \to f$ uniformly on $\mathbb{R}$. However, $f_n, f \notin l_\infty(\mathbb{R})$

(v) Euclidean space $\mathbb{R}^n$ or $\mathbb{C}^n$ with l_2-metric
$$x^{(k)} = (x_1^{(k)}, x_2^{(k)}, \ldots, x_n^{(k)}) \in M \; k \in \mathbb{N}$$
$$x = (x_1, x_2, \ldots, x_n) \in M$$
$$|x_i^{(k)} - x_i| \leq \|x^{(k)} - x\|_2 \leq \sum_{i=1}^n |x_i^{(k)} - x_i|$$
So $x^{(k)} \to x \iff$ for every i, $x_i^{(k)} \to x_i$ coordinate wise convergence

(vi) $f_n(x) = x^n \; x \in [0,1], \; n \in \mathbb{N}$ so (f_n) is a sequence in $C[0,1]$. We know that (f_n) converges pointwise on $[0,1]$ but not uniformly. So not convergent in the uniform metric. However in the L_1-metric:
$$d_1(f_n, 0) = \|f_n\|_1 = \int_0^1 f_n = \frac{1}{n+1} \to 0 \text{ as } n \to \infty$$
So $f_n \to 0$ in $C[0,1]$ in the L_1-metric

(vii) Let N be a subspace of a metric space M. If (x_n) is a convergent sequence in N (ie $x_n \in N \forall n$ and $\exists x \in N$ s.t. $x_n \to x$), then (x_n) is also convergent in M. However, the converse is false e.g. $M = \mathbb{R}, \; N = (0, \infty)$
$$\left(\frac{1}{n}\right)_n \text{ is divergent in } N \text{ but convergent in } M$$

(viii) Let (M, d) and (M', d') be metric spaces and $N = M \oplus_p M'$ ($p = 1, 2$ or ∞). Let $a_n = (x_n, y_n) \in N \; \forall n \in \mathbb{N}$ and$a = (x, y \in N)$ Then
$$a_n \to a \text{ in } N \iff x_n \to x \text{ in } M \text{ and } y_n \to y \text{ in } M'$$
Indeed, we have
$$\begin{aligned} \max\{d(x_n, x), d'(y_n, y)\} &= d_\infty(a_n, a) \\ &\leq d_p(a_n, a) \leq d_1(a_n, a) \\ &= d(x_n, x) + d'(y_n, y) \end{aligned}$$

2.4 Continuity

Definition. Let $f : M \to M'$ be a function between metric spaces (M, d) and (M', d'). For $a \in M$, we say f is **continuous at a** if

$$\forall \varepsilon > 0 \exists \delta > 0 \forall x \in M \ d(x, a) < \delta \implies d'(f(x), f(a)) < \varepsilon$$

We say f is continuous if f is continuous at a for every $a \in M$. I.e.

$$\forall a \forall \varepsilon > 0 \exists \delta > 0 \forall x \in M \ d(x, a) < \delta \implies d'(f(x), f(a)) < \varepsilon$$

Note. δ depends on ε and a (and f)

Prop 2.2. Let $f : M \to M'$ be a function between metric spaces and let $a \in M$. Then TFAE:
(i) f is continuous at a
(ii) if $x_n \to a$ in M then $f(x_n) \to f(a)$ in M'

Proof. (i) $\implies$ (ii): Assume $x_n \to a$ in M. Let $\varepsilon > 0$. We seek $N \in \mathbb{N}$ s.t.

$$\forall n \geq N \ d'(f(x_n), f(a)) < \varepsilon$$

Since f is continuous at a, there is a $\delta > 0$ s.t.

$$\forall x \in M d(x, a) < \delta \implies d'(f(x), f(a) < \varepsilon)$$

Since $x_n \to a$, there is $N \in N$ s.t. $\forall n \geq N \ d(x_n, a) < \delta$. So $\forall n \geq N \ d'(f(x_n), f(a)) < \varepsilon$
(ii) $\implies$ (i): We argue by contradiction. Assume f is not continuous at a. This means:

$$\exists \varepsilon > 0 \forall \delta > 0 \exists x \in M \ d(x, a) < \delta, \ d'(f(x), f(a)) \geq \varepsilon$$

Fix such a "bad" $\varepsilon > 0$. Then $\forall n \in \mathbb{N} \ \exists x_n \in M$

$$d(x_n, a) M \frac{1}{n} \text{ and } d'(f(x_n), f(a)) \geq \varepsilon$$

Then $x_n \to a$ in M but $f(x_n) \not\to f(a)$ in M ※

Prop 2.3. Let f, g be scalar functions on a metric space M. Let $a \in M$. If f, g are constant at a, then so are $f + g$ and $f \cdot g$. Moreover, letting $N = \{x \in M : g(x) \neq 0\}$ and assuming $a \in N$, we gave $f/g : N \to \mathbb{C}$ is continuous at a. So if f, g are continuous, then so are $f + g, f \cdot g$ and f/g

Proof. Assume $x_n \to a$ in M. Then

$$(f \cdot g)(x_n) = f(x_n) \cdot g(x_n) \to f(a) \cdot g(a) = (f \cdot g)(a)$$

This uses previous proposition plus IA analysis. So by Prop 2 again, $f \cdot g$ is continuous at a. Similar argument for $f + g$ and f/g

Note. If $f : M \to M'$ is continuous then for any sequence (x_n) in M, if (x_n) is convergent in M, then $f(x_n))$ is convergent in M' and

$$\lim_{n\to\infty} f(x_n) = f(\lim_{n\to\infty} x_n)$$

Prop 2.4. Let $f : M \to M'$ and $g : M' \to M''$ be functions between metric spaces. Let $a \in M$. If F is continuous at a and g is continuous at $f(a)$ then $g \circ f : M \to M''$ is continuous at a. If f, g are continuous, then so is $g \circ f$

Proof. Fix $\varepsilon > 0$. We seek $\delta > 0$ s.t. $\forall x\ inM$ if $d(x,a) < \delta$ then $d''(g(f(x)), g(f(a))) < \varepsilon$. Since g is continuous at $f(a)$, there is $\eta > 0$ s.t.

$$\forall y \in M' d'(y, f(a)) < \eta \implies d''(g(y), g(f(a))) < \varepsilon$$

Since f is continuous at a, there is $\delta > 0$ s.t.

$$\forall x \in M\ d(x,a) M \delta \implies d'(f(x), f(a)) < \eta$$

So $\forall x \in M, d(x,a) M\delta \implies d''(g(f(x)), g(f(a))) < \varepsilon$

Examples.
(i) Constant functions: $f : M \to M',\ f(x) = b\ \forall x \in M$ Then $d'(f(x), f(a)) = 0\ \forall x \in M$. So $\forall a \in M \forall \varepsilon > 0$, any $\delta > 0$ will do
(ii) Identity functions $f : M \to M, f(x) = x$. Then $d(f(x), f(a)) = d(x,a)$ so $\forall a \in M, \forall \varepsilon > 0\ \delta = \varepsilon$ will do
(iii) Using prop 3 and the two examples above, we get all real and complex polynomials are constant as are rational functions. Using uniform convergence, uniform limits of such functions are also continuous e.g. $\exp, \sin, \cos$ etc.
(iv) Let (M,d) be a metric space. Then d is itseld a function between metric spaces:

$$d : M \oplus_p M \to \mathbb{R}\ (p = 1, 2, \text{ or } \infty)$$

Given $v = (x, x')$ nad $w = (y, y')$ in $M \oplus_p M$,

$$d(v) - d(w) = d(x,x') - d(y,y') \leq d(x,y) + d(x',y') = d_1(v,w) \leq 2d_p(v,w)$$

Definition. Let $f : M \to M;$ be a function between metric spaces. Then f is
(i) **Isometric** if

$$\forall x, y \in M\ d'(f(x), f(y)) = d(x,y)$$

(ii) **Lipscitz** $\exists C \in \mathbb{R}^+\ \forall x, y \in M$

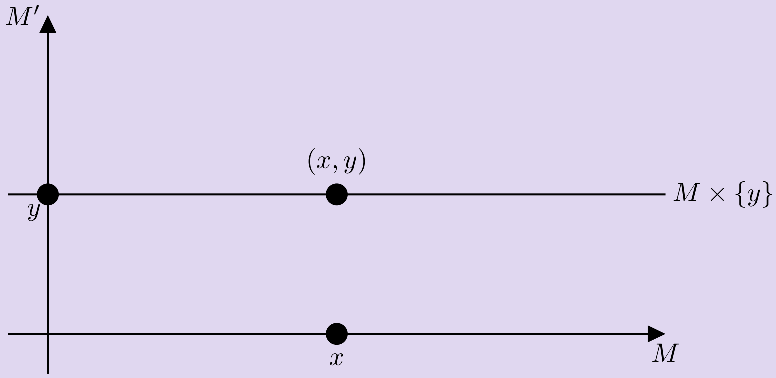
$$d'(f(x), f(y)) \leq Cd(x,y)$$

(iii) **Uniformly continuous** if

$$\forall \varepsilon > 0 \exists \delta > 0\ \forall x, y \in M\ d(x,y) < \delta \implies d'(f(x), f(y)) < \varepsilon$$

Note. (i) Isometric $\implies$ Lipschitz $\implies$ uniformly continuous $\implies$ continuous E.g. if $N \subset M$, the inclusion $i : N \to M,\ i(x) = x$ is isometric but not surjective (unless $N = M$). An isometric and surjective map is an **isometry**. If $\exists$ isometry $f : M \to M'$, say M and M' are **isometric** (or $M;$ is an **isometric copy of M**)

Examples. (v) Let $(M,d),(M',d')$ be metric spaces, fix $y \in M'$. Define $f : M \to M \oplus_p M',\ x \mapsto (x,y)$



$$d_p(f(x), f(z)) = d_p((x,y),(z,y)) = d(x,z)$$

so f is isometric and $M \times \{y\}$ is an isometric copy of M in $M \oplus_p M'$.
E.g. fix $a = (a_1, \dots, a_n) \in \mathbb{R}^n$,

$$x \mapsto (a_1, \dots, a_{k-1}, x, a_{k+1}, \dots, a_n) : \mathbb{R} \to \mathbb{R}^n$$

is isometric

(vi) Consider the projections

$$q : M \oplus_p M' \to M,\ q(x,y) = x$$

$$q' : M \oplus_p M' \to M', q'(x,y) = y$$

$$d(q(x,y), q(x',y')) = d(x,x') \leq d_p((x_y,(x',y')))$$

So q is 1-Lipschitz, as is q'.
E.g. $\mathbb{C}^n \to \mathbb{C},\ (z_1, \dots, z_n) \mapsto z_k$ is continuous therefore polynomials in any number of variables are continuous (prop 3)

2.5 The Topology of a Metric Space

Definition. Let M be a metric space, $x \in M, r > 0$. The **open ball** in M of centre x and radius r is the set

$$D_r(x) = \{y \in M : d(y,x) < r\}$$

Note. $x_n \to x$ in M iff $\forall \varepsilon > 0 \exists N \in \mathbb{N} \forall n \geq N\ x_n \in D_\varepsilon(x)$. Given $f : M \to M'$

$$f \text{ is continuous at } x \iff \forall \varepsilon > 0 \exists \delta > 0\ f(D_\delta(x)) \subset D_\varepsilon(f(x))$$

Definition. The **closed ball** in M of centre x and radius $r (r \geq 0)$ is the set

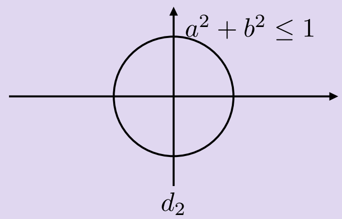
$$B_r(x) = \{y \in M : d(y,x) \leq r\}$$

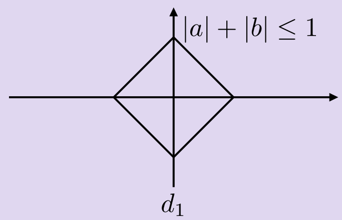
Examples. (i) In $\mathbb{R}$

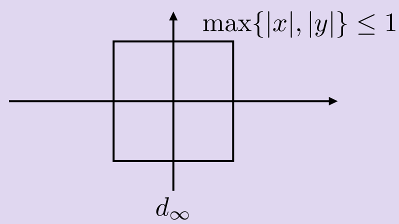
$$D_r(x) = (x - r, x + r)$$
$$B_r(x) = [x - r, x + r]$$

(ii) In $(\mathbb{R}^2, d_p)$ $p = 1, 2$ or ∞) consider

$$B_1(0) = \{x \in \mathbb{R}^2 | d_p(x, 0) = \|x\|_p \leq 1$$







Note.

$$D_r(x) \subset B_r(x) \subset D_s(x) \forall r < s$$

(iii) If M is a discrete metric space then for $x \in M$

$$D_1(x) = \{x\}, \ B_1(x) = M$$

Definition. Let M be a metric space and $U \subset M$. For $x \in M$ say U **is a neighbourhood of** x **in** M if $\exists r > 0$ $D_r(x) \subset U$
$(\iff) \exists r > 0 B_r(x) \subset U$
We say U is **open** in M (or that U is **an open subset of** M) if

$$\forall x \in U \exists r > 0 \ D_r(x) \subset U$$

(i.e. U is neighbourhood of all its points)

Examples. (i) $D_r(x)$ nad $B_r(x)$ are neighbourhoods of x

(ii)

$$H = \{z \in \mathbb{C} : \text{ Im}z \geq 0\}$$

Let $w \in H$ then $\delta =$ Imw.
If $\delta > 0$ then $D_\delta(w) \subset H$.
If $\delta = 0$ then $\forall r > 0$ $D_r(w) \not\subset H$. H is not open.

Lemma 2.5. Open balls are open

Proof. Consider $D_r(x)$ in a metric space M. Need:

$$\forall y \in D_r(x) \exists \delta > 0 : D_\delta(t) \subset D_r(x)$$

Let $y \in D_r(x)$. Set $\delta = r - d(x, y)$. Then $\delta > 0$. If $z \in D_\delta(y)$ then

$$d(z, x) \leq d(z, y) + d(y, x) < \delta + d(y, x) = r$$

So $x \in D_r(x)$. This shows $D_\delta(y) \subset D_r(x)$

Corollary 2.6. Let M be a metric space, $U \subset M, x \in M$. Then U is a neighbourhood of $x \iff \exists$ open subset V of M s.t. $x \in V \subset U$

Proof. $\implies$: B_1 by definition $\exists r > 0$ $D_r(x) \subset U$. By lemma 5 $V = D_r(x)$ is open in M and $x \in V \subset U$.
$\implies$: if $x \in V \subset U$ and V is open, by definition $\exists r > 0$

$$D_r(x) \subset V$$

So $D_r(x) \subset U$ and so U is a neighbourhood of x

Prop 2.7. In a metric space M, TFAE

(i) $x_n \to x$
(ii) $\forall$ neighbourhoods U of $x \in M \exists N \in \mathbb{N} \forall n \geq N$ $x_n \in U$
(iii) $\forall$ open subsets U of M with $x \in U$, $\exists N \in \mathbb{N}$ $\forall n \geq N$ $x_n \in U$

Proof. (i) $\implies$ (ii): Let U be a neighbourhood of x in M. By definition $\exists \varepsilon > 0$ $D_\varepsilon(x) \subset U$. Since $x_n \to x$ $\exists N \in \mathbb{N}$ $\forall n \geq N$ $d(x_n, x) < \varepsilon$ i.e. $x_n \in D_\varepsilon(x)$ so $\forall n \geq N$ $x_n \in U$.
(ii) $\implies$ (iii) Clear since any open set U with $x \in U$ is a neighbourhood of x.
(iii) $\implies$ (i) Fix $\varepsilon > 0$. By lemma 5 $U = D_\varepsilon(x)$ is open and $x \in U$. By (iii) $\exists N \in \mathbb{N}$ $\forall n \geq N$ $x)n \in U$ i.e.

$$d(x_n, x) < \varepsilon$$

Prop 2.8. Let $f : M \to M'$ be a function between metric spaces.

(i) For $x \in M$ TFAE

   (a) f is continuous at x

   (b) $\forall$ neighbourhoods V of $f(x)$ in M' $\exists$ neighbourhood U of x in M s.t. $f(U) \subset V$

   (c) $\forall$ neighbourhoods V of $f(x)$ in M'. $f^{-1}(V)$ is a neighbourhood of x in M.

(ii) TFAE

   (a) f is continuous

   (b) $f^{-1}(V)$ is open in M $\forall$ open subsets V of M'

Proof. (a) (i) $\implies$ (ii) Let V be a neighbourhood of $f(x)$ in M'. By definition $\exists \varepsilon > 0$ $D_\varepsilon(f(x)) \subset V$. Since f is constant at x, $\exists \delta > 0$ $f(D_\delta(x)) \subset D_\varepsilon(f(x))$. Then $U = D_\delta(x)$ is a neighbourhood of $x \in M$ and $f(U) \subset V$.
(ii) $\implies$ (iii): Let V be a neighbourhood of $f(x)$ in M'. By (ii) $\exists$ neighbourhood U of x in M s.t. $f(U) \subset V$. Then $U \subset f^{-1}(V)$ and since U is a neighbourhood of $x \in M$, $\exists r > 0$.

$$D_r(x) \subset U \subset f^{-1}(V)$$

Thus $f^{-1}(V)$ is a neighbourhood of x in M.
(iii) $\implies$ (i): Given $\varepsilon > 0$, $V = D_\varepsilon(f(x))$ is a neighbourhood of $f(x)$ in V. By (iii) $f^{-1}(v)$ is a neighbourhood of x in M. So $\exists \delta > 0$

$$D_\delta(x) \subset f^{-1}(V)$$

Thus

$$f(D_\delta(x)) \subset V = D_\varepsilon(f(x))$$

(b) (i) $\implies$ (ii) Let V be open in M'. Let $c \in f^{-1}(V)$. Then $f(x) \in V$. Since V is open $\exists \varepsilon > 0$ $D_\varepsilon(f(x)) \subset V$. Since f is continuous at x, $\exists \delta > 0$ $f(D_\delta(x)) \subset D_\varepsilon(f(x))$

$$\therefore D_\delta(c) \subset f^{-1}(D_\varepsilon(f(x))) \subset f^{-1}(V)$$

Then $f^{-1}(V)$ is open in M.
(ii) $\implies$ (i): Let $x \in M$, let $\varepsilon > 0$. Then $V = D_\varepsilon(f(x))$ is open in M' by lemma 5.
By (ii) $f^{-1}(V)$ is open in M. Also, $x \in f^{-1}(V)$ as $f(x) \in V$. By definition $\exists \delta > 0$ s.t.

$$D_\delta(x) \subset f^{-1}(V)$$

$$\therefore f(D_\delta(x)) \subset V = D_\varepsilon(f(x))$$

Definition. The **topology of a metric space** M is the family of all open subsets of M

Prop 2.9. The topology of a metric space satisfies the following

(i) $\varnothing$ and M are open

(ii) If U_i is open in M $\forall i \in I$, then $\bigcup_{i\in I} U_i$ is open in M

(iii) U, V open in $M \implies U \cap V$ open in M

Proof. (i) Clear

(ii) Given $x \in \bigcup_{i\in I} U_i$, $\exists i_0 \in I$ s.t. $x \in U_{i_0}$. U_{i_0} is open so by definition, $\exists r > 0$ s.t.

$$D_r(x) \subset U_{i_0} \subset \bigcup_{i\in I} U_i$$

(iii) Given $x \in U \cap V$, since U is open and $x \in U$, $\exists r > 0$ s.t. $D_r(x) \subset U$ and since V is open and $x \in V$, $\exists s > 0$ s.t. $D_s(x) \subset V$. Let $t = \min(r, s)$ then $t > 0$ and

$$D_t(x) = D_r(x) \cap D_s(x) \subset U \cap V$$

Definition. A subset A of a metric space M is **closed in** M (or is a **closed subet of** M) if for every sequence (x_n) in A that is convergent in M, we have $\lim_{n\to\infty} x_n \in A$

Lemma 2.10. Closed balls are closed

Proof. Consider $V_r(x) = \{y \in M : d(y, x) \leq r\}$ in a metric space M and a sequence (x_n) in $B_r(x)$ s.t. $x_n \to z$ in M. We need $z \in B_r(x)$. We need $z \in B_r(x)$

$$d(z, x) \leq d(z, x_n) + d(x_n, x) \leq d(z, x_n) + r \to r \text{ as } n \to \infty$$

$$\therefore d(z, x) \leq r$$

and hence $z \in B_r(x)$

Examples. (i) $[0, 1] = B_{1/2}(1/2)$ is closed in $\mathbb{R}$. $[0, 1]$ is not open e.g. $D_r(0) \not\subset [0, 1]$ for any $r > 0$

(ii) $(0, 1) = D_{1/2}(1/2)$ is open (Lemma 5). $(0, 1)$ is not closed:

$$\frac{1}{n+1} \in (0, 1) \forall n \in \mathbb{N}$$

but

$$\frac{1}{n+1} \to 0 \text{ in } \mathbb{R}, 0 \notin (0, 1)$$

(iii) $\mathbb{R}$ is opena nd closed in $\mathbb{R}$.

(iv) $(0, 1]$ is neither open nor closed. Trivial check

Lemma 2.11. Let A be a subset of a metric space M. Then A is closed in $M \iff M\backslash A$ is open in M

Proof. $\implies$: Assume A closed, $M\backslash A$ is not open. So $\exists x \in M\backslash A\ \forall r > 0$

$$D_r(x) \not\subset M\backslash A,\ D_r(x) \cap A \neq \varnothing$$

Hence $\forall n\ \exists x_n \in D_{1/n}(x) \cap A$. Then $d(x_n, x) < 1/n \to 0$, so $x_n \to x$ on M and $x_n \in A\ \forall n$. Contradiction as A is closed.
$\impliedby$: Assume $M\backslash A$ is open but A is ot closed. So $\exists (x_n)$ in A s.t. $x_n \to x$ in M but $x \notin A$. Since $x \in M\backslash A$ and $M\backslash A$ is open

$$\exists \varepsilon > 0 D_\varepsilon(x) \subset M\backslash A$$

Since $x_n \to x, \exists N \in \mathbb{N}\ \forall n \geq N\ x_n \in D_\varepsilon(x)$ and hence $x_n \in M\backslash A$

Example. Let M be a discrete metric space. Let $A \subset M$. Then $\forall x \in A$

$$D_1(x) = \{x\} \subset A$$

So A is open. So every subset of M is open in M, and hence every subset of M is closed in M by lemma 11

Definition. A map $f : M \to M'$ between metric spaces is a **homeomorphism** if f is a bijection and f and f^{-1} are both continuous. Equivalently, f is a bijection and $\forall$ open sets V in M', $f^{-1}(V)$ is open in M and $\forall$ open sets U in M, $f(U)$ is open in M' (prop 8). If $\exists$ a homeomorphism between M and M', we say M and M' are homeomorphic

Example. $(0, \infty)$ and $(0, 1)$ are homeomorphic. $x \mapsto \frac{1}{x+1}$, $x \mapsto \frac{1}{x} - 1$

Note. (i) Every isometry is a homeomorphism. Converse is false
(ii) Id:($\mathbb{R}$, discrete) $\to$ ($\mathbb{R}$, euclidean) is continuous bijection whose inverse is not continuous

Definition. Let d and d' be metrics on a set M. We say d and d' are **equivalent** (write $d \sim d'$) if they define the same topology on M. (i.e. for $U \subset M$, U is open in (M, d) iff U is open in (M, d')). So $d \sim d' \iff$ Id: $(M, d \to M, d')$ is homeomorphism.

Note. If $d \sim d'$ then (M, d) and (M, d') have the same convregent sequences and the same continuous maps.

Definition. Let d and d' be metrics on a set M. Say d and d' are **uniformly equivalent** if Id:$(M, d) \to (M, d')$ and Id: $(M, d') \to (M, d)$ are uniformly continuous. We write $d \sim_u d'$. Say d and d' are **Lipschitz equivalent** if Id:$(M, d) \to (M, d')$ and Id: $(M, d') \to (M, d)$ are Lipschitz maps. We write $d \sim_{\text{Lip}} d'$

Note. $d \sim_{\text{Lip}} d'$ iff $\exists a > 0, b > 0$ s.t.

$$ad(x, y) \leq d'(x, y) \leq bd(x, y) \ \forall x, y \in M$$

Note. $d \sim_{\text{Lip}} d' \implies d \sim_n d' \implies d \sim d'$

Examples.

(i) Given metric space (M, d)

$$d'(x, y) = \min\{1, d(x, y)\}$$

defines a metric on M and $d' \sim_u d$.

(ii) On a product space $M \times M'$, d_1, d_2 and d_∞ are pairwise Lipschitz equivalent

(iii) On $C[0, 1]$, the L_1-metric and the uniform metric are not equivalent, e.g. $f_n(x) = x^n, n \in \mathbb{N}, x \in [0, 1]$ has $d_n \to 0$ in the L_1-metric but (f_n) is not convergent in $C[0, 1]$ in the uniform metric.

(iv) The discrete and euclidean metrixs on $\mathbb{R}$ are not equivalent

3 Completeness and the Contraction Mapping Theorem

Remark. Recall: in $\mathbb{R}$ and $\mathbb{C}$, every Cauchy sequence is convergent. A sequence (x_n) in $\mathbb{R}$ or $\mathbb{C}$ is bounded if $\exists C \in \mathbb{R}^+ \ \forall n \in \mathbb{N} \ |x_n| \leq C$.

Definition. A sequence (x_n) in a metric space M is **Cauchy** if

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} \forall m, n \geq N \ d(x_m, x_n) < \varepsilon$$

bounded if

$$\exists z \in M \exists r > 0 \forall n \ x_n \in B_r(z)$$

Note. (x_n) is bounded $\iff$ $\forall z \in M \exists r > 0 \forall n x_n \in B)r(z)$. Assume there is $z \in M, r > 0$ s.t. $\forall n x_n \in B_r(z)$. Given $w \in M$, let $R = r + d(z, w)$. By Δ-inequality

$$B_r(z) \subset B_R(w)$$

o e.g. in $\mathbb{R}^n, \mathbb{C}^n$ or $C[a, b]$ if the metric comes from a norm $\|\cdot\|$, then (x_n) is bounded $\iff$ $\exists C \in \mathbb{R}^+ \ \|x_n\| \subseteq C \ \forall n$

Lemma 3.1. Convergent $\implies$ Cauchy $\implies$ bounded

Proof. Let (x_n) be a sequence in a metric space M. If (x_n) is convergent in M, let $x = \lim_{n\to\infty} x_n$. Given $\varepsilon > 0$, we have $N \in \mathbb{N}$ s.t.

$$\forall n \geq N \ d(x_n, x) < \varepsilon$$

Then

$$\forall m, n \geq N \ d(x_m, x_n) \leq d(x_m, x) + d(x, x_n) < 2\varepsilon$$

So (x_n) is Cauchy.
Now assume (x_n) in M, there is $N \in \mathbb{N}$ s.t.

$$\forall m, n \geq N \ d(x_m, x_n) \leq 1$$

Let $r = \max\{d(x, x_N), d(x_2, x_N), \ldots, d(x_{N-1}, x_N), 1\}$. Then $x_n \in B_r(x_N)$ for all $n \in \mathbb{N}$. So (x_n) is bounded

Note. Bounded does not imply Caucy e.h. $0, 1, 0, 1, 0, 1, \ldots$ in $\mathbb{R}$.
Cauchy does not imply convergeny e.g. $x_n = 1/n$ in $(0, \infty)$

Definition. A metric space M is **complete** if every Cauchy sequence in M converges in M

Example. $\mathbb{R}$ and $\mathbb{C}$ are complete

Prop 3.2. If M, M' are complete metric spaces, then so is $M \oplus_p M'$ $(p = 1, 2, \infty)$

Proof. Let (a_n) be a Cauchy sequence in $M \oplus_p M'$. Write $a_n = (x_n, x'_n)$ where $x_n \in M, x'_n \in M'$ $(n \in \mathbb{N})$.
Given $\varepsilon > 0$, there is $N \in \mathbb{N}$ $\forall m, n \geq N$ $d_p(a_m, a_n) < \varepsilon$. Then $\forall m, n \geq N$

$$d(x_m, x_n) \leq \max\{d(x_m, x_n), d'(x'_m, x'_n)\} \leq d_p(a_m, a_n) < \varepsilon$$

So (x_n) is Cauchy in M, similarly (x'_n) is Cauchy is M'.
Since M, M' are complete, (x_n) and (x'_n) are convergent in M, M' respectively to, say, x and x' respectively. Set $a = (x, x')$. Then

$$d_p(a_n, a) \leq d_1(a_n, a) = d(x_n, x) + d'(x'_n, x') \to 0 \text{ as } n \to \infty$$

So $a_n \to a$ in $M \oplus_p M'$

Note. (a_n) is Cauchy in $M \oplus_p M'$ $\iff$ (x_n) Cauchy in M and (x'_n) Cauchy in M'

Corollary 3.3. $\mathbb{R}^n$ and $\mathbb{C}^n$ are complete in the l_p-metric for $p = 1, 2, \infty$. In particular, n-dimensional real or complex euclidean space is complete.

Theorem 3.4. Let S be any set, then $l_\infty(S)$ is complete in the uniform metric D

Proof. Let (f_n) be a Cauchy sequence in $l_\infty(S)$. Given $\varepsilon > 0$, there is $N \in \mathbb{N}$ s.t. $\forall m, n \geq N$

$$D(f_m, f_n) = \sup_{x \in S} |f_m(x) - f_n(x)| < \varepsilon$$

i.e. $\forall m, n \geq N$ $\forall x \in S$ $|f_m(x) - f_n(x)| < \varepsilon$. So (f_n) is uniformly Cauchy as defined in Chapter 1. By Theorem 1.6, (f_n) is uniformly convergent. So there's a calar function f on S s.t. $f_n \to f$ uniformly on S. By lemma 1.2, f is bounded, i.e. $f \in l_\infty(S)$. Given $\varepsilon > 0$, there is $N \in \mathbb{N}$

$$\forall n \geq N \ \forall x \in S \ |f_n(x) - f(x)| < \varepsilon$$

so

$$\forall n \geq N \ \sup_{x \in S} |f_n(x) - f(x)| = D(f_n, f) \leq \varepsilon$$

So $f_n \to f$ in $(l_\infty(S), D)$

Prop 3.5. Let N be a subspace of a metric space M.
(i) If N is complete, then N is closed in M
(ii) If M is complete and N is closed in M, then N is complete
So in a complete metric space, a subspace is complete iff closed.

Proof.
(i) Let (x_n) be a sequence in N and assume $x_n \to x$ in M. We need: $x \in N$. (x_n) is convergent in M, so (x_n) is Cauchy in M (Lemma 1) so (x_n) is Cauchy in N. Since N is complete $x_n \to y$, say in N. So $x_n \to y$ in M. Thus $x = y \in N$
(ii) Let (x_n) be a Cauchy squence in N. Then (x_n) is Cauchy in M. Since M is complete, $x_n \to x$ in M for some $x \in M$. Since N is closed in M, we have $x \in N$ so $x_n \to x$ in N.

Definition. Let (M, d) be a metric space. Define

$$C_b(M) = \{f \in l_\infty(M) : f \text{ is continuous}\}$$

This is a subspace of $l_\infty(M)$ in the uniform metric D.

Theorem 3.6. $C_b(M)$ is complete in the uniform metric

Proof. By Theorem 4 and Prop 5 (ii), it is enough to show that $C_b(M)$ is closed in $l_\infty(M)$. So let (f_n) be a sequence in $C_b(M)$ and assume $f_n \to f$ in $l_\infty(M)$. We need: f is continuous. Fix $a \in M$ and $\varepsilon > 0$. Same 3ε proof works as in section 1.

Corollary 3.7. $C[a, b]$, the space of continuous functions on the closed bounded interval $[a, b]$ is complete in the uniform metric

Proof. $C[a, b] = C_b[a, b]$ from IA Analysis

Definition. Let S be a set and (N, e) be a metric space. Let

$$l_\infty(S, N) = \{f : S \to N : f \text{ is bounded}\}$$

f is **bounded** if $\exists y \in N, r > 0$ s.t. $\forall x \in S\ f(x) \in B_r(y)$.
If $g : S \to N$ is aother bounded function, say $\forall x \in S\ g(x) \in B_s(z)$ for some $z \in N, s > 0$ then $\forall x \in S$

$$e(f(x), g(x)) \leq e(f(x), y) + e(y, z) + e(z, g(x)) \leq r + e(y, z) + s$$

So $\sup_{x \in S} e(f(x), g(x))$ exists and we denote this by $D(f, g)$. It's routine to verify that D is a metric, called the **uniform metric** on $l_\infty(S, N)$.

Definition. Now assume $S = M$, where (M, d) is a metric space. We define

$$C_b(M, N) = \{f : M \to N : f \text{ is continuous and bounded}\}$$

Note that $C_b(M, N)$ is a subspace of $l_\infty(M, N)$ with the uniform metric

Theorem 3.8. Let S be a set, (M, d) and (N, e) be metric spaces. Assume (N, e) is complete. Then

(i) $l_\infty(S, N)$ is complete in the uniform metric D.

(ii) $C_b(M, N)$ is complete in the uniform metric D

Proof. (i) Let (f_n) be a Cauchy sequence in $l_\infty(S, N)$. We show (f_n) is pointwise Cauchy. Fix $x \in S$. Given $\varepsilon > 0$, there is $K \in \mathbb{N}$ s.t. $\forall i, j \geq K$

$$D(f_i, f_j) < \varepsilon$$

In particular, $e(f_i(x), f_j(x)) \leq D(f_i, f_j) < \varepsilon$ for $i, j \geq K$.
So $(f_k(x))_{k\in\mathbb{N}}$ is Cauchy in N. Since N is complete, it's convergent in N. This holds for every $x \in S$, so we can define $f : S \to N$

$$f(x) = \lim_{k\to\infty} f_k(x)$$

First we show f is bounded, i.e. $f \in l_\infty(S, N)$. Since (f_k) is Cauchy in D, there is a $K \in \mathbb{N}$ s.t. $\forall i, j \geq K$ $D(f_i, f_j) < 1$ so $\forall i \geq K$ $D(f_i, f_K) < 1$.
f_K is bounded so $\exists y \in N, r > 0 \forall x \in S f_K(x) \in B_r(y)$. Fix $x \in S$. $\forall i \geq K$ $e(f_i(x), f_K(x)) \leq D(f_i, f_K) < 1$. Letting $i \to \infty$

$$e(f(x), f_K(x)) \leq 1$$

So $e(f(x), y) \leq e(f(x), f_K(x)) + e(f_K(x), y) \leq 1 + r$.
Hence $f(x) \in B_{r+1}(y)$. This holds for every $x \in K$, so f is bounded.
Finally, we prove $f_k \to f$ in D. Given $\varepsilon > 0$,there is $K \in \mathbb{N}$ s.t. $\forall i, j \geq K$, $D(f_i, f_j) < \varepsilon$. Fix $i \geq K$, fix $x \in S$. Then

$$\forall j \geq K \ e(f_j(x), f_i(x)) \leq D(f_i, f_j) < \varepsilon$$

Letting $j \to \infty$,

$$e(f(x), f_i(x)) \leq \varepsilon$$

x was arbitrary so $D(f, f_i) \leq \varepsilon$. This holds for every $i \geq K$.

(ii) By part (i) and prop 5 (ii), enough to show that $C_b(M, N)$ is closed in $l_\infty(M, N)$. So let (f_k) be a sequence in $C_b(M, N)$ and assume $f_k \to f$ in $l_\infty(M, N)$. We need: f is continuous. Fix $a \in M$, and use 3ε proof.

Definition. A function $f : M \to M'$ between metric spaces is a **contraction mapping** if $\exists \lambda, 0 \leq \lambda < 1$ s.t.

$$\forall x, y \in M \quad d'(f(x), f(y)) \leq \lambda d(x, y)$$

i.e. f is λ-Lipschitz, so contraction

Theorem 3.9 (Contraction mapping Theorem, CMT, or Banach's fixed point theorem)**.** Let M be a non-empty, complete metric space and $f : M \to M$ a contraction mapping. Then f has a unique fixed point, i.e., $\exists$ unique $z \in M$ s.t. $f(z) = z$

Proof. Let λ be such that $0 \le \lambda < 1$ and

$$\forall x, y \in M \quad d(f(x), f(y)) \le \lambda d(x, y)$$

. Uniqueness: If $f(z) = z$ and $f(w) = w$, then

$$d(z, w) = d(f(z), f(w)) \le \lambda d(z, w)$$

Since $\lambda < 1$, we have $d(z, w) = 0$ i.e. $z = w$.
Existence: Fix $x_0 \in M$ and set $x_n = f(x_{n-1})$ for $n \in \mathbb{N}$, i.e.

$$x_n = \underbrace{f(f(\ldots(f}_{n \text{ times}}(x_0))))$$

(Our idea is to have $z = f^\infty(x_0)$ then $f(z) = z$)
Fix $n \in \mathbb{N}$

$$d(x_n, x_{n+1}) = d(f(x_{n-1}), f(x_n)) \le \lambda d(x_{n-1}, x_n) \le \cdots \le \lambda^n d(x_0, x_1)$$

For $m > n$

$$d(x_n, x_m) \le \sum_{k=n}^{m_1} d(x_k, x_{k+1}) \le \sum_{k=n}^{m-1} \lambda^j d(x_0, z_1) \le \frac{\lambda^n}{1-\lambda} d(x_0, x_1)$$

Since $\lambda^n/(1-\lambda)d(x_0, x_1) \to 0$, given $\varepsilon > 0$, $\exists N \in \mathbb{N}\ \forall n \ge N$

$$\frac{\lambda^n}{1-\lambda} d(x_0, x_1) < \varepsilon$$

So $\forall m \ge n \ge N$, $d(x_n, x_m) < \varepsilon$. We proved (x_n) is Cauchy.
M is complete so $x_n \to z$, say, in M as $n \to \infty$. f is continuous, so $f(x_n) \to f(z)$
Also $f(x_n) = x_{n+1} \to z$ thus $f(z) = z$.

Remarks.

(i) Letting $m \to \infty$ in the inequality for $d(x_n, x_m)$, we get

$$d(x_n, z) \le \frac{\lambda^n}{1-\lambda} d(x_0, x_1)$$

So $x_n \to z$ exponentially fast.

(ii) $f : \mathbb{R}\backslash\{0\} \to \mathbb{R}\backslash\{0\}$, $x \mapsto \frac{x}{2}$ is a contraction ($\lambda = 1/2$), but has no fixed point

(iii) $f : \mathbb{R} \to \mathbb{R}$, $c \mapsto x + 1$ is isometric ($\lambda = 1$), but not fixed point $f : [1, \infty) \to [1, \infty]$, $x \mapsto x + 1/x$

$$\forall x, y \in [1, \infty), x \ne y,\ |f(x) - f(y)| < |x - y|$$

$[1, \infty)$ is closed in $\mathbb{R}$ therefore complete

Example. An application: let $y_0 \in \mathbb{R}$. The initial value problem

$$f'(t) = f(t^2),\ f(0) = y_0$$

has a unique solution on $[0, 1/2]$ i.e. $\exists$ unique differentiable function $f : [0, 1/2] \to \mathbb{R}$ s.t. $f(0) = y_0$ and $f'(t) = f(t^2)\ \forall t \in [0, 1/2]$.

- If f is a solution, then $f \in C[0, 1/2]$ and FTC, it satisfies

  $$f(t) = y_0 + \int_0^t f(s^2)\,\mathrm{d}s$$

  (note: $f'(s) = f(s^2)$ is continuous)
  Conversely, if

  $$f \in X[0, 1/2] \text{ and } f(t) = t_0 + \int_0^t f(s^2)\,\mathrm{d}s\ \forall t \in [0, 1/2]$$

  then f is a solutoin to the initial value problem.
- Let $M = C[0, 1/2]$ with the uniform metric. This is non-empty and complete (cor 7). Define $T : M \to M,\ g \mapsto Tg$ where

  $$(Tg)(t) = y_0 + \int_0^t g(s^2)\,\mathrm{d}s,\ t \in [0, 1/2]$$

  Tg is well-defined as $s \mapsto g(s^2)$ is continuous and by FTC we have Tg differentiable and $(Tg)'(t) = g(t^2)$. So $Tg \in M$. Step 1 says: f is a solution to the IVO $\iff$ $f \in M$ and $Tf = f$
- T is a contraction. Let $g, h \in M$. For $t \in [0, 1/2]$,

  $$|(Tg)(t) - (Th)(t)| = |\int_0^t g(s^2) - h(s^2)\,\mathrm{d}s| \leq t \cdot \sup_{s \in [0,1/2]} |g(s^2) - h(s^2)| \leq \frac{1}{2} D(g, h)$$

  sup over t yields

  $$D(Tg, Th) \leq \frac{1}{2} D(g, h)$$
- By CMT, T has a unique fixed point, so by step 2, IVP has unique solution

Remark. The above shows that for any $\delta \in (0, 1)$, there is a unique solution to the IVP on $[0, \delta]$ - call this $f\delta$. For $0 < \delta < \mu < 1$, $f_\mu|_{[0,\delta]} = f_\delta$ by uniqueness. So the IVP has unique solution on $[0, 1)$.

Theorem 3.10 (Lindelof-Picard)**.** We are given $n \in \mathbb{N}, a, b, R \in \mathbb{R}$ with $a < b, R > 0$ and a continuous function

$$\varphi : [a, b] \times B_R(y_0) \to \mathbb{R}^n$$

where $y_0 \in \mathbb{R}^n$. We assume that $\exists K > 0$ s.t.

$$\forall t \in [a, b]\ \forall x, y \in B_R(y_0),\ \|\varphi(t, x) - \varphi(t, y)\| \leq K\|x - y\|$$

Then $\exists \varepsilon > 0$ s.t. for any $t_0 \in [a, b]$ the IVP

$$f'(t) = \varphi(t, f(t)) \text{ and } f(t_0) = y_0$$

has a unique solution on $[c, d] = [t_0 - \varepsilon, t_0 + \varepsilon] \cap [a, b]$
In other words, there is a unique differentiable function $f : [c, d] \to \mathbb{R}^n$ s.t. $f'(t) = \varphi(t, f(t))\ \forall t \in [x, d]$ and $f(t_0) = y_0$

Proof. By lemma 2.10, $B_R(y_0)$ is closed subset of $\mathbb{R}^n$ so φ is a continuous function on the closed and bounded set $[a, b] \times B_R(y_0)$, and hence φ is bounded.
Set $C = \sup\{\|\varphi(t, x)\| : t \in [a, b],\ x \in B_R(y_0)\}$ and set $\varepsilon = \min(R/C, 1/(2k))$. We will show this works. Fix $t_0 \in [a, b]$ and let $[c, d] = [t_0 - \varepsilon, t_0 + \varepsilon] \cap [a, b]$ We need: $\exists$ unique differentiable $f : [c, d] \to \mathbb{R}^n$ .s.t $f(t_0 = y_0)$ and $f'(t) = \varphi(t, f(t))\ \forall t \in [c, d]$
Since $V_R(y_0)$ is closed in $\mathbb{R}^n$ and since $\mathbb{R}^n$ is complete, by prop 5 (ii), $B_R(y_0)$ is complete.
By theorem 8, $M = C([c, d], B_R(y_0))$ is complete in the uniform metric D. Also $M \neq \varnothing$. Then f is a solution of the IVP avove iff $f \in M$ and

$$f(t) = y_0 + \int_{t_0}^{t} \varphi(s, f(s))\,\mathrm{d}s$$

This follows from the FTC (applied coordinate wise)
We define $T : M \to M,\ g \mapsto Tg$ where

$$(Tg)(t) = y_0 + \int_{t_0}^{t} \varphi(s, g(s))\,\mathrm{d}s,\ t \in [c, d]$$

We show that T is well defined: $s \mapsto \varphi(s, g(s))$ is continuous so integrable and by FTC, Tg is differentiable and

$$(Tg)'(t) = \varphi(t, g(t))\ \forall t \in [c, d$$

so in particular $Tg : [c, d] \to \mathbb{R}^n$ is continuous. Finally, for $t \in [c, d]$

$$\|(Tg)(t) - y_0\| = \|\int_{t_0}^{t} \varphi(s, g(s))\,\mathrm{d}s\| \leq \|t - t_0\| \sup_{s \in [c,d]} \|\varphi(s, g(s))\| \leq \varepsilon \cdot C \leq R$$

So $Tg \in M$. By the earlier observation, f is a solution of the IVP $\iff$ $f \in M$ and $Tf = f$.
T is a contraction: Let $g, h \in M$. For $t \in [c, d]$

$$(\|Tg)(t) - (Th)(t)\| = \|\int_{t_0}^{t} \varphi(s, g(s)) - \varphi(s, h(s))\,\mathrm{d}s\|$$

Note

$$\|\varphi(s, g(s)) - \varphi(s, h(s))\| \leq K \cdot \|g(s) - h(s)\| \leq K \cdot D(g, h)$$

So

$$\|(Tg)(t) - (Th)(t)\| \leq |t - t_0| \cdot K \cdot D(g, h) \leq \varepsilon LD(g, h)$$

Take sup over all $t \in [c, d]$

$$D(Tg, Th) \leq \varepsilon KD(g, h) \leq \frac{1}{2}D(g, h)$$

Finally by CMT, T has unique fixed point in M.

Notes.

(i) To say f is a soltion of the IVP above implicitly includes the assumption that $f(t) \in B_R(y_0) \forall t \in [c,d]$

(ii) Given a function $f : [c,d] \to \mathbb{R}^n$, let $f_k : [c,d] \to \mathbb{R}$ be the kth component of f, $1 \leq k \leq n$. I.e. $f_k = q_k \circ f$, where $q_k : \mathbb{R}^n \to \mathbb{R}, \ (y_1, \ldots, y_n) \mapsto y_k$.
So $f(t) = (f_1(t), f_2(t), \ldots, f_n(t)) \ \forall t \in [1,d]$.
f is differentiable iff each f_k is differentiable and

$$f'(t) = (f_1'(t), d_2'(t), \ldots, f_n'(t)), \ t \in [c,d]$$

If f is constant, then so are f_k and hence each f_k is differentiable. We define $\int_c^d f(t)\,\mathrm{d}t$ to be the elment $v = (v_1, v_2, \ldots, v_n) \in \mathbb{R}^n$ where

$$v_k = \int_c^d f_k(t)\,\mathrm{d}t$$

$$\|v\|^2 = \sum_{k=1}^n v_k^2 = \sum_{k=1}^n v_k \cdot \int_c^d f_k(t)\,\mathrm{d}t = \int_c^d \sum_{k=1}^n v_k f_k(t)\,\mathrm{d}t$$

$$\implies \|v\|^2 \leq \int_c^d \|v\| \|f(t)\|\,\mathrm{d}t = \|v\| \int_c^d \|f(t)\|\,\mathrm{d}t$$

We proved that

$$\| \int_c^d f(t)\,\mathrm{d}t \| \leq \int_c^d \|f(t)\|\,\mathrm{d}t \leq (d-c) \cdot \sup_{t \in [c,d]} \|f(t)\|$$

Remarks.

(i) For any $S \in (0,1)$, taking $\varepsilon = \min(R/C, \delta/K)$ works. It follows that with $\varepsilon = \min(R/C, 1/K)$, the IVP $f'(t) = \varphi(t, f(t))$ and $f(t_0) = y_0$ has unique solution on $(t_0 - \varepsilon, t_0 + \varepsilon) \cap [a,b]$

(ii) In general there's no solution on $[a,b]$

(iii) Theorem 10 can handle nth order ODEs for any $n \in \mathbb{N}$ (see pdf on lecturer webpage)

4 Topological Spaces

Definition. Let X be a set. A **topology on X** is a family τ of subsets of X (i.e. $\tau \subset \mathbb{P}(X)$) s.t.

(i) $\varnothing, X \in \tau$

(ii) if $U_i \in \tau \ \forall i \in I$ (I is some index set), then

$$\bigcup_{i \in I} U_i \in \tau$$

(iii) If $U, V \in \tau$ then $U \cap V \in \tau$

A **topological space** is a pair (X, τ) where X is a set and τ is a topology on X. Members of τ are called **open** sets of the topology. So $U \subset X$ is open in X (or is an **open subset of X**) if $U \in \tau$. We sometimes say that U is **τ-open**

Note. If $U_i \in \tau$ for $i = 1, \ldots, n$ then

$$\bigcap_{i=1}^{n} U_i \in \tau$$

Examples. (i) Metric topologies: Let (M, d) be a metric space. Recall $U \subset M$ is open in the metric sense if $\forall x \in U \ \exists r > 0 \ B_r(x) \subset U$. We sometimes say U is **d-open**. Prop 2.9 we proved that the family of d-open sets is a topology on M

Definition. Let (X, τ) be a topological space. Say X is **metrisable** (or τ is metrisable) if $\exists$ metric d on X s.t. τ is the metric topology on X induced by d.
I.e. $U \subset X$ is τ-open $\iff$ U is d-open. If d' is another metric equivalent to d then d' also induces the same topology I on X

Examples. (ii) The indiscrete topology on a set X is $\tau = \{\varnothing, X\}$. If $|X| \geq 2$, then this is not metrizable. Let d be a metric on X. Fix $x \neq y$ in X and set $r = d(x, y)$ and $U = D_r(x)$. By lemma 2.5, U is d-open, $x \in U$ but $y \notin U$ so $U \notin \tau$

Definition. If τ_1, τ_2 are two topologies on a set X say τ_1 is **coarser** than τ_2 or that τ_2 is **finer** than τ_1 if

$$\tau_1 \subset \tau_2$$

e.g. The discrete topology on X is the coarsest topology on X.

Examples. (iii) The **discrete topology** on a set X is $\mathbb{P}(X)$. This is the finest topology on X. This is metrisable: by the discrete metric.

(iv) The cofinite topology on a set X is

$$\tau = \{\varnothing\} \cup \{U \subset X : U \text{ is cofinite in } X\}$$

When X is finite, $\tau = \mathbb{P}[X]$. When X is infinitem then τ is not metrisable. Let $x \neq y$ in X and assume $x \in U, y \in V$ and U, V are open in X. Then U, V are cofinite, and hence $U \cap V \neq \varnothing$

Definition. We say a topological space X is **Hausdorff** if $\forall x \neq y$ in X $\exists$ open sets $U, V \subset X$ s.t., $x \in U, y \in V$, $U \cap V = \varnothing$. (We say x, y are separated by open sets)

Note. The cofinite topology on an open set is not Hausdorff

Prop 4.1. Metric spaces are Hausdorff

Proof. Let $x \neq y$ be points in a metric space (M, d). Let $r > 0$ be s.t. $2r < d(x, y)$. Set $U = D_r(x)$, $V = D_r(y)$. Then U, V are open (lemma 2.5)m $x \in U, y \in V$ and if $z \in U \cap V$ then

$$d(x, y) \leq d(x, z) + d(z, y) < r + r = 2r < d(x, y) ※$$

So $U \cap V = \varnothing$

Note. This shows that the cofinite topology on an ∞ is not metrisable

Definition. A subset A of a topological space (X, τ) is **closed in X** (or is a **closed subet of X** or **τ-closed**) if $X \backslash A$ is open in X

Note. In a metric space, this agrees with the earlier definition by Lemma 2.11

Prop 4.2. The collection of closed sets in a topological space X satisfy the following:

(i) $\varnothing, X$ are closed
(ii) If A_i is closed in X $\forall i \in I$, where $I \neq \varnothing$ index set, then $\bigcap_{i \in I} A_i$ is closed in X
(iii) If A, B are closed in X, then $A \cup B$ is closed

Examples. (i) In a discrete topological space, every set is closed
(ii) In the cofinite topology on a set X, a subset A is closed iff $A = X$ or A is finite

Definition. Let X be a topological space, $U \subset X, x \in X$. We say U is a **neighbourhood** of x in X if $\exists$ open set V in X s.t. $x \in V \subset U$

Note. In a metric space, this agrees wiht the earlier definition by Corollary 2.6

Prop 4.3. Let U be a subset of a topological space X. Then U is open iff U is a neighbourhood of x for every $x \in U$

Proof. $\implies$: Let $x \in U$. Then set $V = U$. Then V is open, $x \in V \subset U$.
$\impliedby$: For each $x \in U$, we can take an open set V_x in X s.t. $x \in V_x \subset U$ then

$$U = \bigcup_{x \in U} V_x \text{ is open}$$

Definition. Let (x_n) be a sequence in a top space X and let $x \in X$. We say **(x_n) converges to x** (write $x_n \to x$) if

$$\forall \text{ neighbourhoods } U \text{ of } x \text{ in } X, \exists N \in \mathbb{N}\ \forall n \geq N\ x_n \in U$$

Equivalently (prop 3): $\forall$ open sets U with $x \in U\ \exists N \in \mathbb{N}\ \forall n \geq N\ x_n \in U$

Note. In a metric space, this agrees with the earlier definition by prop 2.7

Examples. (i) Eventually constant sequences: if $\exists z \in X\ \exists N \in N\ \forall n \geq N\ x_n = z$ then $x_n \to z$

(ii) In an indiscrete top. space, every sequence converges to every point

(iii) Consider a set X with the cofinite topology. Assume $x_n \to x$ in X. If $y \neq x$ then $X \backslash \{y\}$ is a neighbourhood of x, so $N_y = \{n \in \mathbb{N} : x_n = y\}$ is finite.
Conversely, assume (x_n) a sequence in X s.t. for some $x \in X\ \forall y \neq x\ N_y$ is finite. Then $x_n \to x$.
Thus, if N_y is finite $\forall y \in X$, then $x_n \to y\ \forall y \in X$

Prop 4.4. If $x_n \to x$ and $x_n \to y$ in a Hausdorff space, then $x = y$.

Proof. Assume $x \neq y$. Choose open sets U, V s.t. $x \in U, y \in V, U \cap V = \varnothing$. Since $x_n \to x$, there is $N_1 \in \mathbb{N}\ \forall n \geq N_1\ x_n \in U$. Since $x_n \to y$, there is $N_2 \in \mathbb{N}\ \forall n \geq N_2\ x_n \in V$. For any $n \geq \max(N_1, N_2)$, we have $x_n \in U \cap V$

Remark. If $x_n \to x$ in a Hausdorff space, then we sometimes write $x = \lim_{n\to\infty} x_n$

Note. In a metric space, for a subset A, we have A is closed $\iff$ whenever $x_n \to x$ in the space with $x_n \in A$ for all n, we have $x \in A$
In a general topological space, " $\implies$ " is true, but " $\impliedby$ " is not

Definition. Let X be a topological space and $A \subset X$. The **interior of A in X** (denoted A^0 or $\text{int}(A)$) ot be

$$A^0 = \text{ int}(A) = \bigcup\{U \subset X : U \text{ is open in } X, U \subset A\}$$

We define the **closure of A in X** (denoted $\bar{A}$ or $\text{cl}(A)$) to be the set

$$\bar{A} = \bigcap\{F \subset X : F \text{ closed in } X, F \supset A\}$$

Note. (i) A^0 is open in X, $A^0 \subset A$, moreover U is open in X and $U \subset A$, then $U \subset A^0$. So A^0 is the largest (wrt inclusion) open set contained in A

(ii) $\bar{A}$ is closed, $\bar{A} \supset A$, moreover if F is closed in X and $F \supset A$, then $F \supset \bar{A}$. So $\bar{A}$ is the smallest closed set containing A

Prop 4.5. Let X be a topological space and $A \subset X$. Then

(i)
$$A^0 = \{x \in X : A \text{ is a neighbourhood of } x\}$$

(ii)
$$\bar{A} = \{x \in X : \forall \text{ neighbourhoods } U \text{ of } x, U \cap A \neq \varnothing\}$$

Proof. (i) A is a neighbourhood of $x \iff \exists$ open set U s.t. $x \in U \subset A \iff x \in A^0$

(ii) Suppose $x \notin \bar{A}$. Then $\exists$ closed set $F \supset A$ s.t. $x \notin F$. Set $U = X \backslash F$. Then U is open and $x \in U$. So U is a neighbourhood of x and $U \cap A = \varnothing$.
Suppose $\exists$ neighbourhood U of x s.t. $U \cap A = \varnothing$. There is open set V s.t. $x \in V \subset U$. Then $V \cap A = \varnothing$. Set $F = X \backslash V$. Then F is closed and $A \subset F$. Then $\bar{A} \subset F$ and so $x \notin \bar{A}$

Examples. In $\mathbb{R}$ let $A = [0, 1) \cup \{2\}$. Then $A^0 = (0, 1)$, $\bar{A} = [0, 1] \cup \{2\}$.
$\mathbb{Q}^0 = \varnothing, \bar{Q} = \mathbb{R}$. $\mathbb{Z}^0 = \varnothing, \bar{Z} = \mathbb{Z}$

Note. In a metric space, for a subset A, we have $x \in \bar{A} \iff \exists (x_n)$ in A s.t. $x_n \to x$. In a general topological space, " $\Longleftarrow$ " is true, but " $\Longrightarrow$ " is false

Definition. A subset A of a topological space X is **dense** in X if $\bar{A} = X$. We say X is **separable** if $\exists$ countable $A \subset X$ s.t. A is dense in X

Examples. $\mathbb{R}$ is separable as $\mathbb{Q}$ is dense in $\mathbb{R}$ and $\mathbb{R}^n$ is separable as $\mathbb{Q}^n$ is dense in $\mathbb{R}^n$. An uncountable discrete topological space is NOT separable

4.1 Subspaces

Definition. Let (X, τ) be a topological space and $Y \subset X$. The **subspace topology** or **relative topology** on Y induced by τ is the topology

$$\{V \cap Y : V \text{ is an open subset of } X\}$$

on Y. We sometimes deonite this by $\tau|_Y$. So for $U \subset Y$, U is open in $U \iff \exists$ open set V in X such that $U = V \cap Y$

Example. $X = \mathbb{R}$, $Y = [0, 2], U = (1, 2]$. $U \subset Y \subset X$. U is open in Y e.g. $(1, 3)$is open in X and $U = V \cap Y$.
U is not open in X:
$$\forall r > 0, \{y \in X : |y - 2| < r\} \not\subset U$$

Remarks.

(i) A subset of a topological space will always be given the subspace topology unless written otherwise stated

(ii) Let (X, τ) be a topological space and $Z \subset Y \subset X$. Two natural topologies on Z: Think $Z \subset X$, Z has $\tau|_Z$. Or think $Z \subset Y$, Z has $(\tau|_Y)|_Z$. These are the same.

(iii) Let (M, d) be a metric space and $N \subset M$. There are two natural topologies on N: think of N on a metric subspace of (M, d) with the metric $d|_{N\times N}$ which induces the metric topology on N. Or, d induces the metric topology on M, which in turn induces the relative topology on N. Reason: for $x \in N, r > 0$

$$\{y \in N : d(y, x) < r\},\ \{y \in M : d(y, x) < r\} \cap N$$

Prop 4.6. Let X be a topological space, $A \subset Y \subset X$.

(i) A is closed in $Y \iff \exists$ closed subset B of X s.t. $A = B \cap Y$

(ii)

$$Cl_Y(A) = Cl_X(A) \cap Y$$

(closure of A in Y)

Remark. (ii) is false for interior in general e.g. $X = \mathbb{R}, A = Y = \{0\}$, $\text{int}_Y(A) = A$, $\text{int}_X(A) = \varnothing$

Proof. (i) If A is closed in Y, $Y \backslash A$ is open in Y. So by def $Y \backslash A = V \cap Y$ for some open V in X. Then $B = X \backslash V$ is closed in X and $A = B \cap Y$.
If $A = B \cap Y$, B closed in X, then $X \backslash B$ is open in X, and hence $Y \backslash A = (X \backslash B) \cap Y$ is open in Y.

(ii) $Cl_X(A)$ is closed in X, so by (i), $Cl_X(A) \cap Y$ is clsoed in Y. Also, $A \subset Cl_X(A) \cap Y$. So $Cl_Y(A) \subset CL_X(A) \cap Y$.
Also, $Cl_Y(A)$ is closed in Y, so by (i), $Cl_Y(A) = B \cap Y$ for some closed set B in X. Then $A \subset B$ and B is closed in X, so $\text{Cl}_X(A) \subset B$, and hence $Cl_Y(A) = B \cap Y \supset Cl_X(A) \cap Y$

Note. $U \subset Y \subset X$, Y is open in X. Then U is open in $Y \iff U$ is open in X

4.2 Continuity

Definition. A function $f : X \to Y$ between topological spaces is **continuous** if $\forall$ open sets V in Y, $f^{-1}(V)$ is open in X.

Note. For functions between metric spaces, this agreeswith the ε-δ definition of continuity by prop 2.8

Examples. (i) Constant functions $f : X \to Y$, for some $y_0 \in Y$, we have $\forall x \in X, f(x) = y_0$. For any $V \subset Y$, we have

$$f^{-1}(V) = \begin{cases} \varnothing & y_0 \notin V \\ X & y_0 \in V \end{cases}$$

so f is continuous

(ii) Identity $f : X \to X, f(x) = x$. For $V \subset X, f^{-1}(V) = V$

(iii) Inclusion $Y \subset X,\ i : Y \to X,\ i(y) = y\ \forall y \in Y$. For open set V in X, $i^{-1}(V) = V \cap Y$ which is open in Y by definition. If $g : X \to Z$ is continuous, then $g|_Y = g \circ i$ is continuous (see next prop)

Prop 4.7. Let $f : X \to Y$ be a function between topological spaces

(i) f is continuous $\iff$ $\forall$ clsoed sets B in Y, $f^{-1}(B)$ is closed in X

(ii) If f is continuous and $g : Y \to Z$ is another continuous function, then $g \circ f$ is continuous

Proof. (i) For any $D \subset Y$, $f^{-1}(Y \backslash D) = X \backslash f^{-1}(D)$. Now use the fact that $A \subset X$ (or Y) is open in X (resp Y) $\iff$ $X \backslash A$ (resp $Y \backslash A$) is closed in X (resp Y)

(ii) If W is an open subset of Z, then $g^{-1}(W)$ is open in Y since g is continuous, and $f^{-1}(g^{-1}(W))$ is open in X since f is continuous. So $f^{-1}(g^{-1}(W)) = (g \circ f)^{-1}(W)$ is open in X. Thus $g \circ f$ is continuous

Remark. There is a notion of continuity at a point (Kelly: General Topology)

Definition. A function $f : X \to Y$ between topological spaces is a **homeomorphism** if f is a bijection and both f and f^{-1} are continuous. If such f exits, we say X and Y are homeomorphic.
A property $\mathcal{P}$ of topological spaces is a **topological property** or **topological invariant** if $\forall$ pairs X, Y of homeomorphic topological spaces X has $\mathcal{P} \iff Y$ has $\mathcal{P}$

Examples. (i) Being metrizable

(ii) Being Hausdorff

(iii) Being complete metrixable is NOT a topological invariant. E.g. on $\mathbb{R} \exists$ metrics d, d' s.t. $d \sim d'$, d is complete, d' is not

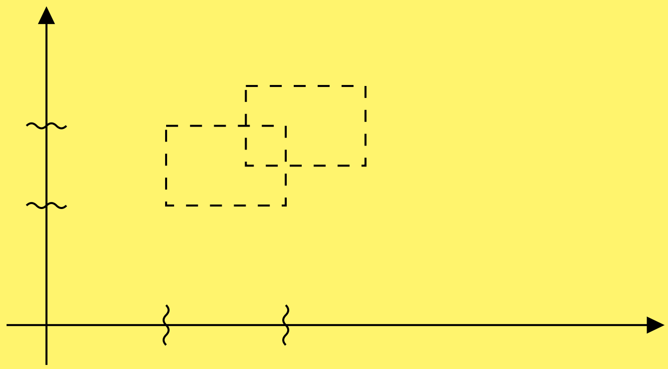
Note. If $f : C \to Y$ is a homeomorphism, then for an open set U in X, $f(U) = (f^{-1})^{-1}(U)$ is open in Y since $f^{-1} : Y \to X$ is continuous

Definition. A function $f : X \to Y$ between topological spaces is an **open map** if $\forall$ open sets U in X, $f(U)$ is open in Y

Note. $f : X \to Y$ is a homeomorphism $\iff$ f is a continuous and open bijection

4.3 Product Topology

Moral. Let X, Y be topological spaces, we want to define a topology on $X \times Y$. We want if U open in X, V open in Y, then $U \times V$ open in $X \times Y$ Have $\varnothing = \varnothing \times \varnothing$, $X \times Y = X \times Y$, $U \times V \cap U' \times V' = (U \cap U') \times (V \cap V')$



We also declare unions $\bigcup_{i \in I} U_I \times V_i$ where U_i open in X, V_i open in Y $\forall i \in I$ open in $X \times Y$

Definition. The **product topology** on $X \times Y$ consists of all sets of the form $\bigcup_{i \in I} U_i \times V_i$, where I is arbitrary, $\forall i \in I$ U_i open in X and V_i open in Y. This is a topology on $X \times Y$

Note. For $W \subset X \times Y$, we have W is open $\iff$ $\forall z \in W \exists$ open sets U in X, V in Y s.t. $Z \in U \times V \subset W$. For $W \subset X \times Y$ and $z = (x, y) \in X \times Y$, W is a neighbourhood of z $\iff$ $\exists$ neighbourhood U of x in X, V of y in Y s.t. $U \times V \subset W$

Example. Let (M, d), (M', d') be metric spaces. We have a metric d_∞ on $M \times M'$:

$$d_\infty((x, x'), (y, y')) = \max\{d(x, y), d'(x', y')\}$$

This induces the metric topology on $M \times M'$. Also M and M' are topological spaces with their metric topologies, which in turn gives the product topology on $M \times M'$. For $z = (x, x') \in M \times M'$ and $r > 0$

$$\begin{aligned} D_r(z) &= \{(y, y') \in M \times M' : d_\infty((y, y'), (x, x')) < r\} \\ &= \{(y, y') \in M \times M' : d(x, y) < r, d'(x', y') < r\} \\ &= D_r(x) \times D_r(x') \end{aligned}$$

Remark. Let $W \subset M \times M'$. Then W is open in the product topology $\iff$ $\forall z = (x, x') \in W$ $\exists$ open sets U in M, U' in M' s.t. $(x, x') \in U \times U' \subset W$ $\iff$ $\forall z = (x, x') \in W$ $\exists r > 0$ s.t.

$$D_r(x) \times D_r(x') \subset W$$

$\iff$ W is d_∞-open
E.g. the product topology on $\mathbb{R} \times \mathbb{R}$ is the Euclidean topology on $\mathbb{R}^2$

Prop 4.8. Let X, Y be topological spaces and let $X \times Y$ be given the product topology. Then the coordinate projections

$$q_X : X \times Y \to X, (x, y) \mapsto x$$

and

$$q_Y : X \times Y \to Y, (x, y) \mapsto y$$

satisfy the following:

(i) q_X, q_Y are continuous

(ii) if Z is any topoogical space and $g : Z \to X \times Y$ is a function, then g is continuous $\iff$ $q_X \circ g, q_Y \circ g$ are continuous

Proof. (i) If U is open in X then $q_X^{-1}(U) = U \times Y$ is open in $X \times Y$ so q_X is continuous. Similarly, q_Y is continuous

(ii) " $\implies$ " follows from (i) and the fact that composite of continuous functions are continuous.

" $\impliedby$ " Let $h = q_X \circ g : Z \to X$, $k = q_Y \circ g : Z \to Y$ so

$$g(x) = (h(x), k(x)),\ x \in Z$$

we assume that h, k are continuous. For open sets U in X, V in Y, we have

$$\begin{aligned} z \in g^{-1}(U \times V) &\iff g(z) \in U \times V \\ &\iff h(z) \in U, k(z) \in V \\ &\iff z \in h^{-1}(U) \cap k^{-1}(V) \end{aligned}$$

So $g^{-1}(U \times V) = h^{-1}(U) \cap k^{-1}(V)$ is open in z as hk are continuous. Given an arbitrary open set S in $X \times Y$, we have $W = \cup_{i \in I} U_i \times V_i$ where U is an index set, U_i is open in X, V_i is open in Y $\forall i \in I$

$$g^{-1}(W) = \bigcup_{i \in I} g^{-1}(U_i \times V_i \text{ is open in } X \times Y \text{ by above}$$

Remark. Given $n \in \mathbb{N}$ and topological spaces $X_1, \dots, X_n$, the product topology on $X = X_1 \times \cdots \times X_n$ consists of all unions of set sof the form $U_1 \times \cdots \times U_n$ where U_j is open in X_j for all $j = 1, \dots, n$. If X_j is metrisable with metric$e_j, 1 \leq j \leq n$, then the prfuct topology on X is metrisable e.g. with

$$d_\infty((x_j), (y_j)) = \max_{1 \leq j \leq n} e_j(x_j, y_j)$$

Tje ama;pgie prop 8 holds

4.4 Quotient Spaces

Definition. Let X be a set and R an **equivalence relation** on X. This means $R \subset X \times X$ (write $x\tilde{y}$ instead of $(x, y) \in R$) s.t.
(i) R is reflexive $\forall x \in X, x \sim x$
(ii) R is symmetric: $\forall x, y \in X, x \sim y \implies y \sim x$
(iii) R is transitive: $\forall x, y, z \in X, x \sim y, y \sim z \implies x \sim z$
For $x \in X$, let $q(x) = \{y \in X : y\tilde{x}\}$ called the **equivalence class** of x. These partition X. Let X/R denote the **set of all equivalence classes**. The $q : X \to X/R$, $x \mapsto q(x)$ is called the **quotient map**.

Definition. Now assume X is a topological space. The **quotient topology** on X/R is

$$\{V \subset X/R : q^{-1}(V) \text{ is open in } X\}$$

Indeed this is a topology
(i) $q^{-1}(\varnothing) = \varnothing$ is open in X, so $\varnothing$ is open in X/R

$$q^{-1}X/R = X \text{ is open in } X \implies X/R \text{ is open in } X/R$$

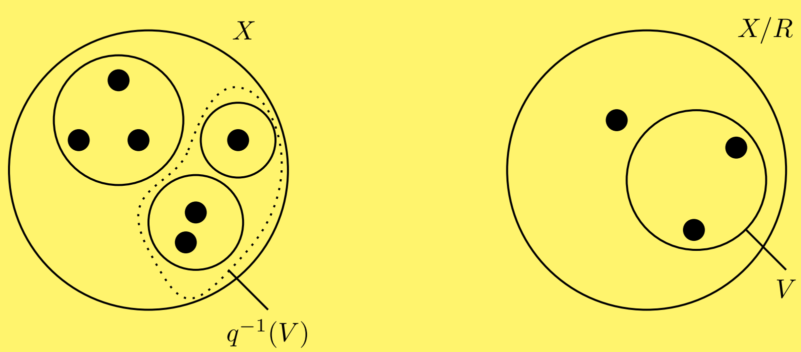
(ii) Suppose V_i is an open subset of X/R $\forall i \in I$, then

$$q^{-1}(\bigcup_{i \in I} V_i) = \bigcup_{i \in I} q^{-1}(V_i) \text{ is open in } X$$

since by definition, each $q^{-1}(V_i)$ is open in X
(iii) $q^{-1}(U \cap V) = q^{-1}(U) \cap q^{-1}(V)$ is open in V if U, V are open in X/R

Remarks.
(i) $q : X \to X/R$ is continuous
(ii) Let $x \in X, t \in X/R$. $x \in t \iff t = q(x)$. For $V \subset X/R$

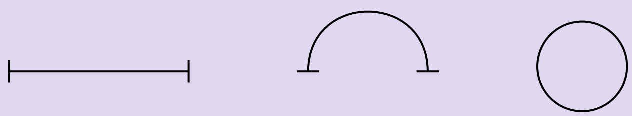
$$\begin{aligned} q^{-1}(V) &= \{x \in X :: q(x) \in V\} \\ &= \{x \in X : \exists t \in V \ t = q(x)\} \\ &= \{x \in X : \exists t \in V \ x \in t\} \\ &= \bigcup_{t \in V} t \end{aligned}$$



Examples. (i) $\mathbb{R}$ is also a group under $+$. $\mathbb{Z} \le \mathbb{R}$, have the quotient group $\mathbb{R}/\mathbb{Z}$. This is the set of equivalence classes where $x \sim y \iff x - y \in \mathbb{Z}$. What is $\mathbb{R}/\mathbb{Z}$ with the quotient toplogy?

$$\forall x \in \mathbb{R}\ \exists y \in [0,1]\ x \sim y$$

$$\forall x, y \in [0,1] x \sim y \text{ iff } x = y \text{ or } \{x, y\} = \{0, 1\}$$



we "glue" 0,1 together

So $\mathbb{R}/\mathbb{Z}$ is homeomorphic to

$$S^1 = \{(x,y) \in \mathbb{R}^2 : \|(x,y)\| = \sqrt{x^2 + y^2} = 1\}$$

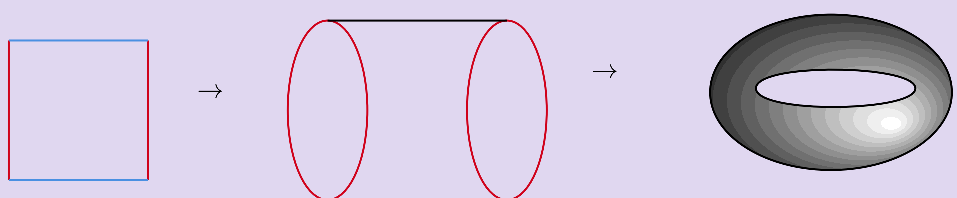
This requires proof.

(ii) $\mathbb{Q} \le \mathbb{R}$, $\mathbb{R}/\mathbb{Q}$. What is the quotient topology?
Let $V \subset \mathbb{R}$, W, V open, $V \neq \varnothing$. Then $q^{-1}(V)$ is open and $\neq \varnothing\varnothing$ (q surjective). $\exists a < b$ s.t. $(a,b) \subset q^{-1}(V)$ $(a, b \in \mathbb{R})$. Given $x \in \mathbb{R}$ choose $r \in (a-x, b-x) \cap \mathbb{Q}$, then $r + x \in (a,b) \subset q^{-1}(V)$ so $q(x) = q(r+x) \in V$. So $V = \mathbb{R}/\mathbb{Q}$. So $\mathbb{R}/\mathbb{Q}$ has the indiscrete topology which is not metrisable and not Hausdorff.

(iii) $Q = [0,1] \times [0,1] \subset \mathbb{R}^2$. Define

$$(x_1, x_2) \sim (y_1, y_2) \iff \begin{cases} (x_1, x_2) = (y_1, y_2) \text{ or} \\ x_1 = y_1, \{x_2, y_2\} = \{0,1\} \text{ or} \\ x_2 = y_2, \{x_1, y_1\} = \{0,1\} \end{cases}$$

and $(0,0), (1,0), (0,1), (1,1)$ are equivalent.

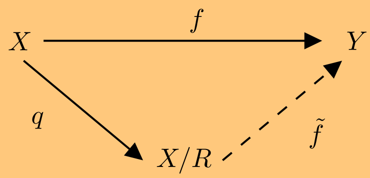


$(\mathbb{R}^2/\mathbb{Z}^2)$

Claim. Let X be a set, R an equivalence relation on X, $q : X \to X/R$ the quotient map. Let$Y$ be another set, $f : X \to Y$ a function. Assume f **respects** R:

$$\forall x, y \in X \ x \sim y \implies f(x) = f(y)$$

Then $\exists$ unique map $\tilde{f} : X/R \to Y$ s.t. $f = \tilde{f} \circ q$, i.e. the diagram



Proof. For $z \in X/R$, write $z = q(x)$ for some $x \in X$ and define $\tilde{f} = f \circ q^{-1}$

Note. (i) $\operatorname{im} f = \operatorname{im} \tilde{f}$ (as q is surjective)
(ii) $\tilde{f}$ is injective if $\forall x, y \in X \ \tilde{f}(q(x)) = \tilde{f}(q(y))$ implies $q(x) = q(y)$ So $\forall x, y \in X \ f(x) - f(y) \implies x \sim y$

Definition. Say f **fully respects** R if

$$\forall x, y \in X \ x \sim y \iff f(x) = f(y)$$

In this case $\tilde{f}$ is injective

Prop 4.9. Let X be a topological space, R an equivalence relation on X, $q : X \to X/R$ the quotient map with X/R given the quotient topology. Let Y be another topological space, $f : X \to Y$ a function that respects R. Let $\tilde{f} : X/R \to Y$ be the unique map s.t. $f = \tilde{f} \circ q$. Then

(i) f continuous $\implies \tilde{f}$ continuous
(ii) f an open map $\implies \tilde{f}$ is an open map

In particular, if f is a continuous, surjective map that **fully respects** R, then $\tilde{f}$ is a continuous bijection. If in addition, f is an open map, then $\tilde{f}$ is a homeomorphism

Proof. (i) Let V be an open set in Y. Is $\tilde{f}^{-1}(V)$ open ni X/R
Look at $q^{-1}(\tilde{f}^{-1}(V)) = ()\tilde{f} \circ q)^{-1}(V) = f^{-1}(V)$ is open in X as f is continuous. So by definition $\tilde{f}^{-1}(V)$ is open in X/R

(ii) Let V be an open set in X/R. Is $\tilde{f}(V)$ open in Y?
Let $U = q^{-1}(V)$. Then U is open in X by definition. As q is surjective, $q(U) = q(q^{-1}(V)) = V$ so

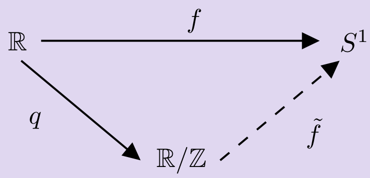
$$\tilde{f}(V) = \tilde{f}(q(U)) = (\tilde{f} \circ q)(U) = f(U)$$

is open in $U = Y$ since f is an open map.

Example. $\mathbb{R}/\mathbb{Z}$ is homeomorphic to

$$S^1 = \{x \in \mathbb{R}^2 : \|x\| = 1\}$$

Define $f(t) = (\cos(2\pi t), \sin(2\pi t)),\ t \in \mathbb{R}$. Then $s - t \in \mathbb{Z} \iff f(s) = f(t)$. f is surjective, and continuous.



By prop 9, $\exists$ unique $\tilde{f} : \mathbb{R}/\mathbb{Z} \to S^1$ s.t. $f = \tilde{f} \circ q$ and $\tilde{f}$ is a continuous bijection. It remains to show that f is an open map. Assume not: there is an open set U in $\mathbb{R}$ s.t. $f(U)$ is not oen in S^1. So $S^1 \backslash f(U)$ is not closed, so $\exists (z_n)$ in $S^1 \backslash f(U)$ and $z \in f(U)$ s.t. $z_n \to z$.
$\forall n \in \mathbb{N}$ choose $x_n \in [0,1]$ s.t. $f(x_n) = z_n$ By B-W wlog $x_n \to x \in [0,1]$ (after passing to subsequence). f is continuous so $z_n = f(x_n) \to f(x) = z$. Since $z_n \notin f(U)$, we have $x_n \in \mathbb{R}\backslash U$. Since $\mathbb{R}\backslash U$ is closed and $x_n \to x$, we have $x \notin U$. Since $z \in f(U)$, $\exists y \in U$ s.t. $z = f(y)$ sp $k = y - x \in \mathbb{Z}$. Now

$$f(x_n + k) = f(x_n) = z_n \to z$$

Also

$$x_n + k \to x + k = y \in U$$

Since $z_n \notin f(U),\ x_n + k \notin U$. Since $\mathbb{R}\backslash U$ is closed and $x_n + k \to y$, we have $y \in \mathbb{R}\backslash U$ ※.

Prop 4.10. Let X be a topological space and R an equivalence relation on X

(i) If X/R is Hausdorff, then R is closed in $X \times X$
(ii) If R is closed in $X \times X$ and $q : X \to X/R$ (the quotient map) is an open map, then X/R is Hausdorff.

Proof. Set $W = X \times X \backslash R$

(i) Given $(x, y) \in W$, we have $x \not\sim y$, i.e. $q(x) \neq q(y)$. Since X/R is Hausdorff, there are open sets S, T s.t. $S \cap Y = \varnothing$ and $q(x) \in S,\ q(y) \in T$. Set $U = q^{-1}(S),\ V = q^{-1}(T)$. Thn U, V are open in X and $x \in U$ and $y \in V$.

$$\forall (a, b) \in U \times V,\ q(a) \in S, q(b) \in T$$

so $q(a) \neq q(b)$, i.e. $(a, b) \notin R$. So $(x, y) \in U \times V \subset W$. So W open in $X \times X$ so R is closed

(ii) Let $z \neq w$ be in X/R. Choose $x, y \in X$ s.t. $q(x) = z,\ q(y) = w$. Then $(x, y) \in W$. Since R is closed, W is open, so $\exists$ open sets U, V in Xs.t.

$$(x, y) \in U \times V \subset W$$

Since q is an open map, $q(U), q(V)$ are open in $X/R,\ z = q(x) \in q(U),\ w = q(y) \in q(V)$

$$\forall a \in U, b \in V,\ (a, b) \in U \times V \subset W$$

so $(a, b) \notin R$ i.e. $q(a) \neq q(b)$. So $q(U) \cap q(V) = \varnothing$

5 Connectedness

Recall the intermeiate value theorem (IVT): If $f : I \to \mathbb{R}$ is continuous, I is an interval and $x < y$ in $I,\ c \in \mathbb{R}$ is strictly between $f(x)$ and $f(y)$ then $\exists z,\ x < z < y$ s.t. $f(z) = c$

Note. I an interval means $\forall x < y < z$ in $\mathbb{R}$ if $x, z \in I$ then $y \in I$. So IVT says: continuous image of an interval is an interval

Example. $f : [0,1) \cup (1,2] \to \mathbb{R}$

$$f(x) = \begin{cases} 0 & x \in [0,1) \\ 1 & x \in (1,2] \end{cases}$$

is continuous but $\mathrm{im}(f)$ is not an interval.

Definition. A topological space X is **disconnected** if $\exists$ subsets U, V of X s.y.

$$U \cap V = \varnothing$$

$$U \cup V = X$$

U, V are $\neq \varnothing$ and U, V are open.
We say U, V disconnect X.
Say X is **connected** if X is not disconnected.

Theorem 5.1. For a topological space X, TFAE:

(i) X is connected

(ii) $f : X \to \mathbb{R}$ continuous $\implies f(X)$ is an interval

(iii) $f : X \to \mathbb{Z}$ continuous $\implies f$ is constant

Proof. (i) $\implies$ (ii): assume $f(X)$ not an interval: $\exists a < b < c$ in $\mathbb{R}$ s.t. $a, c \in f(X),\ b \notin f(X)$. Choose $x, y \in X$ s.t. $f(x) = a,\ f(y) = c$. Let $U = f^{-1}(-\infty, b)$. Then U, V are open as f is continuous, U, V are $\neq \varnothing$ as $x \in U, y \in V$. $U \cap V = \varnothing$ as $(-\infty, b) \cap (b, \infty) = \varnothing$, $U \cup V = f^{-1}(\mathbb{R} \backslash \{b\}) = X$ as $b \notin f(X)$. So U, V disconnect X ※.
(ii) $\implies$ (iii): immediate.
(iii) $\implies$ (i): Assume U, V disconnect X. Define $f : X \to \mathbb{Z}$

$$f(x) = \begin{cases} 0 & x \in U \\ 1 & x \in V \end{cases}$$

for any $Y \subset \mathbb{R}$

$$f^{-1}(Y) = \begin{cases} \varnothing & 0, 1 \notin Y \\ U & 0 \in Y, 1 \notin Y \\ V & 0 \in Y, 1 \in Y \\ X & 0, 1 \in Y \end{cases}$$

is always open. So f is continuous, but f is not constant ※

Corollary 5.2. Let $X \subset \mathbb{R}$. Then x is connected $\iff$ X is an interval

Proof. "$\implies$": The inclusion map $i : X \to \mathbb{R}$ is continuous so by theorem 1, its image, X is an interval
"$\impliedby$": $\forall$ continuous $f : X \to \mathbb{R}$, $f(X)$ is an interval by the IVT. So by Theorem 1, X is connected

Note. Direct proof of "$\impliedby$": Assume U, V disconnect X. Fix $x \in U, y \in V$. Wlog $x < y$. Set $z = \sup U \cap [x, y]$, which contains x and bounded above by u. Note $z \in [x, y] \subset X$. We'll show $z \in U \cap V$, which is a contradiction.
$\forall n \in \mathbb{N}$ $z - 1/n < z$, so $\exists x_n \in U \cap [x, y]$ s.t. $z - 1/n < x_n \le z$ so $x_n \to z$. Also, $U = X \backslash V$ is closed, so $z \in U$. Thus, $z < y$. Choose $N \in \mathbb{N}$ with $z + 1/N < y$. Then $\forall n \ge N$ $z < z + 1/n < y$ hence $z + 1/n \in V$. Now $z + 1/n \to z$ and $V = X \backslash U$ is closed, so $z \in V$ ※

Examples. (i) Any indiscrete topological space is connected
(ii) Any cofinite topology on an ∞ set is connected
(iii) The discrete topology on a set of size ≥ 2 is disconnected.

Lemma 5.3. Let Y be a subspace of a topological space X. Y is disconnected $\iff$ $\exists$ open subsets U, V of X s.t. $U \cap V \cap Y = \varnothing$, $U \cup V \supset Y$, $U \cap Y \ne \varnothing$ and $V \cap Y \ne \varnothing$

Proof. $\implies$: Assume U', V' are open subsets of Y that disconnect Y. Then $\exists$ open sets U, V in X s.t. $U' = U \cap Y$ and $V' = V \cap Y$. These U and V work.
$\impliedby$: Assume U, V are as given. Then $U' = U \cap Y$, $V = V \cap Y$ are open sets in Y and they disconnect Y

Remark. In the above situation, we say that the open subsets U, V of X disconnect Y

Prop 5.4. Let Y be a subspace of a topological space X. Then if Y is connected, then so is $\bar{Y}$, the closure of Y in X

Proof. Assume $\bar{Y}$ is disconnected: $\exists$ open sets U, V in X that disconnect $\bar{Y}$. Then

$$U \cap V \cap Y \subset U \cap V \cap \bar{Y} = \varnothing$$

so

$$U \cap V \cap Y = \varnothing$$

Also

$$U \cup V \supset \bar{Y} \supset Y$$

So U, V would disconnect Y unless $U \cap Y = \varnothing$ or $V \cap Y = \varnothing$. But Y is connected, so wlog $V \cap Y = \varnothing$. Then $Y \subset X \backslash V$ and $X \backslash V$ is closed, so $\bar{Y} \subset X \backslash V$. So $V \cap \bar{Y} = \varnothing$, which is a contradiction since U, V disconnect $\bar{Y}$.

Remark. More generally, if $Y \subset Z \subset \bar{Y}$ and Y is connected, then Z is connected. This follows from Prop 4

$$\text{Cl}_Z(Y) = \text{Cl}_X(Y) \cap Z = Z$$

by Prop 4.6

Theorem 5.5. Let $f : X \to Y$ be a continuous function between topological spaces. If X is connected, then so is $f(X)$

Proof. Let U, V be open subsets of Y and assume they disconnect $f(X)$. For $x \in X$, $f(x) \in f(X) \subset U \cup V$ so

$$f^{-1}(U) \cup f^{-1}(V) = X$$

Also if $x \in f^{-1}(U) \cap f^{-1}(V)$ then

$$f(x) \in U \cap V \cap f(X) = \varnothing \text{ ※}$$

so

$$f^{-1}(U) \cap f^{-1}(V) = \varnothing$$

Since f is continuous, $f^{-1}(U)$ and $f^{-1}(V)$ are open in X. Since $U \cap f(X) \neq \varnothing$ and $V \cap f(X) \neq \varnothing$, we have $f^{-1}(U) \neq \varnothing$ and $f^{-1}(V) \neq \varnothing$. So $f^{-1}(U), f^{-1}(V)$ disconnect X ※

Remarks.

(i) Connectedness is a topological property: if X, Y are homeomorphic topological spaces, then X is connected $\iff$ Y is connected

(ii) If $f : X \to Y$ is continuous and $A \subset X$ and A is connected, then $f(A)$ is connected. Apply Theorem 5 to $f|_A : A \to Y$

Corollary 5.6. Any quotient of a connected topological space is connected

Example. Let $Y = \{(x, \sin(1/x) : x > 0)\} \subset \mathbb{R}^2$. The function $f : (0, \infty) \to \mathbb{R}^2$

$$f(x) = (x, \sin(1/x))$$

is continuous, so by Theorem 5 (and Corollary 2), we have $Y = \text{ im}f$ is connected.
By Prop 4, $\bar{Y}$ is also connected. Let $Z = Y \cup \{(0, y) : -1 \leq y \leq 1\}$
Claim: $\bar{Y} = Z$
Proof of claim:
Given $y \in [-, 1, 2], \forall n \in \mathbb{N}$ $(0, 1/n)$ is mapped to (n, ∞) by $x \mapsto 1/x$, so by IVT $\exists x_n \in (0, 1/n)$ s.t. $\sin(1/x_n) = y$

$$(x_n, \sin(1/x_n)) = (x_n, y) \to (0, y) \in \bar{Y}$$

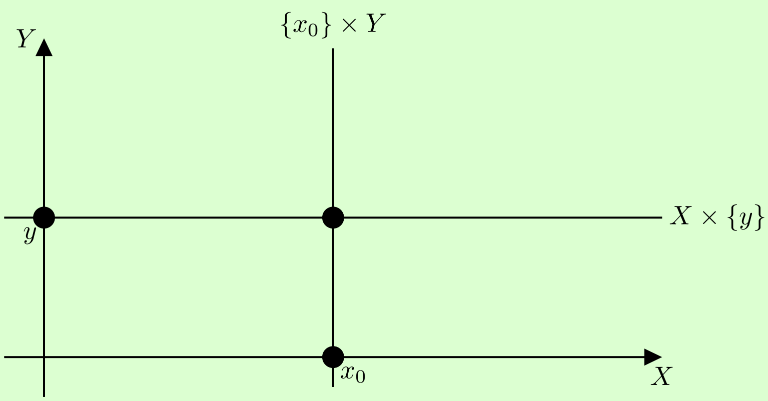
So $Y \subset Z \subset \bar{Y}$. So enough to show Z is closed. Assume $(x_n, y_n) \in Z$ $\forall n \in \mathbb{N}$ and $(x_n, y_n) \to (x, y)$ in $\mathbb{R}^2$. Since $y_n \in [-1, 1]$ $\forall n$ and $y_n \to y$ so $y \in [-1, 1]$. So if $x = 0$ then $(x, y) \in Z$. If $x \neq 0$, then since $x_n \to x$, we have $x + n \neq 0$ $\forall$ large n, so $\sin(1/x_n) = y_n$ $\forall$ large n so $(x_n, y_n) \to (x, \sin(1/x)) \in Z$

Lemma 5.7. Let X be a topological space and $\mathcal{A}$ be a family of connected subsets of X. Assume $A \cap B \neq \varnothing$ $\forall A, B \in \mathcal{A}$ Then $\bigcup_{A \in \mathcal{A}} A$ is also connected

Proof. Set $Y = \bigcup_{A \in \mathcal{A}} A$. Let $f : Y \to \mathbb{Z}$ be a continuous function. $\forall A \in \mathcal{A}$ $f|_A : A \to \mathbb{Z}$ is continuous and hence constant by Theorem 1 as A is connected. $\forall A, B \in \mathcal{A}$ $A \cap B \neq \varnothing$ so $f|_A$ and $f|_B$ have the same constant value. So f must be constant. By theorem 1, Y is connected

Theorem 5.8. Let X, Y be connected topological spaces, then $X \times Y$ is connected in the product topology

Proof. Wlog $X \neq \varnothing, Y \neq \varnothing$.



Fix $x_0 \in X$. Define $f : Y \to X \times Y$, $y \mapsto (x_0, y)$. The components of f are the functions $y \mapsto x_0, Y \to X$ is continuous as it's constant. $y \mapsto y : Y \to Y$ is continuous as it's the identity. So f is continuous by Prop 4.8. By Theorem 5, im $f = \{x_0\} \times Y$ is connected. Similarly, $\forall y \in Y$ $X \times \{y\}$ is connected. For $y \in Y$

$$\{x_0\} \times Y \cap X \times \{u\} = \{(x_0, y)\} \neq \varnothing$$

is connected. So by lemma 7, $A_y = \{x_0\} \times Y \cup X \times \{y\}$ is connected

$$\forall y, z \in Y \;\; A_z \supset \{x_0\} \times Y \therefore A_Y \cap A_Z \neq \varnothing$$

By lemma 7

$$\bigcup_{y \in Y} A_y = X \times Y$$

is conntected

Example. $\mathbb{R}^n$ is connected $\forall n \in \mathbb{N}$

5.1 Components

Definition. Let X be a topological space. We define a relation $\sim$ on X on X

$$x \sim y \iff \exists \text{ connected subset } A \text{ of } X \text{ s.t. } x, y \in A$$

$\forall x \in X$ $x \sim x$ as $\{x\}$ is connected. Symmetry is clear from definition. If $x \sim y, y \sim z$ then $\exists$ connected sets A, B in X s.t. $x, y \in A$, $y, z \in B$. Then $A \cap B \neq \varnothing$, so by lemma 7, $A \cup B$ is connected. Since $x, z \in A \cup B$, we have $x \sim z$

Notation. For $x \in X$, write C_x for the equivalence class containing x. It's called the **connected component of x in X**. The equivalence classes are called **connected components of X**

Prop 5.9. The connected components of a topological space X are $\neq \varnothing$, maximal connected subsets of X, are closed and they partition X

Proof. Let C be a connected component of X. So $C = C_x$ for some $x \in X$. Then $x \in C$, so $C \neq \varnothing$. Assum4 $C \subset A \subset X$, A is connected. Then $\forall y \in A$, since $x, y \in A$, we have $y \sim x$, so $y \in C$. So $A \subset C$ and so $A = C$. $\forall y \in C$, we have $y \sim x$, so there is a connected subset A_y of X s.t. $x, y \in A_y$. Then, $A = \bigcup_{y \in C} A_u$ is connected by Lemma 7 and $A \supset C$ so $A = C$ and C is connected. By Prop 4, $\bar{C}$ is connected and $\bar{C} \supset C$, so $C = \bar{C}$ is closed

Definition. Let X be a topological space. For $x, y \in X$, **a path from x to y** in X is a continuous function $\gamma : [0,1] \to X$ s.t. $\gamma(0) = x$, $\gamma(1) = y$. X is **path-connected** if $\forall x, y \in X$ $\exists$ a path from x to y in X.

Example. In $\mathbb{R}^n$, $D_r(x)$ is path-connected: given $y, z \in D_r(x)$, let

$$\gamma(t) = (1-t)y + tz, \ t \in [0,1]$$

Then γ is continuous (components are continuous) and takes values in $D_r(x)$ since

$$\|\gamma(t) - x\| = \|(1-t)y + tz - x\| = \|(1-t)y + tz - ((1-t)x + tx)\| \leq (1-t)\|y - x\| + t\|z - x\| < r$$

Similarly, every convex subset of $\mathbb{R}^n$ is path-connected

Theorem 5.10. Path-connected $\implies$ connected

Proof. Assume X is not connected and let U, V disconnect X. Fix $x \in U, y \in V$. Assume $\gamma : [0,1] \to X$ is continuous with $\gamma(0) = x$ and $\gamma(1) = y$. Then $\gamma^{-1}(U) \times \gamma^{-1}(V)$ disconnect $[0,1]$

Example. Converse is false in general. Recall

$$X = \{(x, \sin(1/x)) : x > 0\} \cup \{(0, y) : -1 \leq y \leq 1\}$$

is connected. We show X is not path connected. Assume $\gamma[0,1] \to X$ is continuous, $\gamma(0) = (0,0)$ nad $\gamma(1) = (1, \sin 1)$. Write $\gamma = (\gamma_1, \gamma_2)$. Assume $t \in (0,1]$ is s.t. $\gamma_1(t) > 0$ e.g. $t = 1$. Then $\gamma_1((0,t)) \supset (0, \gamma_1|t|)$ by IVT.

$$\exists n \in \mathbb{N}\ \frac{1}{2\pi n} \in (0, \gamma_1(t)) \implies \exists s \in (0,t)\ \gamma_1(s) = \frac{1}{2\pi n}$$

and so $\gamma_1(s) = 0$. Similarly, $1/(2\pi n + \pi/2) \in (0, \gamma_1(t))$ so $\exists s \in (0,t)$ s.t.

$$\gamma_1(s) = \frac{1}{2\pi n + \frac{\pi}{2}} \implies \gamma_2(s) = 1$$

In both cases, $\gamma_1(s) > 0$. We inductively find

$$1 > t_1 > t_2 > \dots$$

s.t.

$$\gamma_2(t_n) = \begin{cases} 0 & n \text{ odd} \\ 1 & n \text{ even} \end{cases}$$

$t_n \to t$, some $t \in [0,1]$, γ_2 is continuous so $\gamma_2(t_n) \to \gamma_2(t)$ ※

Lemma 5.11 (Gluing lemma)**.** Let X be a topological space. Assume $X = A \cup B$ where A, B are closed in X. We are given continuous functions $g : A \to Y$, $h : B \to Y$ (where Y is a topological space) s.t. on $A \cap B$, $g = h$. Then $f : X \to Y$

$$f(x) = \begin{cases} g(x) & x \in A \\ h(x) & x \in B \end{cases}$$

is well defined and continuous

Proof. First observe: if $F \subset A$ and F is closed in A, then F is closed in X. Indeed by Proposition 4.6, $\exists$ closed set G in X s.t. $F = A \cap G$. Since A is also closed in X, it follows that F is closed in X (same holds for $F \subset B$).
Now let V be a closed set in Y. Then

$$\begin{aligned} f^{-1}(V) &= (f^{-1}(V) \cap A) \cup (f^{-1}(V) \cap B) \\ &= g^{-1}(V) \cup h^{-1}(V) \end{aligned}$$

and $g^{-1}(V)$ is closed in A, $h^{-1}(V)$ is closed in B by continuity of g, h thus $f^{-1}(V)$ is closed in X. Hence f is continuous by Proposition 4.7.

Definition. Let X be a topological space. For x, y in X. Write $x \sim y$ is $\exists$ path from x to y in X. This is an equivalence relation:

- The continuous function shows that $x \sim x \ \forall x \in X$
- If $\gamma : [0,1] \to X$ is continuous and $\gamma(0) = x,\ \gamma(1) = y$ then $t \mapsto \gamma(1-t)$ is a path from y to x.
- Assume $x \sim y,\ y \sim z$. Let $\gamma, \delta : [0,1] \to X$ be continuous functions s.t. $\gamma(0) = x,\ \gamma(1) = y$ and $\delta(0) = y,\ \delta(1) = z$. Define
-
$$\eta(t) = \begin{cases} \gamma(2t) & t \in [0, \frac{1}{2}] \\ \delta 2t - 1 & t \in [\frac{1}{2}, 1] \end{cases}$$
$$[0,1] = [0, \frac{1}{2}] \cup [\frac{1}{2}, 1]$$
at $1/2$, $\eta(1/2) = \gamma(1) = \delta(0) = y$. By lemma 11, η is continuous and $\eta(0) = x,\ \eta(1) = z$ so $x \sim z$

We call equivalence classes, **path-connected components** of X

Theorem 5.12. Let U be an open subset of $\mathbb{R}^n$. Then U is connected $\iff$ U is path-connected

Proof. $\Longleftarrow$: Theorem 10.
$\Longrightarrow$: wlog, $U \neq \varnothing$. Fix $x_0 \in U$. Let
$$P = \{x \in U : x \sim x_0\}$$
We will show that P is both open and closed in U. Then, P and $U \backslash P$ disconnect U unless $P = \varnothing$ or $U \backslash P = \varnothing$. Since $x_0 \in P$, we have $U = P$ and so we are done.
Fix $x \in U$. Since U is open $\exists r > 0\ D_r(x) \subset U$. Recall $\forall y \in D_r(x)\ y \sim x$. If $x \in P$, then $\forall y \in D_r(x)$, $y \sim x, x \sim x_0$ so $y \sim x_0$. So $D_r(x) \subset P$. So P is open.
If $x \in U \backslash P$ and $y \in D_r(x)$ has $y \sim x_0$ then since $y \sim x$, we have $x \sim x_0$ ※. So $D_r(x) \subset U \backslash P$. So $U \backslash P$ is open, and P is closed

Claim. For $n \geq 2$, $\mathbb{R}$ and $\mathbb{R}^n$ are not homeomorphic

Proof. Assume $f : \mathbb{R} \to \mathbb{R}^n$ is a homeomorphism and $g = f^{-1} : \mathbb{R}^n \to \mathbb{R}$. Then $f|_{\mathbb{R}\backslash\{0\}}$ is a homeomorphism
$$\mathbb{R}\backslash\{0\} \to \mathbb{R}^n \backslash \{f(0)\}$$
with inverse $g|_{\mathbb{R}\backslash\{f(0)\}}$. $\mathbb{R}\backslash\{0\}$ is disconnected, $\mathbb{R}^n\backslash\{f(0)\}$ is connected (e.g. because it is path connected) ※

6 Compactness

Recall: continuous function on a closed bounded interval is bounded and attains its bounds. We ask for what topological spaces X is every continuous function $f : X \to \mathbb{R}$ bounded?
Some answers:

(i) If X is finite

(ii) If $\forall$ continuous $f : X \to \mathbb{R}$ $\exists n \in \mathbb{N}$ and subsets $A_1, \dots, A_n$ of X s.t.

$$X = \bigcup_{j=1}^{n} A_j$$

and ff is bounded on A_j $\forall j$, then the property holds

Note. If $f : X \to \mathbb{R}$ is continuous then $\forall x \in X$, $U_x = f^{-1}((f(x) - 1, f(x) + 1))$ is open, $x \in U_x$ and f is bounded on U_x. $X = \bigcup_{x \in X} U_x$. If $\exists$ finite $F \subset X$ s.t. $\bigcup_{x \in F} U_x = X$ then f is bounded on X.

Definition. Let X be a topological space. An **open cover** for X is a family $\mathcal{U}$ of open subsets of X s.t. $\bigcup_{u \in \mathcal{U}} U = X$. A **subcover** of $\mathcal{U}$ is a subset $\mathcal{V} \subset \mathcal{U}$ s.t. $\bigcup_{U \in \mathcal{V}} U = X$. This is called a **finite subcover** if $\mathcal{V}$ is a finite set. X is **compact** if every open cover for X has a finite subcover.

Theorem 6.1. Let X be a compact topological space and $f : X \to \mathbb{R}$ continuous. Then f is bounded and attains its bounds

Proof. For $n \in \mathbb{N}$, let $U_n = \{x \in X : |f(x)| < n\}$, then U_n is open since $x \mapsto |f(x)|$ is continuous and $(-n, n)$ is open. It is clear that $X = \bigcup_{n \in \mathbb{N}} U_n$. So $\{U_n : n \in \mathbb{N}\}$ is an open cover for X. Since X is compace, $\exists$ finite $F \subset \mathbb{N}$ s.t.

$$X = \bigcup_{n \in F} U_n = U_N$$

where $N = \max F$. So $\forall x \in X$ $|f(x)| < N$, so f is bounded.
Let $\alpha = \inf_X f$ (exists as f is bounded). Assume $\nexists x \in X$ $f(x) = a$. Then $\forall x \in X$ $f(x) > a$ so $\exists n \in \mathbb{N}$ $f(x) > \alpha + \frac{1}{n}$. So letting

$$V_n = \{x \in X : f(x) > \alpha + \frac{1}{n}\} = f^{-1}((\alpha + \frac{1}{n}, \infty))$$

we have V_n is open and $\bigcup_{n \in \mathbb{N}} V_n = X$. So $\exists$ finite $F \subset \mathbb{N}$ such that $\bigcup_{n \in F} X = V_N$, $N = \max F$. So $\forall x \in X$, $f(x) > \alpha + 1/N$ so $\inf_X f \geq \alpha + \frac{1}{N}$ ※. Similarly, $\exists x \in X$ $f(x) = \sup_X f$

Lemma 6.2. Let Y be a subspace of a topological space X. Then Y is compact $\iff$ wherever $\mathcal{U}$ is a family of open sets in X satisfying $\bigcup_{U \in \mathcal{U}} \supset Y$, there is a finite $\mathcal{V} \subset \mathcal{U}$ s.t. $\bigcup_{U \in \mathcal{V}} U \supset Y$

Theorem 6.3. $[0,1]$ is compact

Proof. Let $\mathcal{U}$ be a family of open sets in $\mathbb{R}$ s.t. $[0,1] \subset \bigcup_{U \in \mathcal{U}} U$. For $A \subset [0,1]$, say $\mathcal{U}$ finitely covers A if $\exists$ finite $\mathcal{V} \subset \mathcal{U}$ st. $\bigcup_{U \in \mathcal{V}} U \supset A$. Note: if $A = B \cup C$, $A, B, C \subset [0,1]$ and $\mathcal{U}$ finitely coveres B and C, then U finitely covers A.
Assume that $\mathcal{U}$ does not finitely cover $[0,1]$. Then one of $[0,1/2]$ and $[1/2,1]$ is not finitely covered by $\mathcal{U}$, call that $[a_1,b_1]$. Let $C = \frac{1}{2}(a_1+b_1)$. Then one of $[a_1,c]$ and $[c,b_1]$ is not finitely covered by $\mathcal{U}$ – call it $[a_2,b_2]$. Continue inductively to obtain

$$[a_1,b_1] \supset [a_2,b_2] \supset [a_3,b_3] \supset \dots$$

s.t. $\forall n$ $[a_n,b_n]$ is not finitely covered by $\mathcal{U}$ and $b_n - a_n = 2^{-n}$. Then $a_n \to x$ for some $x \in [0,1]$ and so $b_n = a_n + 2^{-n} \to x$. Can choose $U \in \mathcal{U}$ s.t. $x \in U$. U is open in $\mathbb{R}$ so $\exists \varepsilon > 0$ $(x-\varepsilon, x+\varepsilon) \subset U$. Since $a_n, b_n \to x$, can choose n s.t. $a_n, b_n \in (x-\varepsilon, x+\varepsilon)$, then $[a_n,b_n] \subset U$ ※

Examples.

(i) Any finite set is compact

(ii) On any set X, the cofinite topology is compact. Wlog $X \neq \varnothing$. Let $\mathcal{U}$ be an open cover for X. Choos $U \in \mathcal{U}$ s.t. $U \neq \varnothing$. Then $F = X \backslash U$ is finite. For $x \in F$ pick $U_x \in \mathcal{U}$ s.t. $x \in U_x$. Then $\{U_x : x \in F \cup \{U\}$ is a finite subcover.

(iii) Assume $x+n \to x$ in a topological space X. Let $Y = \{x_n : n \in \mathbb{N}\} \cup \{x\}$. Then Y is compact. Let $\mathcal{U}$be a family of oepn sets in X s.t. $\bigcup_{U \in \mathcal{U}} U \supset Y$. Choose $U \in \mathcal{U}$ s.t. $x \in U$. Since U is open and $x_n \to x$, we have $N \in \mathbb{N}$ $\forall n \geq N$ $x_n \in U$. As in (ii), it is clear $\exists$ finite subcover

(iv) The indiscrete topology on any set is compact

(v) An infinite set X with the discrete topology is not compact:

$$\{\{x\} : x \in X\}$$

is an open coer with no finite subcover

(vi) $\mathbb{R}$ is not compact:

$$\{(-n,n) : n \in \mathbb{N}\}$$

is an open cover with no finite subcover

Theorem 6.4. Let Y be a subspace of a toplogical space X.

(i) X compact, Y closed in $X \implies Y$ compact

(ii) X Hausdorff, Y compact $\implies Y$ closed in X.

Proof. (i) Let $\mathcal{U}$ be a family of open sets in $s.t.$

$$\bigcup_{U \in \mathcal{U}} U \supset Y$$

Then $\mathcal{U} \cup \{X \backslash Y\}$ is an oepn cover for X. So $\exists$ a finite $\mathcal{V} \subset \mathcal{U}$ s.t.

$$\bigcup_{U \in \mathcal{V}} \cup (X \backslash Y) = X$$

Then $\bigcup_{U \in \mathcal{V}} \supset Y$

(ii) Fix $x \in X \backslash Y$. For $y \in Y$, we have $x \neq y$ so $\exists$ open sets U_y, V_y in X s.t. $x \in U_y, y \in V_y$ and $U_y \cap V_y = \varnothing$ (X is Hausdorff)

$\{V_y : y \in Y\}$ is a cover of Y by open sets in X. So $\exists$ finite $F \subset Y$ s.t. $\bigcup_{y \in F} V_y \supset Y$. ($Y$ compact). Then $U = \bigcap_{y \in F} U_y$ is open and $x \in U$ and

$$U \cap Y \subset (\bigcap_{y \in F} U_y) \cap (\bigcup_{y \in F} V_y) = \varnothing$$

So $x \in U \subset X \backslash Y$. So $X \backslash Y$ is a neighbourhood of all of its points, so it's open. Hence Y is closed

Theorem 6.5. Let $f : X \to Y$ be a continuous function between topological spaces with X compact. Then $f(X)$ is compact

Proof. Let $\mathcal{U}$ be a family of open sets in Y. s.t. $\bigcup_{U \in \mathcal{U}} \supset f(X)$. Then $\bigcup_{U \in \mathcal{U}} f^{-1}(U) = X$, and $f^{-1}(U)$ is open in X $\forall U \in \mathcal{U}$ as f is continuous. X is compact so $\exists$ finite $\mathcal{V} \subset \mathcal{U}$ s.t.

$$X = \bigcup_{U \in \mathcal{V}} f^{-1}(U)$$

Hence

$$f(X) \subset \bigcup_{U \in \mathcal{V}} U$$

Remarks.

(i) Compactness is a topological property

(ii) If $f : X \to Y$ is continuous and $A \subset X$, A is compact, then $f(A)$ is compact

Corollary 6.6. Any quotient of a compact space is compact

Corollary 6.7. If $a < b$ in $\mathbb{R}$, then $[a, b]$ is homeomorphic to $[0, 1]$, and hence compact.

Theorem 6.8 (The topological inverse function theorem, TIFT)**.** Let $f : X \to Y$ be a continuous bijction form a compact space X to a Hausdorff space Y. Then f^{-1} is continuous (i.e., f is an open map, or f is a homeomorphism)

Proof. Let U be an open subset of X. Then $K = X \backslash U$ is closed. By Theorem 4, K is compact. By Theorem 5, $f(K)$ is compact. By Theorem 4, $f(K)$ is closed in Y. So $f(U) = Y \backslash f(K)$ is open in Y. So $f(U) = Y \backslash f(K)$ is open in Y

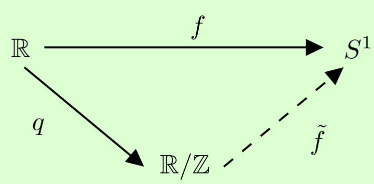
Example. $\mathbb{R}/\mathbb{Z}$ is homeomorphic to $S^1 = \{x \in \mathbb{R}^2 : \|x\| = 1\}$

Proof. Define

$$f : \mathbb{R} \to S^1, \quad f(t) = (\cos(2\pi t), \sin(2\pi t))$$

$$\forall s, t \ f(s) = f(t) \iff s - t \in \mathbb{Z}$$

f is continuous and surjective. Let $\tilde{f} : \mathbb{R}/\mathbb{Z} \to S^1$ be the unique map s.t. $\tilde{f} \circ q = f$.



By prop 4.9, $\tilde{f}$ is a continuous bijection.

$$\mathbb{R}/\mathbb{Z} = q(\mathbb{R}) = q([0, 1])$$

is compact by Thoerem 5. S^1 is Hausdorff since it's a metic space. By Thoerem 7, $\tilde{f}$ is a homeomorphism.

Theorem 6.9 (Tychonov's theorem)**.** If X, Y are compact topological spaces, then so is $X \times Y$ in the product topology

Proof. Let $\mathcal{U}$ be an open cover for $X \times Y$

- WLOF every member of $\mathcal{U}$ is of the form $U \times V$ where U is open in X and V is open in Y. Indeed for $z \in X \times Y$, choose $W_z \in U$ s.y. $z \in W_z$, and in turn $\exists$ open sets U_z in X and X_z in Y s.t. $z \in U_z \times V_z \subset W_z$ so $\{U_z \times \times V_z : z \in X \times Y\}$ is an open cover for $X \times Y$. If $\exists$ finite $F \subset X \times Y$ s.t. $\bigcup_{z \in F} U_z \times V_z = X \times Y$, then $W_z : z \in F$ is a finite subcover of U.
- Fix $x \in X$. Recall $\{x\} \times Y$ is the continuous image of Y under

$$y \mapsto (x, y)$$

And hence $\{x\} \times Y$ is compact by Theorem 5. Since $\{x\} \times Y \subset X \times Y = \bigcup_{w \in \mathcal{U}} W$, $\mathcal{U}$ finitely covers $\{x\} \times Y$. So $\exists n_x \in \mathbb{N}$ open sets $U_{x,1}, U_{x,2}, \ldots, U_{x,n_x}$ in X and $V_{x,1}, v_{x,2}, \ldots, V_{x,n_x}$ in Y s.t.

$$U_{x,j} \times V_{x,j} \in \mathcal{U} \text{ and } \{x\} \times Y \subset \bigcup_{j=1}^{n_x} U_{x,j} \times V_{x,j}$$

Wlog $x \in U_{x,j}$ $\forall j$. Set $U_x = \cap_{j=1}^{n_x} U_{x,j}$. Then $x \in U_x$, U_x is open in X, and

$$U_x \times Y \subset \bigcup_{j=1}^{n_x} U_{x,j} \times V_{x,j}$$

- $\{U_x : x \in X\}$ is an open cover for X. So $\exists$ finite $F \subset X$ s.t. $X = \bigcup_{x \in F} U_x$ so

$$X \times Y = \bigcup_{x \in F} U_x \times Y \subset \bigcup_{x \in F} \bigcup_{j=1}^{n_x} U_{x,j} \times V_{x,j}$$

So $\{U_{x,j} \times V_{x,j} : x \in F,\ 1 \leq j \leq n_x\}$ is a finite subcover of U

Remark. More generally, if $X_1, \ldots, X_n$ are compact spaces, then so is $X_1 \times \cdots \times X_n$

Theorem 6.10 (Heine-Borel)**.** A subset K of $\mathbb{R}^n$ is compact $\iff$ K is closed and bounded

Proof. "$\implies$": $\mathbb{R}^n$ is a metric space, and hence Hausdorff. By Theorem 4,K is closed in $\mathbb{R}^n$. $x \mapsto \|x\|$ is continuous ($|\|x\| - \|y\|| \leq \|x - y\|$), $\therefore$ by Theorem 1, bounded on K. So K is bounded. "$\impliedby$": As K is bounded, $\exists M \geq 0$ $\forall x \in K$ $\|x\| \leq M$. So $K \subset [-M, M]^n$. $[-M, M]^n$ is compact (it's homeomorphic to $[0, 1]$). By Tychonov, $[-M, M]^n$ is compact. Now K is a closed subspace of a compact space and hence compact by Theorem 4.

Example. $[0, 1]^2, B_r(x) \subset \mathbb{R}^n$. Now the start of the proof of Linelof-Picard makes sense.

6.1 Sequential Compactness

Definition. A topological space X is **sequentially compact** if every sequence in X has a convergent subsequece. I.e. given (x_n) in X, $\exists k_1 < k_2 < \dots$ in $\mathbb{N}$ $\exists x \in X$ s.t. $x_{k_n} \to x$

Notation. Given a sequence $(x_n)_{n=1}^{\infty}$ and an infinite set $M \subset \mathbb{N}$, we write $(x_m)_{m\in M}$ for the subsequence $(x_{m_n})_{n=1}^{\infty}$ where $m_1 < m_2 < m_3 < \dots$ are the elements of M. Note that if $L \subset M \subset \mathbb{N}$, L, M infinite, then $(x_n)_{n\in L}$ is a subsequecne of $(x_n)_{n\in M}$

Examples. (i) Any closed, bounded subset of $\mathbb{R}^n$ is sequentially compact by Bolzano-Weierstass

(ii) Similarly, a closed, bounded subset K of $\mathbb{R}^n$ is sequentially compact.
Let $(x_m)_{m=1}^{n}$ be a sequence in K. Write $x_m = (x_{m,1}, \dots, x_{m,n})$. K bounded $\implies$ (x_m) bounded $\implies$ $\forall j$ $(x_{m,j})_{m=1}^{\infty}$ is bounded. By Bolzano Weierstass pplied to $(x_{m,1})_{m=1}^{\infty}$, $\exists$ infinite $M_1 \subset \mathbb{N}$ s.t. $(x,1)_{m\in M_1}$ convergent in $\mathbb{R}$. $(x_{m,2})_{m\in M_1}$ is bounded in $\mathbb{R}$ so by B-W exists infinite $M_2 \subset M_1$ s.t. $(x_{m,2})_{m\in M_2}$ convergent in $\mathbb{R}$. Note that $(x_{m,1})_{m\in M_2}$ still converges. Continue: $\exists M_1 \supset M_2 \supset \dots \supset M_n$ infinte sets such that $(x_{m,j})_{m\in M_j}$ converges in $\mathbb{R}$ for $j = 1, \dots, n$. Then $(x_{m,j})_{m\in M_n}$ converges $\forall j$ and h4nce $(x_m)_{m\in M_n}$ converges in $\mathbb{R}^n$ and the limit is in K as K is closed

Remark. This shows that in $\mathbb{R}^n$, compact $\implies$ sequentially compact. Converse is also true.

Aim: compactness and sequential compactness are the same in metric spaces. For the rest fo the section, we fix a metric space (M, d)

Definition. For $\varepsilon > 0$ and $F \subset M$, say F is an **ε-net** for M if $\forall x \in M$ $\exists y \in F$ $d(y, x) \le \varepsilon$ (i.e. $M = \bigcup_{y\in F} V_\varepsilon(y)$). This is called a **finite ε-net** if F is finite. Say M is **totally bounded** if $\forall \varepsilon > 0$ $\exists$ finite ε-net for M

Example. Given $\varepsilon > 0$, choose n s.t. $1/n < \varepsilon$. Then $\{1/n, 2n, \dots, (n-1)/n\}$ is an ε-net for $(0, 1)$

Definition. For non-empty $A \subset M$, the diameter of A is

$$\text{diam}A = \sup\{d(x, y) : x, y \in A\}$$

(infinite if set not bounded in $\mathbb{R}$)
So diam $A < \infty \iff A$ is bounded

Example. diam $B_r(x) \le 2r$

Lemma 6.11. Assume M is totally bounded, and let $A \subset M, A \neq \varnothing$, and closed, and let $\varepsilon > 0$. then $\exists K \in \mathbb{N}, \neq \varnothing$ closed set $B_1, B_2, \ldots, B_K$ s.t. $A = \bigcup_{k=1}^{K} B_k$ and diam $B_k < \varepsilon \; \forall k$

Proof. Let F be a finite $\varepsilon/2$-net for M. So $M = \bigcup_{x \in F} B_{\varepsilon/2}(x)$ and hence $A = \bigcup_{x \in F}[A \cap B_{\varepsilon/2}(x)]$. Let $G = \{x \in F : A \cap B_{\varepsilon/2}(x) \neq \varnothing\}$ and for $x \in G$, let $B_x = A \cap B_{\varepsilon/2}(x)$. Then for $x \in G,\; B_x \neq \varnothing$, $B_x \subset B_{\varepsilon/2}(x)$ and so diam $B_x \leq \varepsilon$ and B_x is closed. Finally

$$\bigcup_{x \in G} B_x = A$$

Theorem 6.12. For a metric space (M,d), TFAE

(i) M is compact

(ii) M is sequentially compact

(iii) M is complete and totally bounded

Proof. (i) $\implies$ (ii): Let (x_n) be a sequence in M. For $n \in \mathbb{N}$ let $T_n = \{x_k : k > n\}$. Note the limit of any convergent subsequence is in $\bigcap_{n\in\mathbb{N}} \bar{T}_n$. First we prove $\bigcap_{n\in\mathbb{N}} \bar{T}_n \neq \varnothing$. Assume otherwise. Then

$$\bigcup_{n\in\mathbb{N}} (M \backslash \bar{T}_n) = M$$

Since M is compact, $\exists N \in \mathbb{N}$ s.t. $M\backslash \bar{T}_N = M$ (we are using $\forall m \leq n\; T_m \supset T_n$). Contradiction, as $T_N \neq \varnothing$.

Fix $x \in \cap_{n\in\mathbb{N}} \bar{T}_n$. $x \in \bar{T}_1$, so $D_1(x) \cap T_1 \neq \varnothing$ so $\exists k_1 > 1$ s.t. $d(x_{k_1}, x) < 1$

$x \in \bar{T}_{k_1}$, so $D_{1/2}(x) \cap T_{k_1} \neq \varnothing$ so $\exists k_2 > k_1$ s.t. $d(x_{k_2,x}) < 1/2$

$x \in \bar{T}_{k_2}$ so $D_{1/3}(x) \cap T_{k_2} \neq \varnothing$ so $\exists k_3 > k_2$ s.t. $d(x_{k_3}, x) < 1/3$. Continue inductively to get $k_1 < k_2 < \dots$ s.t. $d(x_{k_n}, x) < 1/n\ \forall n$, so $x_{k_n} \to x$.

(ii) $\implies$ (iii): To show M is complete, let (x_n) be a Cauchy sequence in M. Choose $k_1 < k_2 < \dots$ s.t. (x_{k_n}) is convergent in M and let

$$x = \lim_{n\to\infty} x_{k_n}$$

We show $x_n \to x$. Fix $\varepsilon > 0$. There is $N \in \mathbb{N}\ \forall m,n \geq N\ d(x_m, x_n) < \varepsilon$. Then $\forall m \geq N,\ k_m \geq m \geq N$ and $\forall m \geq N$ have

$$d(x_n, x) \leq d(x_n, x_{k_m}) + d(x_{k_m}, x) \leq \varepsilon + d(x_{k_m}, x)$$

Let $, \to \infty$: $d(x_n, x) \leq \varepsilon$. So $x_n \to x$.

Assume M is not totally bounded, then $\exists \varepsilon > 0$ s.t. M has no finite ε-set. Fix $x_1 \in M$. Assume we picked $x_1, \dots, x_{n-1}$ in M. Then

$$\bigcup_{j=1}^{n-1} B_\varepsilon(x_j) \neq M$$

Can pick $x_n \in M_n \backslash \bigcup_{j=1}^{n-1} B_\varepsilon(x_j)$. Inductively obtain $(x_n)_{n=1}^\infty$ s.t. $d(x_m, x_n) > e\ \forall n > m$ in $\mathbb{N}$. So (x_n) has no Cauchy subsequence therefore no convergent subsequence.

(iii) $\implies$ (i): Let $\mathcal{U}$ be an open cover for M. Assume that $\mathcal{U}$ does not finitely cover M. We construct non-empty closed subsets

$$A_0 \supset A_1 \supset A_2 \supset \dots \text{ of } M$$

such that $\forall n \geq 0\ \mathcal{U}$ does not finitely cover A_n, and that $\forall n \geq 1$ diam $A_n < 1/n$. Set $A_0 = M$. Suppose for some $n \geq 0$ we have already found A_{n-1}. By Lemma 10 (since M is totally bounded) we can write $A_{n-1} = \bigcup_{k=1}^K B_k$ where $K \in \mathbb{N}$, $B_1, \dots, B_K$ are non-empty, closed and diam $B_k < 1/n\ \forall k = 1, \dots, K$.

Since $\mathcal{U}$ does not finitely cover A_{n-1}, $\exists k$ s.t. $\mathcal{U}$ does not finitely cover B_k. Set $A_n = B_k$. Now for each n pick $x_n \in A_n$

$$\forall N\ \forall m,n \geq N\ x_m, x_n \in A_N$$

so

$$d(x_m, x_n) \leq \text{diam } A_N < \frac{1}{N}$$

Proof. It follows that (x_n) is Cauchy. M is complete, so $x_n \to x$ for some $x \in M$. Choose $U \in \mathcal{U}$ st, $x \in U$. U is open, so $\exists r > 0$ $D_r(x) \subset U$. Choose n s.t. $d(x_n, x) < r/2$ and diam $A_n < r/2$ $\forall y \in A_n$

$$\begin{aligned} d(y,x) &\leq d(y,x_n) + d(x_n,x) \\ &\leq \text{diam } A_n + \frac{r}{2} < r \end{aligned}$$

$A_n \subset D_r(x) \subset U$※ as U does not finitely cover A_n

Remarks.

(i) We can deduce Heine-Borel (closed and bounded $\implies$ compact only) from B-W

(ii) The product of sequentially compact topological space is sequetially compact in the product space. This yields a new proof of Tychonov for metric spaces

(iii) There exists topological spaces that are compact but not sequentially compact. There exists topological spaces that are sequentially compact but not comapct

7 Differentiation

Let $m, n \in \mathbb{N}$

$$L(\mathbb{R}^m, \mathbb{R}^n) = \{T : \mathbb{R}^m \to \mathbb{R}^n : T \text{ linear}\} \cong M_{n,m} \cong \mathbb{R}^{mn}$$

Let $e_1, \ldots, e_m$ be the standard basis (S.B.) of $\mathbb{R}^m$. Let $e'_1, \ldots, e'_m$ be the standard basis (S.B.) of $\mathbb{R}^n$. Then $T \in L(\mathbb{R}^m, \mathbb{R}^n)$ is identified with the $n \times m$ matrix $(T_{j,i})_{1 \leq j \leq n, 1 \leq i \leq m}$ where

$$T_{j,i} = \langle Te_i, e'_j \rangle$$

here $\langle \cdot, \cdot \rangle$ is the standard inner product in $\mathbb{R}^n$)

$$\langle \sum_{j=1}^{n} x_j e'_j, \sum_{j=1}^{n} y_j e'_j \rangle = \sum_{j=1}^{n} x_j y_j$$

We can view $L(\mathbb{R}^m, \mathbb{R}^n)$ as the (mn) dimentional vector space $\mathbb{R}^{mn}$ which has the euclidean norm:

$$\|T\| = (\sum_{i=1}^{m} \sum_{j=1}^{n} T_{j,i}^2)^{1/2} = (\sum_{i=1}^{m} \|Te_i\|^2)^{1/2}$$

So $L(\mathbb{R}^m, \mathbb{R}^n)$ becomes a metric space with the euclidean distance

$$d(S,T) = \|S - T\| \text{ for } S, T \in L(\mathbb{R}^m, \mathbb{R}^n)$$

Lemma 7.1. (i) For $T \in L(\mathbb{R}^m, \mathbb{R}^n)$ and $x \in \mathbb{R}^m$

$$\|Tx\| \leq \|T\| \cdot \|x\|$$

So T is a Lipschitz map and henc continuous

(ii) If $S \in L(\mathbb{R}^n, \mathbb{R}^p),\ T \in L(\mathbb{R}^m, \mathbb{R}^n)$ then

$$\|ST\| \leq \|S\| \cdot \|T\|$$

Proof. (i) Write $X = \sum_{i=1}^m x_i e_i$. Then

$$\begin{aligned}\|Tx\| &= \|\sum_{i=1}^m x_i T e_i\| \\ &\leq \sum_{i=1}^m |x_i| \cdot \|Te_i\| \\ &\leq (\sum_{i=1}^m x_i^2)^{1/2} \cdot (\sum_{i=1}^m \|Te_i\|^2)^{1/2} \\ &\leq \|T\| \cdot \|x\|\end{aligned}$$

For $x, y \in \mathbb{R}^m$

$$d(Tx, Ty) = \|Tx - Ty\| = \|T(x-y)\| \leq \|T\| \cdot \|x - y\| = \|T\| d(x, y)$$

So T is Lipschitz and hence continuous

(ii)

$$\|ST\| = (\sum_{i=1}^m \|STe_i\|^2)^{1/2} \leq (\sum_{i=1}^m \|S\|^2 \|Te_i\|^2)^{1/2} = \|S\| \|T\|$$

Remark. Recall from IA: A function $f : \mathbb{R} \to \mathbb{R}$ is differentiable at $a \in \mathbb{R}$ if the limit

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

exists. The limit is called the derivative of f at a and denoted $f'(a)$. Have f differentiable at $a \iff \exists \lambda \in \mathbb{R}\ \exists \varepsilon : \mathbb{R} \to \mathbb{R}$ s.t. $\varepsilon(0) = 0$ ε is continuous at 0 and

$$f(a+h) = f(a) + \lambda h + h + h \cdot \varepsilon(h)$$

(trivial to show)

Note. If f is continuous at a, then

$$f(a+h) = f(A) + \eta(g)$$

where $\eta(h) \to 0$ as $h \to 0$. More generally, in IA Analysis, we showed that if f is n-times differentiable at a then

$$f(a+h) = f(a) + \sum_{k=1}^n \frac{f^{(k)(a)}}{k!} h^k + o(h^n)$$

Definition. Let $m, n \in \mathbb{N}$ and $f : \mathbb{R}^m \to \mathbb{R}^n$ a function, $a \in \mathbb{R}^m$. We say **f is differentiable at a** if

$$\exists T \in L(\mathbb{R}^m, \mathbb{R}^n)\ \exists \varepsilon : \mathbb{R}^m \to \mathbb{R}^n \text{ s.t. } \varepsilon(0) = 0 \text{ and } \varepsilon \text{ is continuous at } 0$$

and

$$f(a+h) = f(a) + T(h) + \|h\|\varepsilon(h)$$

Note.

$$\varepsilon(h) = \begin{cases} 0 & h = 0 \\ \frac{f(a+h)-f(a)-T(h)}{\|h\|} & h \neq 0 \end{cases}$$

So f is differentiable at $a \iff \exists T \in L(\mathbb{R}^m, \mathbb{R}^n)$ s.t.

$$\frac{f(a+h) - f(a) - T(h)}{\|h\|} \to 0 \text{ as } h \to 0$$

Notation. Can also write

$$f(a+h) = f(A) + T(h) + o(\|h\|)$$

Claim. T is unique

Proof. If $S, T \in L(\mathbb{R}^m, \mathbb{R}^n)$ both satisfy

$$\frac{f(a+h) - f(a) - T(h)}{\|h\|} \to 0$$

and

$$\frac{f(a+h) - f(a) - S(h)}{\|h\|} \to 0 \text{ as } h \to 0$$

Then

$$\frac{S(h) - T(h)}{\|h\|} \to 0 \text{ as } h \to 0$$

Fix $x \in \mathbb{R}^m,\ x \neq 0$, then $x/k \to 0$ as $k \to \infty$. So

$$\frac{Sx - Tx}{\|x\|} = \frac{S(x/k) - T(x/k)}{\|x/k\|} \to 0$$

So $Sx = Tx$. It follows that $S = T$.

Definition. If f is differentiable at a then the unique $T \in L(\mathbb{R}^m, \mathbb{R}^n)$ s.t.

$$\frac{f(a+h) - f(a) - Th}{\|h\|} \to 0$$

is called the **derivative of f at a** denoted $f'(a)$ or $Df(a)$ or $Df|_a$

Definition. If $f : \mathbb{R}^m \to \mathbb{R}^n$ is differentiable at $a \in \mathbb{R}^m$ for every $a \in \mathbb{R}^m$ then say **f is differentiable on $\mathbb{R}^m$**. The function

$$f' = D : \mathbb{R}^m \to L(\mathbb{R}^m, \mathbb{R}^n), \quad \mapsto f'(a)$$

is called the **derivative of f on $\mathbb{R}^m$**

Examples. (i) Constant functions $f : \mathbb{R}^m \to \mathbb{R}^n,\ f(x) = b\ \forall x \in \mathbb{R}^n$ (some $b \in \mathbb{R}^n$). At $a \in \mathbb{R}^m$:

$$f(a+h) = b = f(a) + 0 + 0$$

So f is differentiable at a and $f'(a) = 0$. So $f : \mathbb{R}^m \to L(\mathbb{R}^m, \mathbb{R}^n),\ a \mapsto 0$

(ii) Linear maps. If $f : \mathbb{R}^m \to \mathbb{R}^n$ is linear, then for $a \in \mathbb{R}^m$

$$f(a+h) = f(a) + f(h) + 0$$

So f is differentiable at a and $f'(a) = f$. So $f' : \mathbb{R}^m \to L(\mathbb{R}^m, \mathbb{R}^n),\ a \mapsto f$. So f' is a constant function

(iii) $f : \mathbb{R}^m \to \mathbb{R},\ f(x) = \|x\|^2$. For $a \in \mathbb{R}^m$:

$$f(a+h) = \|a+h\|^2 = \underbrace{\|a\|^2}_{f(a)} + \underbrace{2\langle a, h\rangle}_{\text{linear in } h} + \underbrace{\|h\|^2}_{\text{error}}$$

It follows that f is differentiable at a and

$$f'(a)(h) = 2\langle a, h\rangle$$

Note that $f' : \mathbb{R}^m \to L(\mathbb{R}^m, \mathbb{R})$ is linear.

(iv) $M_n = M_{n,n} \cong \mathbb{R}^{n^2}$. $f : M_n \to M_n,\ f(A) = A^2$. Fix $A \in M_n$

$$f(A+H) = (A+H)^2 = \underbrace{A^2}_{f(A)} + \underbrace{AH + HA}_{\text{linear in } H} + H^2$$

BY lemma 1, $\|H^2\| \leq \|H\|^2$ and so

$$\frac{\|H\|^2}{\|H\|} \leq \|H\| \text{ as } H \to 0$$

So f is differentiable at A and $f'(A)(H) = AH + HA$

Examples. (v) Suppose $f : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^p$ is bilinear. Fix $(a, b) \in \mathbb{R}^m \times \mathbb{R}^n$

$$f((a,b) + (h,k)) = f(a+h, b+k) = f(a,b) + f(a,k) + f(h,b) + f(h,k)$$

The map $\mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^p$, $(h,k) \mapsto f(a,k) + f(h,b)$ is linear. We show that $f(h,k) = o(\|(h,k)\|)$. Write

$$h = \sum_{i=1}^n h_i e_i, \quad k = \sum_{j=1}^n k_j e'_j$$

Then

$$f(h,k) = \sum_{i=1}^m \sum_{j=1}^n h_i k_j f(e_i, e'_j)$$

$$\|f(h,k)\| \le \sum_{i=1}^m \sum_{j=1}^n |h_i||k_j|\|f(e_i, e'_j)\| \le C \cdot \|(h,k)\|^2$$

Where we use that $|h_i| \le \|(h,k)\|$ for all i and $|k_j| \le \|(h,k)\|$ for all j and

$$C = \sum_{i=1}^m \sum_{j=1}^n \|f(e_i, e'_j)$$

So

$$\frac{\|f(h,k)\|}{\|(h,k)\|} \le C\|(h,k)\| \to 0 \text{ as } (h,k) \to (0,0)$$

So f is differentiable at (a,b) and $f'(a,b)(h,k) = f(a,k) + f(h,b)$

Remark. So far our maps had domain the whole of $\mathbb{R}^m$ or M_n etc

Definition. Let U be an open subset of $\mathbb{R}^m$, $f : U \to \mathbb{R}^n$ a function and let $a \in U$. Say f is **differentiable at a** if $\exists T \in L(\mathbb{R}^m, \mathbb{R}^n)$ s.t.

$$f(a+h) = f(a) + T(h) + \|h\|\varepsilon(h)$$

where $\varepsilon(0) = 0$, ε is continuous at 0 (i.e. $\varepsilon(h) \to 0$ as $h \to 0$)

Notes.

(i) ε is defined on $\{h \in \mathbb{R}^m : a + h \in U\} = U - a$ which is open and $0 \in U - a$ so $\exists r > 0$ $D_r(0) \subset U - a$. Then

$$\varepsilon(h) = \begin{cases} 0 & h = 0 \\ \frac{f(a+h) - f(a) - T(h)}{\|h\|} & h \neq 0, \ a + h \in U \end{cases}$$

(ii) f is differentiable $\iff$ $\exists T \in L(\mathbb{R}^m, \mathbb{R}^n)$ s.t.

$$\frac{f(a+h) - f(a) - T(h)}{\|h\|} \to 0 \text{ as } h \to 0$$

(iii) The T above is unique and is called the derivative of f at a denoted $f'(a)$. So $f(a+h) = f(a) + f'(a)(h) + o(\|h\|)$

Remark. For $m = 1$, $L(\mathbb{R}, \mathbb{R}^n) \cong \mathbb{R}^n$, $T \leftrightarrow T(1)$ putting $v = T(1)$, we have $T(\lambda) = \lambda v\ \forall \lambda \in \mathbb{R}$. Let $U \subset \mathbb{R}$ be open, $f : U \to \mathbb{R}^n$ a function, $a \in U$. f is differentiable at $a \iff \exists v \in \mathbb{R}^n$ s.t.

$$\frac{f(a+h) - f(a) - hv}{|h|} \to 0 \iff \exists v \in \mathbb{R}^n \frac{f(a+h) - f(a)}{h} \to v$$

$\iff \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$ exists, then this limit is the derivative of f at a

Prop 7.2. We have open set $U \subset \mathbb{R}^m, f : U \to \mathbb{R}^n, a \in U$. f differentiable at $a \implies f$ continuous at a

Proof. Have

$$f(a+h) = f(a) + f'(a)(h) + \|h\| \cdot \varepsilon(h)$$

So for $x \in U$

$$f(x) = f(a) + f'(a)(x-a) + \|x-a\| \cdot \varepsilon(x-a)$$

$x \mapsto f(a)$ is constant, so continuous. $x \mapsto x - a$ is continuous. $f'(a)$ is linear, so continuous and $\|\cdot\|$ is continuous so $f'(a)(x-a)$ and $\|x-a\|$ are continuous in x. Finally, ε is continuous at 0, so $x \mapsto \varepsilon(x-a)$ is continuous at a by composition

Prop 7.3 (Chain rule)**.** We have open set U in $\mathbb{R}^m$ and V in $\mathbb{R}^n$ functions $f : U \to \mathbb{R}^n$ with $f(U) \subset V$, $g : V \to \mathbb{R}^p, a \in U$. Assume f is differentiable at a, g is differentiable at $b = f(a)$ then $g \circ f$ is differentiable at a and

$$(g \circ f)'(a) = g'(f(a)) \circ f'(a)$$

Proof. Let $S = f'(a),\ T = g'(f(a))$. We have

$$f(a+h) = f(a) + S(h) + \|h\| \cdot \varepsilon(h)$$
$$g(b+k) = g(b) + T(k) + \|k\| \cdot \zeta(k)$$

for suitable ε, ζ.

$$(g \circ f)(a+h) = g(\underbrace{f(a)}_{b} + \underbrace{S(h) + \|h\| \cdot \varepsilon(h)}_{k})$$

We put $k = k(h) = S(h) + \|h\| \cdot \varepsilon(h)$ so

$$\begin{aligned}(g \circ f)(a+h) &= g(b) + T(S(h) + \|h\| \cdot \varepsilon(h)) + \|k\| \cdot S(k) \\ &= (g \circ f)(a) + TS(h) + \underbrace{\|h\| T(\varepsilon(h)) + \|k\| \cdot \zeta(k)}_{\eta(h)}\end{aligned}$$

We claim

$$\frac{\eta(h)}{\|h\|} \to 0 \text{ as } h \to 0$$

Then this shows $g \circ f$ is differentiable at a and

$$(g \circ f)'(a) = TS = g'(f(a)) \circ f'(a)$$

$$\frac{\|h\| T(\varepsilon(h))}{\|h\|} = T(\varepsilon(h)) \to 0 \text{ as } h \to 0$$

as

$$\|T(\varepsilon(h))\| \leq \|T\| \cdot \|\varepsilon(h)\| \to 0 \text{ as } h \to 0$$

by Lemma 1.

$$\frac{\|k\|}{\|h\|} \leq \frac{\|S(h)\| + \|h\| \cdot \|\varepsilon(h)\|}{\|h\|} \leq \|S\| + \|\varepsilon(h)\|$$

$k = S(h) + \|h\|\varepsilon(h) \to 0$ as $h \to 0$ and hence $\zeta(k) \to 0$ as $k \to 0$. So

$$\frac{\eta(h)}{\|h\|} = T(\varepsilon(h)) + \frac{\|k\|}{\|h\|}\zeta(k) \to 0 \text{ as } h \to 0$$

Prop 7.4. U, f, a as before. Let f_j be the jth component of f $(1 \leq j \leq n)$. Then f is differentiable at $a \iff$ each f_j is differentiable at a and then

$$f'(a)(h) = \sum_{j=1}^{n} f_j'(a)(h) e_j'$$

Proof. Let $q_j : \mathbb{R}^n \to \mathbb{R}$ be the jth coordinate projection $q_j(y_1, \ldots, y_n) = y_j$ so $f_j = q_j \circ f$ and $f(x) = (f_1(x), \ldots, f_n(x))$.
$\implies$: Assume f differentiable at a. So by chain rule $f_j = q_j \circ f$ is differentiable at a and

$$f_j'(a) = q_j'(f(a)) \circ f'(a) = q_j \circ f'(a)$$

So

$$\begin{aligned} f'(a)(h) &= \sum_{j=1}^{n} q_j(f'(a)(h)) e_j' \\ &= \sum_{j=1}^{n} f_j'(a)(h) e_j' \end{aligned}$$

$\impliedby$: We have

$$f_j(a+h) = f_j(a) + f_j'(a)(h) + \|h\| \varepsilon_j(h)$$

for suitable ε_j.

$$\begin{aligned} f(a+h) &= \sum_{j=1}^{n} f_j(a+h) e_j' \\ &= \sum_{j=1}^{n} (f_j(a) + f_j'(a)(h) + \|h\| \cdot \varepsilon_j(h)) \varepsilon_j' \\ &= \sum_{j=1}^{n} f_j(a) e_j' + \sum_{j=1}^{n} f_j'(a)(h) e_j' + \|h\| \sum_{j=1}^{n} \varepsilon_j(h) e_j' \end{aligned}$$

Since $\varepsilon_j(h) \to 0$ as $h \to 0$ $\forall j$, we have $\varepsilon(h) \to 0$ as $h \to 0$ so f is differentiable at a

Prop 7.5. We have open set $U \subset \mathbb{R}^m$, functions $f, g : U \to \mathbb{R}^n, \varphi : U \to \mathbb{R}$, $a \in U$. Assume f, g, φ are differentiable at a. Then so are $f+g$ and $\varphi \cdot f$ and $(f+g)'(a) + f'(a) + g'(a)$ and

$$(\varphi \cdot f)'(a)(h) = \varphi(a) \cdot [f'(a)(h)] + [\varphi'(a)(h)] \cdot f(a)$$

Proof. Have

$$\begin{aligned} f(a+h) &= f(a) + f'(a)(h) + \|h\|\varepsilon(h) \\ g(a+h) &= g(a) + g'(a)(h) + \|h\|\zeta(h) \\ \varphi(a+h) &= \varphi(a) + \varphi'(a)(h) + \|h\|\eta(h) \end{aligned}$$

$$\begin{aligned} (f+g)(a+h) &= f(a+h) + g(a+h) \\ &= (f+g)(a) + (f'(a) + g'(a))(h) + \|h\| \cdot (\varepsilon(h) + \zeta(h)) \end{aligned}$$

Since $h \mapsto \varepsilon(h) + \zeta(h)$ is 0 at 0, continuous at 0, it follows that $f+g$ is differentiable at a and

$$(f+g)'(a) = f'(a) + g'(a)$$

$$\begin{aligned} (\varphi \cdot f)(a+h) &= \varphi(a+h) \cdot f(a+h) \\ &= (\varphi \cdot f)(a) + [\varphi(a) \cdot [f'(a)(h)] + [\varphi'(a)(h)] \cdot f(a)] \\ &\quad + f'(a)(h) \cdot \varphi(a)(h) + \|h\|\delta(h) \end{aligned}$$

where $\delta(h) = (f'(a)(h) \cdot \eta(h) + \varphi'(a)(h)\varepsilon(h) + \eta(h)f(a) + \varphi(a)\varepsilon(h) + \|h\| \cdot \eta(h)\varepsilon(h))$

$$\begin{aligned} \frac{|\varphi'(a)(h) \cdot f'(a)(h)}{\|h\|} &= \frac{|\varphi'(a)(h)| \cdot \|f'(a)(h)\|}{\|h\|} \\ &\le \frac{\|\varphi'(a)\| \cdot \|h\| \cdot \|f'(a)\| \cdot \|h\|}{\|h\|} \\ &= \|\varphi'(a)\| \cdot \|f'(A)\| \cdot \|h\| \to 0 \text{ as } h \to 0 \end{aligned}$$

$\delta(h) \to 0$ as $h \to 0$ since the same is true for $\varepsilon(h), \eta(h), f'(a)(h), \varphi'(a)(h)$

7.1 Partial Derivatives

Definition. We have an open set $U \subset \mathbb{R}^m$, a function $f : U \to \mathbb{R}^n$ and $a \in U$. Fix a direction u in $\mathbb{R}^m$, i.e. $u \in \mathbb{R}^m \backslash \{0\}$. If

$$\lim_{t \to 0} \frac{f(a+tu) - f(a)}{t}$$

exists, we call it the **directional derivative of f at a in direction u** and denote it $D_u f(a)$

Notes.

(i) $D_u f(a) \in \mathbb{R}^n$ and $f(a+tu) = f(a) + tD_u f(a) + o(t)$

(ii) Let $\gamma : \mathbb{R} \to \mathbb{R}^m$ by $\gamma(t) = a + tu$. Then $f \circ \gamma$ is defined on $\gamma^{-1}(U)$ which is open since γ is continuous and $0 \in \gamma^{-1}(U)$

$$\frac{f(a+tu) - f(a)}{t} = \frac{(f \circ \gamma)(t) - (f \circ \gamma)(0)}{t}$$

So $D_u f(a)$ exists $\iff$ $f \circ g$ is differentiable at 0 and then

$$D_u f(a) = (f \circ \gamma)'(0)$$

Special case: $u = e_i$, $1 \leq i \leq m$. If $D_{e_i} f(a)$ exists, we call it the **ith partial derivative of f at a** denoted $D_i f(a)$

Prop 7.6. Let U, f, a be as before. If f is differentiable at a, then $D_u f(a)$ exists $\forall u \in \mathbb{R}^m \backslash \{0\}$ and $D_u f(a) = f'(a)(u)$. Moreover

$$f'(a)(h) = \sum_{i=1}^{m} h_i D_i f(a) \quad \forall h = \sum_{i=1}^{m} h_i e_i \in \mathbb{R}^m$$

Proof. Have

$$f(a+h) = f(A) + f'(a)(h) + \|h\| \cdot \varepsilon(h)$$

for suitable ε. Put $h = tu$:

$$f(a+tu) = f(a) + t \cdot f'(a)(u) + |t| \|u\| \varepsilon(tu)$$

So

$$\frac{f(a+tu) - f(a)}{t} = f'(a)(u) + \frac{|t|}{t} \cdot \|u\| \varepsilon(tu) \to f'(a)(u)$$

So $D_u f(a) = f'(a)(u)$. Now for $h = \sum_{i=1}^{m} h_i e_i \in \mathbb{R}^m$, we have

$$f'(a)(h) = \sum_{i=1}^{m} h_i f'(a)(e_i) = \sum_{i=1}^{m} h_i D_i f(a)$$

Proof (Alternative)**.** Let $\gamma(t) = a + tu$. Then $f \circ \gamma$ is defined on the open set $\gamma^{-1}(U)$. Note that γ is differentiable and $\gamma'(t) = u \; \forall t$. BY Chain rule, $f \circ \gamma$ is differentiable at 0. So $D_u f(a)$ exists and

$$D_u f(a) = (f \circ \gamma)'(0) = f'(\gamma(0))(\gamma'(0)) = f'(a)(u)$$

Remarks.

(i) If $D_u f(a)$ exists, then so does $D_u f_j(a)$ ($f_j = q_j \circ f$ as in Prop 4). Indeed,

$$\frac{f_j(a_t u) - f_j(a)}{t} = q_j(\frac{f(a+tu) - f(a)}{t}) \to q_j(D_u f(a))$$

(ii) Converse of Prop 6 is false in general

7.2 Jacobian Matrix

Definition. Let U, f, a be as before. Assume f is differentiable at a. Then the **Jacobian matrix of f at a**, denoted $Jf(a)$, is the matrix of $f'(a)$ w.r.t. the SBs of $\mathbb{R}^m$ and $\mathbb{R}^n$. For $1 \le i \le m$, the ith column of $Jf(a)$ is

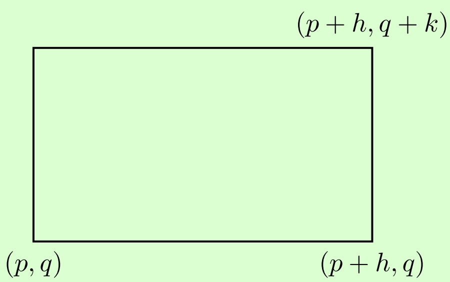
$$f'(a)(e_i) = D_i f(a)$$

For $1 \le j \le n$, the (j, i) entry of $Jf(a)$ is

$$[Jf(a)]_{j,i} = \langle D_i f(a), e'_j \rangle = q_j(D_i f(a)) = D_i f_j(a) = \frac{\partial f_j}{\partial x_i}$$

Theorem 7.7. U, f, a as before. Suppose $\exists$ open neighbourhood V of a with $V \subset U$ s.t. $D_i f(x)$ exists $\forall x \in V$ $\forall 1 \le i \le m$, and moreover $x \mapsto D_i f(x) : V \to \mathbb{R}^n$ is continuous at a $\forall 1 \le i \le m$. Then f is differentiable at a

Proof. By considering components of f, wlog $n = 1$. We now take $m = 2$ purely for notational convenience. Let $a = (p, q)$



Want

$$f'(p, q)(h, k) = hD_1 f(p, q) + kD_2(p, q)$$

Let

$$\psi(h, k) = f(p + h, q + k) - f(p, q) - hD_1 f(p, q) - kD_2 f(p, q)$$

] We need $\psi(h, k) = o(\|(h, k)\|)$. Then done.

$$\begin{aligned}\psi(h, k) &= f(p + h, q + k) - f(p + h, q) - kD_2 f(p, q) && \text{(I)}\\ &+ f(p + h, q) - f(p, q) - hD_1 f(p, q) && \text{(II)}\end{aligned}$$

II: this is $o(h)$ and hence $o(\|(h, k)\|)$ by def of $D_1 f(p, q)$
I: Let $\varphi(t) = f(p + h, q + tk)$ (fix (h, k)). Then φ is differentiable and

$$\varphi'(t) = D_2 f(p + h, q + tk) \cdot k$$

(Chain Rule). By MVT, $\exists t = t(h, k) \in (0, 1)$ s.t.

$$\varphi(1) - \varphi(0) = \varphi'(t)$$

So (I) $= \varphi(1) - \varphi(0) - kD_2 f(p, q) = kc[D_2 f(p + h, q + tk) - D_2 f(p, q)]$. As $(h, k) \to (0, 0)$, $(p + h, q + tk) \to (p, q)$ so by continuity of $D_2 f$ at a, I is $o(k)$ and hence $o(\|(h, k)\|)$

Theorem 7.8 (Mean Value Inequality, MVI). Let $U \in \mathbb{R}^m$ be open, $f : U \to \mathbb{R}^n$ be differentiable at every z in U. Let $a, b \in U$ s.t. the line segment

$$[a, b] = \{(1-t)a + tb : 0 \le t \le 1\} \subset U$$

Assume $\exists M \ge 0\ \forall z \in [a, b]\ \|f'(z)\| \le M$. Then $\|f(b) - f(a)\| \le M \cdot |b - a|$

Proof. Let $u = b - a,\ v = f(b) - f(a)$. Wlog $u \ne 0$. Let $\gamma(t) = a + tu,\ t \in \mathbb{R}$. Then $f \circ \gamma$ is defined on $\gamma^{-1}(U)$ and is differentiable by Chain Rule:

$$(f \circ \gamma)'(t) = f'(\gamma(t))(\gamma'(t)) = f'(a + tu)(u)$$

$$\begin{aligned} \|f(b) - f(a)\|^2 &= \langle f(b) - f(a), v\rangle \\ &= \langle (f \circ \gamma)(1) - (f \circ \gamma)(0), v\rangle \end{aligned}$$

Let us define $\varphi(t) = \langle (f \circ \gamma)(t), v\rangle$. Since $y \mapsto \langle y, v\rangle : \mathbb{R}^n \to \mathbb{R}$ is linear, by Chain Rule φ is differentiable and

$$\varphi'(t) = \langle (f \circ \gamma)'(t), v\rangle = \langle f'(a + tu)(u), v\rangle$$

By MVT $\exists \theta \in (0, 1)$ s.t. $\varphi(1) - \varphi(0) = \varphi'(\theta)$ so

$$\begin{aligned} \|f(b) - f(a)\|^2 &= \varphi(1) - \varphi(0) = \varphi'(\theta) \\ &= \langle f'(a + \theta u)(u), v\rangle \\ &\le \|f'(a + \theta u)(u)\| \cdot \|v\| \\ &\le \|f'(a + \theta u)\| \cdot \|u\| \cdot \|v\| \le M \cdot \|b - a\| \cdot \|v\| \end{aligned}$$

Hence $\|f(b) - f(a)\| \le M\|b - a\|$

Corollary 7.9. Let U be an open, connected subset of $\mathbb{R}^m$ and $f : U \to \mathbb{R}^n$ be differentiable at every $a \in U$. If $f'(a) = 0\ \forall a \in U$ then f is constant

Proof. If $a, b \in U$ satisfy $[a, b] \subset U$ then by MVI (Thm 8)

$$\|f(b) - f(a)\| \le (\sup_{z \in [a,b]} \|f'(z)\|) \cdot \|b - a\| = 0$$

So $f(a) = f(b)$. For $x \in U\ \exists r > 0$ s.t. open ball $D_r(x) \subset U\ \forall y \in D_r(x)\ [x, y] \subset D_r(x) \subset U$ so $f(y) = f(x)$. So f is locally constant and hence constant since U is connected

Remark. Let $V \subset \mathbb{R}^m$, $W \subset \mathbb{R}^m$ be open sets and $f : V \to W$ be a bijection. Let $a \in V$. Assume f is differentiable at a and $f^{-1} : W \to V$ is differentiable at $f(a)$. Let $S = f'(a),\ T = (f^{-1})^{-1}(f(a))$. By Chain Rule:

$$TS = (f^{-1} \circ f)'(a) = I_m$$

$$ST = (f \circ f^{-1})'(f(a)) = I_n$$

So $m = \operatorname{tr}(TS) = \operatorname{tr}(ST) = n$ and so $f'(a)$ is invertible. Aim to prove an inverse.

Definition. Let $U \subset \mathbb{R}^m$ open and $f : U \to \mathbb{R}^n$ function. Say f is **differentiable on U** if f is differentiable at a for every $a \in U$. Then the **derivative of f on U** is $f : U \to L(\mathbb{R}^m, \mathbb{R}^m)$ $a \mapsto f'(a)$. Say f is a **C^1-function** on U if f is continuously differentiable on U, i.e. f is differentiable on U and $f' : U \to L(\mathbb{R}^m, \mathbb{R}^n)$ is continuous

Theorem 7.10 (Inverse Function Theorem, IFT)**.** Let $U \subset \mathbb{R}^n$ be open and $f : U \to \mathbb{R}^n$ be a C^1-function. Let $a \in U$ and assume $f'(a)$ is invertible. Then $\exists$ open sets V, W s.t. $a \in V,\ f(a) \in W,\ V \subset U$ and $f|_V : V \to W$ is a bijection with inverse function $g : W \to V$ also a C^1-function. Moreover

$$g'(y) = [f'(g(y))]^{-1} \quad \forall y \in W$$

Proof. (i) We show that WLOG $a = f(a) = 0, f'(a) = I$. Let $T = f'(a)$ and define $h(x) = T^{-1}(f(x+a) - f(a))$. The domain of h is $U - a$ and by Chain Rule h is differentiable:

$$h'(x) = T^{-1} \circ f'(a+x)$$

So $x, y \in U - a$:

$$\begin{aligned} \|h^{-1}(x) - h^{-1}(y)\| &= \|T^{-1} \circ (f'(a+x) - f'(a+y))\| \\ &\leq \|T^{-1}\| \cdot \|f'(a+x) - f'(a+y)\| \end{aligned}$$

It follows that h is a C^1-function. Also $h(0) = 0$ and $h'(0) = I$. If we prove te result for h, it will follow for f since

$$f(x) = T(h(x-a)) + f(a)$$

(ii) We now assume $f(0) = 0,\ f'(0) = I$. Since f' os continuous, $\exists r > 0$ s.t. $B_r(0) \subset U$ and $\forall x \in B_r(0) \|f'(x) - I\| \leq 1/2$. We will show that $\forall x, y \in B_r(0)\ \|f(x) - f(y)\| \geq 1/2 \cdot \|x - y\|$. To see this, let $p : U \to \mathbb{R}^n,\ p(x) = f(x) - x$. Then $p'(x) = f'(x) - I$, so $\|p'(x)\| \leq 1/2\ \forall x \in B_r(0)$. By MVI $\|p(x) - p(y)\| \leq 1/2 \cdot \|x - y\|\ \forall x, y \in B_r(0)$. Hence

$$\begin{aligned} \|f(x) - f(y)\| &= \|(p(x) + x) - (p(y) + y)\| \\ &\geq \|x - y\| - \|p(x) - p(y)\| \geq \frac{1}{2}\|x - y\| \end{aligned}$$

(iii) Let $s = r/2$. We show $f(D_r(0)) \supset D_s(0)$. More precisely, $\forall w \in D_s(0)\ \exists$ unique $x \in D_r(0)$ s.t. $f(x) = w$. Fix $w \in D_s(0)$. Define for $x \in B_r(0)$

$$q(x) = w - f(x) + x = w - p(x)$$

(Note $f(x) = w \iff q(x) = x$)
Since $p(0) = 0$, we have for $x \in B_r(0)$

$$\begin{aligned} \|q(x)\| &\leq \|w\| + \|p(x)\| = \|w\| + \|p(x) - p(0)\| \\ &\leq \|w\| + \frac{1}{2}\|x\| < s + \frac{1}{2}r = r \end{aligned}$$

So $q(B_r(0)) \subset D_r(0) \subset B_r(0)$. For $x, y \in B_r(0)$

$$\|q(x) - q(y)\| = \|p(x) - p(y)\| \leq \frac{1}{2}\|x - y\|$$

So $q : B_r(0) \to B_r(0)$ is a contraction mapping on the nonempty complete metric space $B_r(0)$. By CMT, $\exists$ unique $x \in B_r(0)$ s.t. $q(x) = x$. Note $x = q(x) \in D_r(0)$ by above

Proof. (iv) Let $W = D_s(0),\ V = D_r(0) \cap f^{-1}(W)/$ Then $f|_V : V \to W$ is a bijection with inverse $g : W \to V$ continuous. W is open and $f(0) = 0 \in W$ and since f is continuous $f^{-1}(W)$ is open, so V is open and $0 \in V$. (iii) says that $f|_V : V \to W$ is a bijection. Finally let $u, v \in W$ and let $x = g(u),\ y = g(v)$. Then

$$\begin{aligned} \|g(u) - g(v)\| = \|x - y\| &\le 2\|f(x) - f(y)\| \\ &= 2\|u - v\| \end{aligned}$$

So g is Lipschitz with constant 2 so continuous

(v) (Non-examinable) g in C^1 and $\forall y \in W\ g'(y) = [f'(g(y))]^{-1}$

7.3 Second Derivative

Definition. We are given open set $U \in \mathbb{R}^m : f : U \to \mathbb{R}^n$ and $a \in U$. Assume $\exists$ open set V s.t. $a \in V \subset U$ and f is differentiable on V. Say f is **twice differentiable at a** if $f' : V \to L(\mathbb{R}^m, \mathbb{R}^n)$ is differentiable at a. Let $f''(a) = (f')'(a)$ - called the **second derivative of f at a**

Note.

$$f''(a) \in L(\mathbb{R}^m, L(\mathbb{R}^m, \mathbb{R}^n))$$

7.3.1 Second derivative as a bilinear map

Remark.

$$L(\mathbb{R}^m, L(\mathbb{R}^m, \mathbb{R}^n)) \cong \text{ Bil}(\mathbb{R}^m \times \mathbb{R}^m, \mathbb{R}^n)$$

$$T \leftrightarrow \tilde{T}$$

For $h, k \in \mathbb{R}^m$ $T(h)(k) = \tilde{T}(h, k)$. From now we identify T and $\tilde{T}$

Prop 7.11. Have $U \subset \mathbb{R}^m$ open, $f : U \to \mathbb{R}^n$, $a \in U$. Assume f is differentiable on V where $a \in V \subset U$, V open. Then f is twice differentiable at $a \iff \exists T \in \text{ Bil}(\mathbb{R}^m \times \mathbb{R}^m, \mathbb{R}^n)$ s.t. for every fixed $k \in \mathbb{R}^m$

$$f'(a+h)(k) = f'(a)(k) + T(h,k) + o(\|h\|)$$

Then $T = f''(a)$

Proof. "$\implies$": Assume f twice differentiable at a:

$$f'(a+h) = f'(a) + f''(a)(h) + \|h\| \cdot \varepsilon(h)$$

where $\varepsilon : V - a \to L(\mathbb{R}^m, \mathbb{R}^n)$ s.t. $\varepsilon(0) = 0$ and ε is continuous at 0. Fix $k \in \mathbb{R}^m$ and evaluate at k:

$$f'(a+h)(k) = f'(a)(k) + f''(a)(h,k) + \|h\| \cdot \varepsilon(h)(k)$$

Here $f''(a) \in \text{ Bil}(\mathbb{R}^m \times \mathbb{R}^m, \mathbb{R}^n)$ and

$$\|\varepsilon(h)(k)\| \leq \|\varepsilon(h)\| \cdot \|k\| \to 0 \text{ as } h \to 0$$

so $\|h\| \cdot \varepsilon(h)(k) = o(\|h\|)$
"$\impliedby$": Assume $T \in \text{ Bil}(\mathbb{R}^m \times \mathbb{R}^m, \mathbb{R}^n)$ and

$$\frac{f'(a+h)(k) - f'(a)(t) - T(h,k)}{\|h\|} \to 0 \text{ in } \mathbb{R}^n \text{ as } h \to 0$$

with k fixed. We need

$$\varepsilon(h) = \frac{f'(a+h-f'(a)) - T(h)}{\|h\|} \to 0 \text{ in } L(\mathbb{R}^m, \mathbb{R}^n) \text{ as } h \to 0$$

We know for fixed $k \in \mathbb{R}^m, \varepsilon(h)(k) \to 0$ in $\mathbb{R}^n$ as $h \to 0$. It follows that

$$\|\varepsilon(h)\| = (\sum_{i=1}^{m} \|\varepsilon(h)(e_i)\|^2)^{1/2} \to 0 \text{ as } h \to 0$$

Examples. (i) $f : \mathbb{R}^m \to \mathbb{R}^n$ linear. Then f is differentiable on $\mathbb{R}^m$ and $f'(a) = f\ \forall a \in \mathbb{R}^m$. So $f : \mathbb{R}^m \to L(\mathbb{R}^m, \mathbb{R}^n) a \mapsto f\ \forall a \in \mathbb{R}^m$. So f' is constant so f' is differentiable on $\mathbb{R}^m$ and $f''(a) = 0\ \forall a \in \mathbb{R}^m$

(ii) $f : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^p$ bilinear. Then f is differentiable on $\mathbb{R}^m \times \mathbb{R}^n$ and for $(a, b) \in \mathbb{R}^m \times \mathbb{R}^n$

$$f'(a,b)(h,k) = f(a,k) + f(h,b)$$

Note: this is linear in (a, b) with (h, k)-fixed. So $f : \mathbb{R}^m \times \mathbb{R}^n \to L(\mathbb{R}^m \times \mathbb{R}^n, \mathbb{R}^p)\ (a, b) \mapsto f'(a, b)$ is itself linear so differentiable on $\mathbb{R}^m \times \mathbb{R}^n$ and

$$f''(a,b) = f' \in L(\mathbb{R}^m \times \mathbb{R}^n, L(\mathbb{R}^m \times \mathbb{R}^n, \mathbb{R}^p)) \cong \text{ Bil}((\mathbb{R}^m \times \mathbb{R}^n) \times (\mathbb{R}^m \times \mathbb{R}^n), \mathbb{R}^p)$$

(iii) $f : M_n \to M_n\ f(A) = A^3$. Fix $A \in M_n$

$$f(A+H) = (A+H)^3 = A^3 + \underbrace{A^2H + AHA + HA^2}_{\text{linear in } H} + \underbrace{AH^2 + HAH + H^2A + H^3}_{o(\|H\|)}$$

So f is differentiable at A and $f'(A)(H) = A^2H + AHA + HA^2$ so f is differentiable on M_n. Fix $A \in M_n$ and $K \in M_n$

$$\begin{aligned} f'(A+H)(K) &= (A+H)^2K + (A+H)K(A+H) + K(A+H)^2 \\ &= A^2K + AKA + KA^2 + [AHK + HAK + AKH + HKA + KAH + KHA] \\ &\quad + [\underbrace{H^2K + HKH + KH^2}_{o(\|H\|)}] \end{aligned}$$

Note that $T : M_n \times M_n \to M_n$

$$T(H,K) = AHK + HAK + AKH + HKA + KAH + KHA$$

is bilinear. So the above shows that f is twice differentiable at A and $f''(A) = T$ (prop 11)

7.3.2 Second Derivative and Partial Derivatives

Have open $U \subset \mathbb{R}^m$, funtion $f : U \to \mathbb{R}^n$, $a \in U$. Assume f is twice differentiable at a: so f is differentiable on some open set V with $a \in V \subset U$ and $f' : V \to L(\mathbb{R}^m, \mathbb{R}^n)$, $x \mapsto f'(x)$ is differentiable at a

$$f'(a+h) = f'(a) + f''(a)(h) + o(\|h\|)$$

So

$$f'(a+h)(k) = f'(a)(k) + f''(a)(h,k) + o(\|h\|)$$

with $k \in \mathbb{R}^m$ fixed. Fix $u, v \in \mathbb{R}^m \backslash \{0\}$. Put $k = v$:

$$D_v f(a+h) = D_v f(a) + f''(a)(h,v) + o(\|h\|)$$

So

$$D_v f : V \to \mathbb{R}^n, x \mapsto D_v f(x) = f'(x)(v)$$

is differentiable at a and $(D_v f)'(a)(h) = f''(a)(h,v)$ so

$$\begin{aligned} D_u D_v f(a) &= D_u(D_v f)(a) \\ &= (D_v f)'(a)(u) = f''(a)(u,v) \end{aligned}$$

In particular

$$D_i D_j f(a) = f''(a)(e_i, e_j)$$

for $1 \leq i, j \leq m$

Theorem 7.12 (Symmetry of mixed directional derivatives)**.** Let U, f, a be as above. Assume f is twice differentiable on some open set V wit h$a \in V \subset U$. Assume $f'' : V \to \text{Bil}(\mathbb{R}^m \times \mathbb{R}^m, \mathbb{R}^n)$, $x \mapsto f''(x)$ is continuous at a. Then $\forall u, v \in \mathbb{R}^m \backslash \{0\}$

$$D_u D_v f(a) = D_v D_u f(a)$$

equivalently

$$f''(a)(u, v) = f''(a)(v, u)$$

i.e. $f''(a)$ is a symmetric bilinear map

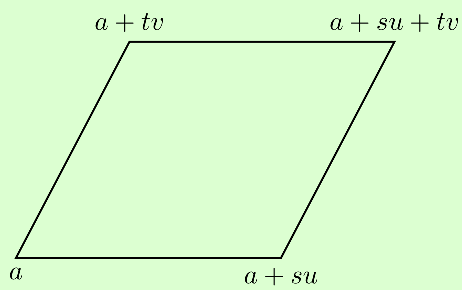
Proof. Wlog $n = 1$. For $1 \leq j \leq n$

$$(D_u f)_j(x) = [D_u f(x)]_j = [f'(x)(u)]_j = f'_j(x)(u) = D_u f_j(x)$$

So $(D_u f)_j = D_u f_j$. Repeat:

$$(D_v D_u f)_j = D_v (D_u f)_j = D_v D_u f_j$$

Enough to show that $D_v D_u f_j(a) = D_u D_v f_j(a)$



$s, t \in \mathbb{R}$

Consider

$$\varphi(s, t) = f(a + su + tv) - f(a + tv) - f(a + su) + f(a)$$

Fix x, t. Consider $\psi(y) = f(a + yu + tv) - f(a + yu)$. Note $\varphi(s, t) = \psi(s) - \psi(0)$. By MVT $\exists \alpha = \alpha(s, t) \in (0, 1)$ s.t.

$$\varphi(s, t) = \psi(s) - \psi(0) = s \cdot \psi'(\alpha \cdot s) = s(D_u f(a + \alpha su + tv) - D_u f(a + \alpha su))$$

Apply MVT to $y \mapsto D_u f(a + \alpha su + yv)$

$$\varphi(s, t) = s \cdot t \cdot D_v D_u f(a + \alpha su + \beta tv)$$

for some $\beta = \beta(s, t) \in (0, 1)$. So

$$\begin{aligned} \frac{\varphi(s, t)}{st} &= D_v D_u f(a + \alpha su + \beta tv) \\ &= f''(a + \alpha su + \beta tv)(u, v) \\ &= \to f''(a)(u, v) \end{aligned}$$

since f'' is continuous at a.
Repeat above with $\psi(y) = f(a + su + yv) - f(a + yv)$ to get

$$\frac{\varphi(s, y)}{st} \to f''(a)(v, u)$$