# Analysis

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## Contents

# 1 Limits and Convergence

## 1.1 Review of Numbers and Sets

**Notation.** Write sequences as: $a_n$, $(a_n)_{n=1}^{\infty}$, $a_n \in \mathbb{R}$

**Definition.** We say that $a_n \to a$ as $n \to \infty$ if given $\varepsilon > 0$, $\exists N$ s.t. $|a_n - a| < \varepsilon$ for all $n \geq N$

**Note.** $N = N(\varepsilon)$

**Definition** (increasing sequence)**.** $a_n \leq a_{n+1}$

**Definition** (decreasing sequence)**.** $a_n \geq a_{n+1}$

**Definition** (strictly increasing sequence)**.** $a_n < a_{n+1}$

**Definition** (strictly decreasing sequence)**.** $a_n > a_{n+1}$

**Note.** Say monotone if stays increasing or stays decreasing

## 1.2 Fundamental Axiom of the real numbers

**Axiom.** If $a_n \in \mathbb{R}, \forall n \geq 1, A \in \mathbb{R}$ and $a_1 \leq a_2 \leq a_3 \leq \dots$ with $a_n \leq A$ for all $n$, there exists $a \in \mathbb{R}$ s.t. $a_n \to a$ as $n \to \infty$
i.e. an increasing sequence of real numbers bounded above converges.

**Note.** Equivalently: a decreasing sequence of real numbers bounded below converges
Equivalent also to: every non-empty set of real numbers bounded above has a supremum

**Notation.** Say LUBA = Least Upper Bound Axiom.

**Definition** (supremum)**.** For $S \subseteq \mathbb{R}$, $S \neq \varnothing$, $\sup S = K$ if

(i) $x \leq K$, $\forall x \in S$
(ii) given $\varepsilon > 0, \exists x \in S$, s.t. $x > K - \varepsilon$

**Note.** Supremum is unique (see N&S notes), infinimum defined similarly.

**Lemma 1.1.**

(i) The limit is unique. That is, if $a_n \to a$, and $a_n \to b$, then $a = b$

(ii) If $a_n \to a$ as $n \to \infty$ and $n_1 < n_2 < n_3 < \dots$, then $a_{n_j} \to a$ as $j \to \infty$ (subsequences converge to the same limit)

(iii) If $a_n = C \; \forall n$, then $a_n \to C$ as $n \to \infty$

(iv) If $a_n \to a$ & $b_n \to b$, then

$$a_n + b_n \to a + b$$

(v) If $a_n \to a$ & $b_n \to b$, then

$$a_n b_n \to ab$$

(vi) If $a_n \to a$, $a_n \neq 0 \; \forall n \,\&\, a \neq 0$ then

$$\frac{1}{a_n} \to \frac{1}{a}$$

(vii) If $a_n \leq A \; \forall n$ and $a_n \to a$, then $a \leq A$

**Proof.**

(i) given $\varepsilon > 0$, $\exists n_1$ s.t. $|a_n - a| < \varepsilon \, \forall n \geq n_1$
and $\exists n_2$ s.t. $|a_n - b| < \varepsilon \, \forall n \geq n_2$
Let $N = \max\{n_1, n_2\}$. Then $\forall n \geq N$

$$|a - b| \leq |a_n - a| + |a_n - b| < 2\varepsilon \, \forall n \geq N$$

If $a \neq b$, take

$$\varepsilon = \frac{|a-b|}{3} \implies |a - b| < \frac{2}{3}|a - b| \text{ ※}$$

(ii) Given $\varepsilon > 0, \exists N$ s.t. $|a_n - a| < \varepsilon \, \forall n \geq N$. Since $n_j \geq j$ (induction),

$$|a_{n_j} - a| < \varepsilon \, \forall j \geq N$$

i.e. $a_{n_j} \to a$ as $j \to \infty$

(iii) Exercise.

(iv) Exercise.

(v)

$$\begin{aligned} |a_n b_n - ab| &\leq |a_n b_n - a_n b| + |a_n b - ab| \\ &= |a_n||b_n - b| + |b||a_n - a| \end{aligned}$$

As $a_n \to a$, given $\varepsilon > 0$, $\exists N_1$ s.t. $|a_n - a| < \varepsilon \, \forall n \geq N_1$ (*)
As $b_n \to b$, given $\varepsilon > 0, \exists N_2$ s.t. $|b_n - b| < \varepsilon \, \forall n \geq N_2$
(*) $\implies$ if $n \geq N_1(1)$, $|a_n - a| < 1$, so:

$$|a_n| \leq |a| + 1$$

$$\implies |a_n b_n - ab| \leq \varepsilon(|a| + 1 + |b|) \, \forall n \geq N_3 = \max\{N_1(1), N_1(\varepsilon), N_2(\varepsilon)\}$$

(vi) Exercise.

(vii) Exercise.

**Lemma 1.2.**

$$\frac{1}{n} \to 0 \text{ as } n \to \infty$$

**Proof.** $1/n$ is a decreasing sequence bounded below so by the fundamental Axiom it has limit $a$.

**Claim.** $a = 0$

**Proof.**

$$\frac{1}{2n} = \frac{1}{2} \times \frac{1}{n} \to \frac{a}{2}$$

by lemma 1.1(v)
But $\frac{1}{2n}$ is a subsequence, so by 1.1(ii) $\frac{1}{2n} \to a$. By uniqueness of limits, lemma 1.1(i), we have

$$a = \frac{a}{2} \implies a = 0 \;\square$$

**Remark.** The definition of limit of a sequence makes perfect sence for $a_n \in \mathbb{C}$

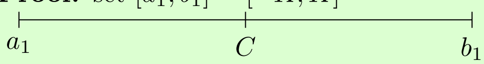
**Definition.** $a_n \to a$ if given $\varepsilon > 0$, $\exists N$ s.t. $\forall n \geq N$, $|a_n - a| < \varepsilon$.
First six parts of Lemma 1.1 are the same over $\mathbb{C}$.
The last one does not makes sense (over $\mathbb{C}$) since it uses the order of $\mathbb{R}$.

## 1.3 Bolzano-Weierstass Theorem

**Theorem 1.3** (Bolzano-Weierstass)**.** If $x_n \in \mathbb{R}$ and there exists $K$ s.t. $|x_n| \leq K \ \forall n$, then we can find $n_1 < n_2 < n_3 < \dots$ and $x \in \mathbb{R}$ s.t. $x_{n_j} \to x$ as $j \to \infty$
In other words: every bounded sequence has a convergent subsequence.

**Remark.** We say nothing about uniqueness of limit, $x_n = (-1)^n$, $x_{2n+1} \to -1$, $x_{2n} \to 1$

**Proof.** set $[a_1, b_1] = [-K, K]$




$C$ = mid point
Consider the following cases:
(i) $x_n \in [a_1, c]$ for $\infty$ many values of $n$
(ii) $x_n \in [c, b_1]$ for $\infty$ many values of $n$
(i) & (ii) could both hold at the same time.
If (i) holds then we set $a_2 = a_1$ and $b_2 = C$. If (i) fails, we have that (ii) must hold and we set $a_2 = C$ & $b_2 = b_1$
Proceed inductively to construct sequences $a_n$, $b_n$ s.t. $x_m \in [a_n, b_n]$ for infinitely many values of $m$.

$$a_{n-1} \leq a_n \leq b_n \leq b_{n-1}$$
$$b_n - a_n = \frac{b_{n-1} - a_{n-1}}{2} \tag{*}$$

**Note.** Called 'bijection method' or "lion hunting"

$a_n$ increasing sequence and bounded
$b_n$ decreasing sequence and bounded
By the Fundamental Axiom,

$$a_n \to a \in [a_1, b_1]$$
$$b_n \to b \in [a_1, b_1]$$

Use (*),

$$b - a = \frac{b-a}{2}$$
$$\implies b - a$$

Since $x_m \in [a_n, b_n]$ for $\infty$ many values of $m$, having chosen $n_j$ s.t. $x_{n_j} \in [a_j, b_j]$, there is $n_{j+1} > n_j$ s.t. $x_{j+1} \in [a_{j+1}, b_{j+1}]$
(I have an "unlimited supply"!)
Hence

$$a_j \leq x_{n_j} \leq b_j$$
$$\implies x_{n_j} \to a \square$$

## 1.4 Cauchy Sequences

**Definition.** $a_n \in \mathbb{R}$ is called a **Cauchy sequence** if given $\varepsilon > 0$, $\exists N > 0$ s.t. $|a_n - a_m| < \varepsilon \ \forall n, m \geq N$

**Lemma 1.4.** A convergent sequence is a Cauchy sequence.

**Proof.** if $a_n \to a$, given $\varepsilon > 0$, $\exists N$ s.t. $\forall n \geq N$, $|a_n - a| < \varepsilon$
Take $m, n \geq N$,

$$|a_n - a_m| \leq |a_n - a| + |a_m - a| < 2\varepsilon \square$$

**Theorem 1.5.** Every Cauchy sequence is convergent.

**Proof.**

**Claim.** If $a_n$ is Cauchy, then it is bounded.

**Proof.** Take $\varepsilon = 1, N = N(1)$, in the Cauchy property, then

$$|a_n - a_m| < 1, \forall n, m \geq N(1)$$

$$|a_m| \leq |a_m - a_N| + |a_N| < 1 + |a_N| \, \forall m \geq N$$

Let $K = \max\{1 + |a_N|, |a_n|, n = 1, 2, \dots N - 1\}$
Then $|a_n| \leq K \, \forall n$ ✓
By the Bolzano-Weierstrass theorem,

$$a_{n_j} \to a$$

**Claim.** $a_n \to a$

**Proof.** Given $\varepsilon > 0$, $\exists j_0$ s.t. $\forall j \geq j_0$

$$|a_{n_j} - a| < \varepsilon$$

Also, $\exists N(\varepsilon)$ s.t. $|a_m - a_n| < \varepsilon \, \forall m, n \geq N(\varepsilon)$
Take $j$ s.t. $n_j \geq \max\{N)\varepsilon), n_{j_0}\}$
Then if $n \geq N(\varepsilon)$,

$$|a_n - a| \leq |a_n - a_{n_j}| + a_{n_j} - a| < 2\varepsilon \square$$

**Remark.** Thus on $\mathbb{R}$ a sequence is convergent iff it is Cauchy.
"Old-fashioned name": "the general principle of convergence"

**Note.** This is a useful property since we do not need to know what the limit is.

### 1.5 Series

**Definition.** $a_n \in \mathbb{R}, \mathbb{C}$. We say that $\sum_{j=1}^{\infty} a_j$ **converges to** $s$ if the sequence of partial sums

$$S_N = \sum_{j=1}^{N} a_j \to s$$

as $N \to \infty$
We write $\sum_{j=1}^{\infty} a_j = s$

If $S_N$ does not converge, we say that $\sum_{j=1}^{\infty} a_j$ diverges.

**Remark.** Any problem on series can be turned into a problem on sequences just by considering the sequence of partial sums.

**Lemma 1.6.**

(i) If $\sum_{j=1}^{\infty} a_j$ & $\sum_{j=1}^{\infty} b_j$ converge, then so does $\sum_{j=1}^{\infty} (\lambda a_j + \mu b_j)$ where $\lambda, \mu \in \mathbb{C}$

(ii) Suppose $\exists N$ s.t. $a_j = b_j \, \forall j \geq N$, then either $\sum_{j=1}^{\infty} a_j$ & $\sum_{j=1}^{\infty} b_j$ both converge or both diverge (initial terms do not matter)

**Proof.**
(i)

$$\begin{aligned} S_N &= \sum_{j=1}^{N} a(\lambda a_j + \mu b_j) \\ &= \lambda \sum_{j=1}^{N} a_j + \mu \sum_{j=1}^{N} b_j \\ &= \lambda c_N + \mu d_N \end{aligned}$$

$c_N \to c \& d_N \to d$ so by lemma 1.1 (version $\mathbb{C}$), $s_N \to \lambda c + \mu d$

(ii) $n \geq N$

$$s_n = \sum_{1}^{n} a_j = \sum_{1}^{N-1} a_j + \sum_{N}^{n} a_j$$

$$d_n = \sum_{1}^{n} b_j = \sum_{1}^{N-1} b_j + \sum_{N}^{n} b_j$$

$$\implies s_n - d_n = \sum_{1}^{N-1} a_j - \sum_{1}^{N-1} b_j$$

(as $a_j = b_j$ for $j \geq N$)
so $s_n$ converges iff $d_n$ does. $\square$

### 1.5.1 The Geometric Series

**Claim.** The geometric series converges iff $|x| < 1$

**Proof.** Set $a_n = x^n - 1\, n \geq 1$

$$S_n = \sum_1^n a_g = 1 + x^2 + \cdots + x^{n-1}$$

Then

$$s_n = \begin{cases} \frac{1-x^n}{1-x} & \text{for } x \neq 1 \\ n & \text{for } x = 1 \end{cases}$$

$$xS_n = x + x^2 + \cdots + x^n = S_n - 1 + x^n$$

$$\implies S_n(1-x) = 1 - x^n$$

if $|x| < 1$, $x^n \to 0$ and $S_n \to \frac{1}{1-x}$
if $x > 1$, $x^n \to \infty \,\&\, S_n \to \infty$
if $x < -1$, $S_n$ does not converge (oscillates)
if $x = -1$, $s = \begin{cases} 1 \text{ for } n \text{ odd} \\ 0 \text{ for } n \text{ even} \end{cases}$

**Note.** Say $S_n \to \infty$ if given $A$, $\exists N$ s.t. $S_n > A$, $\forall n \geq N$
$S_n \to -\infty$, if given $A$, $\exists N$ s.t. $S_n < -A$ for all $n \geq N$
If $S_n$ does not converge or tend to $\pm\infty$, we say that $S_n$ oscillates.

**Claim.** $x^n \to 0$ if $|x| < 1$

**Proof.** Consider the case $0 < x < 1$ and we write $\frac{1}{x} = 1\delta$, $\delta > 0$
So:

$$x^n = \frac{1}{(1+\delta)^n} \leq \frac{1}{1+\delta n} \to 0$$

because $(1+\delta)^n \geq 1 + n\delta$ (from the binomial expansion)

**Lemma 1.7.** If $\sum_{j=1}^{\infty} a_j$ converges, then:

$$\lim_{j\to\infty} a_j = 0$$

**Proof.**

$$S_n = \sum_1^n a_j$$

$$a_n = S_n - S_{n-1}$$

So if $S_n \to a$ then $a_n \to 0$ (since $S_{n-1} \to a$ also)

**Remark.** The converse of 1.7 is false! Shown by example below:

**Claim.** $\sum\limits_1^\infty \frac{1}{n}$ diverges (harmonic series)

**Proof.**

$$S_n = \sum_1^\infty \frac{1}{j}$$

$$S_{2n} = S_n + \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n} > S_n + \frac{1}{2}$$

Since $\frac{1}{n+k} \geq \frac{1}{2n}$ for $k = 1, 2, \ldots, n$
So if $S_n \to a$, then $S_{2n} \to a$ also and thus

$$a \geq a + \frac{1}{2} ※$$

### 1.5.2 Series of Positive/ Non-negative terms

**Theorem 1.8** (The Comparison Test)**.** Suppose $0 \leq b_n \leq a_n \forall n$
Then if $\sum\limits_1^\infty a_n$ converges, so does $\sum\limits_1^\infty b_n$

**Proof.** Let $S_N = \sum\limits_1^N a_n$
$d_N = \sum\limits_1^N b_n$
$b_n \leq a_n \implies d_N \leq S_N$
But $S_N \to S$, then

$$d_N \leq S_N \leq S \,\forall N$$

and $d_N$ is an increasing sequence bounded above $\implies d_N$ converges $\square$

An example using this below:

**Claim.** $\sum_1^n \frac{1}{n^2}$ converges

**Proof.**

$$\frac{1}{n^2} < \frac{1}{n(n-1)} = \frac{1}{n-1} - \frac{1}{n} = a_n$$

$$\sum_2^N a_n = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \cdots + \frac{1}{N-1} - \frac{1}{N}$$
$$= 1 - \frac{1}{N} \to 1 \text{ as } N \to \infty$$

By comparison, $\sum_1^n \frac{1}{n^2}$ converges

In fact, we get $\sum_1^n \frac{1}{n^2} \leq 1 + 1 = 2$

**Note.** Converges to $\frac{\pi^2}{6}$ but we do not prove that here.

**Theorem 1.9** (Root test/ Cauchy's test for convergence)**.** Assume $a_n \geq 0$ and $a_n^{1/n} \to a$ as $n \to \infty$. Then if $a < 1$, $\sum a_n$ converges; if $a > 1$, $\sum a_n$ diverges

**Proof.** If $a < 1$, choose $a < r < 1$.
By definition of limit,
$\exists N$ s.t. $\forall n \geq N$

$$a_n^{1/n} < r \implies a_n < r^n$$

But since $r < 1$, the geometric series $\sum r^n$ converges $\implies$ by Theorem 1.8, $\sum a_n$ converges.
If $a > 1$, then for $n \geq N$,

$$a^{1/n} > 1 \implies a_n > 1$$

Thus $\sum a_n$ diverges (since $a_n$ does not tend to zero). □

**Remark.** Nothing can be said if $a = 1$, see examples later.

**Theorem 1.10** (Ratio test/ D'Alanbert's test)**.** Suppose $a_n > 0$ and $\frac{a_{n+1}}{a_n} \to l$
If $l < 1$, $\sum a_n$ converges.
If $l > 1$, $\sum a_n$ diverges

**Proof.** Suppose $l < 1$ and choose $r$ with $l < r < 1$
Then $\exists N$ s.t. $\forall n \geq N$,

$$\frac{a_{n+1}}{a_n} < r$$

Therefore

$$a_n = \frac{a_n}{a_{n-1}} \frac{a_{n-1}}{a_{n-2}} \dots \frac{a_{N+1}}{a_N} a_N < a_N r^{n-N}, \; n > N$$

$$\implies a_n < K r^n$$

with $K$ independent of $n$
Since $\sum r^n$ converges, so does $\sum a_n$ by Theorem 1.8
If $l > 1$, choose $1 < r < l$
Then $\frac{a_{n+1}}{a_n} > r \, \forall n \geq N$
And as before:

$$a_n = \frac{a_n}{a_{n-1}} \frac{a_{n-1}}{a_{n-2}} \dots \frac{a_{N+1}}{a_N} a_N > a_N r^{n-N}, \; n > N$$

$$a_N r^{n-N} \to \infty \text{ as } n \to \infty$$

So $\sum a_n$ diverges. □

**Remark.** Nothing can be said if $a = 1$.

Examples: Consider ratio test for series $\sum_1^\infty \frac{n}{2^n}$

$$\frac{n+1}{2^{n+1}} \frac{2^n}{n} = \frac{n+1}{2n} \to \frac{1}{2} < 1$$

So we have convergence by the ratio test.

The following examples show limit 1 inconclusive:
$\sum_1^n \frac{1}{n}$ diverges,
$\sum_1^n \frac{1}{n^2}$ converges,
Since $n^{1/n} \to 1$ as $n \to \infty$, root test is also inconclusive when limit $= 1$.
To see this limit, write

$$n^{1/n} = 1 + \delta_n, \, \delta > 0$$

$$n = (1 + \delta_n)^n > \frac{n(n-1)}{2} \delta_n^2$$

(binomial expansion)

$$\implies \delta_n^2 < \frac{2}{n-1} \implies \delta_n \to 0$$

Another root test example:
$\sum_1^n \left[\frac{n+1}{3n+5}\right]^n$, root test gives:

$$\frac{n+1}{3n+5} \to \frac{1}{3} < 1$$

so converges.

**Theorem 1.11** (Cauchy's Condensation Test)**.** Let $a_n$ be a decreasing sequence of positive terms. Then $\sum_1^\infty a_n$ converges iff
$\sum_1^\infty 2^n a_{2^n}$ converges.

**Proof.** First we observe that if $a_n$ is decreasing:

$$a_{2^k} \underset{(*_1)}{\leq} a_{2^{k-1}+i} \underset{(*_2)}{\leq} a_{2^{k-1}},\ 1 \leq i \leq 2^{k-1} \text{ (any } k \geq 1)$$

Assume now that $\sum_1^\infty a_n$ converges with sum let's say $A$
Then,

$$2^{n-1}a_{2^n} = \underbrace{a_{2^n} + \cdots + a_{2^n}}_{2^{n-1} \text{ times}} \underset{(*_1)}{\leq} a_{2^{n-1}+1} + a_{2^{n-1}+2} + \cdots + a_{2^n} = \sum_{m=2^{n-1}+1}^{2^n} a_m$$

Thus

$$\sum_{n=1}^N 2^{n-1}a_{2^n} \leq \sum_{n=1}^N \sum_{m=2^{n-1}+1}^{2^n} a_m$$

$$\implies \sum_{n=1}^N 2^n a_{2^n} \leq 2\sum_{m=2}^{2^N} a_m \leq 2(A - a_1)$$

Thus $\sum_{n=1}^N 2^n a_{2^n}$ increasing and bounded above, converges.
Conversely, assume $\sum 2^n a_{2^n}$ converges.

$$\sum_{m=2}^{2^N} a_m = \sum_{n=1}^N \sum_{m=2^{n-1}+1}^{2^N} a_m \leq \sum_{n=1}^N 2^{n-1}a_{2^{n-1}} \leq B$$

$\implies \sum_{m=1}^N a_m$ is a bounded increasing sequence and thus it converges $\square$

Example/ Application

$\sum_{1}^{\infty} \underbrace{\frac{1}{n^k}}_{a_n}$ converges iff $k > 1$ (for $k > 0$)

Decreasing sequence of positive terms as:

$$\frac{1}{(n+1)^k} < \frac{1}{n^k} \iff \left(\frac{n}{n+1}\right)^k < 1 \iff \frac{n}{n+1} < 1$$

$$2^n a_{2^n} = 2^n \left[\frac{1}{2^n}\right]^k = 2^{n-nk} = (\underbrace{2^{1-k}}_{r})^n$$

And $\sum r^n$ converges iff $r < 1$.

$\implies \sum \frac{1}{n^k}$ converges iff $2^{1-k} < 1$ iff $k > 1$

### 1.5.3 Alternating Series

**Theorem 1.12** (The alternating series test)**.** If $a_n$ decreases and tends to zero as $n \to \infty$, then the series $\sum_{1}^{\infty}(-1)^{n+1}a_n$ converges

**Proof.**

$$S_n = a_1 - a_2 + \cdots + (-1)^{n+1}a_n$$

$$S_{2n} = (a_1 - a_2) + (a_3 - a_4) + \cdots + (a_{2^n-1} - a_{2^n}) \geq S_{2n-2}$$

$$S_{2n} = a_1 - (a_2 - a_3) - (a_4 - a_5) - \cdots - (a_{2n-2} - a_{2n-1}) - a_{2n} \leq a_1$$

So $S_{2n}$ is increasing and bounded above $\implies S_{2n} \to S$

$$S_{2n+1} = S_{2n} + a_{2n+1} \to S + 0 = S$$

This implies that $S_n$ converges to $S$ as:

given $\varepsilon > 0$, $\exists N_1$ s.t. $\forall n \geq N_1$, $|S_{2n} - S| < \varepsilon$

$\exists N_3$ s.t. $\forall n \geq N_2$, $|S_{2n+1} - S| < \varepsilon$

Take $N = 2\max\{N_1, N_2\} + 1$

Then if $k \geq N \implies$

$$|S_k - S| < \varepsilon, \text{ so } S_k \to S$$

**Note.** e.g. $\sum_{1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges

### 1.5.4 Absolute Convergence

**Definition.** Take $a_n \in \mathbb{C}$. If $\sum_{n=1}^{\infty} |a_n|$ is convergent, then the series is **absolutely convergent**

**Note.** Since $|a_N| \geq 0$ we can use th previous tests to check absolute convergence; this is particularly useful for $a_n \in \mathbb{C}$.

**Theorem 1.13.** IF $\sigma a_n$ is absolutely convergent, then it is convergent.

**Proof.** Suppose first that $a_n \in \mathbb{R}$
Let

$$v_n = \begin{cases} a_n \text{ if } a_n \geq 0 \\ 0 \text{ if } a_n < 0 \end{cases}$$

$$w_n = \begin{cases} 0 \text{ if } a_n \geq 0 \\ -a_n \text{ if } a_n < 0 \end{cases}$$

$$v_n = \frac{|a_n| + a_n}{2}, \; w_n = \frac{|a_n| - a_n}{2}$$

Clearly, $v_, w_n \geq 0$,

$$a_n = v_n - w_n, \; |a_n| = v_n + w_n \geq v_n, w_n$$

If $\sum |a_n|$ converges, by comparison, $\sum v_n, \sum w_n$ also converge

$$\implies \sum a_n \text{ converges}$$

If $a_n \in \mathbb{C}$, write $a_n = x_n + iy_n$

$$|x_n|, |y_n| \leq |a_n|$$

$\implies \sum x_n, \sum y_n$ are absolutely convergent, $\implies \sum x_n, \sum y_n$ converge, since $a_n = x_n + iy_n \implies \sum a_n$ converges as well $\square$

**Examples.**

(i) $\sum \frac{(-1)^n}{n}$ converges, but not absolutely convergent

(ii)

$$\sum_{n=1}^{\infty} \frac{z^n}{2^n}, \; \sum \left(\frac{|z|}{2}\right)^n \qquad (*)$$

$\implies$ if $|z| < 2$, convergence of (*) and hence absolute convergence.
if $|z| \geq 2$, then $|a_n| \geq 1$, so $a_n$ foes not tend to zero $\implies \sum \frac{z^n}{2^n}$ diverges

**Definition.** If $\sum a_n$ converges but $\sum |a_n|$ does not, it is said sometimes that $\sum a_n$ is **conditionally convergent**.

**Note.** "conditional": because the sum to which the series converges is conditional on the order in which the elements of the sequence are taken.
If rearranged, the sum is altered.

**Example.**

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots \tag{I}$$

$$1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \dots \tag{II}$$

Let $s)n$ be the partial sum fo (I) and $t_n$ be the sumpartial sum of (II)

$$s_n \to s > 0$$
$$t_n \to \frac{3s}{2}$$

**Definition.** Let $\sigma$ be a bijection of the positive integersm

$$a'_n = a_{\sigma(n)}$$

is a **rearrangement**.

**Theorem 1.14.** If $\sum_1^\infty a_n$ is absolutely convergent, every series consisting of the same terms in any order (i.e. a rearrangement) has the same sum.

**Proof.** We do the proof first for $a_n \in \mathbb{R}$.
Let $\sum a'_n$ be a rearrangement of $\sum a_n$. Let

$$S_n = \sum_1^n a_n$$

$$t_n = \sum_1^n a'_n$$

Suppose first that $a_n \geq 0$
Given $n$, we can find $q$ s.t. $S_q$ contains every term of $t_n$
Since $a_n \geq 0$,

$$t_n \leq s_q \leq s$$

As $n \to \infty$, $t_n \to t$ (increasing sequence bounded above) $\implies t \leq s$. By symmetry, $s \leq t \implies s = t$
If $a_n$ has any sign $v_n$ and $w_n$ from Theorem 1.13

$$v_n = \frac{|a_n| + a_n}{2},\ w_n = \frac{|a_n| - a_n}{2}$$

Consider, $\sum a'_n, \sum v'_n, \sum w'_n$
Since $\sum |a_n|$ converges, both $\sum v_n, \sum w_n$ converge, now use the case $v_n, w_n \geq 0$ to deduce that

$$\sum v'_n = \sum v_n, \sum w'_n = \sum w_n$$

and the claim follows since $a_n = v_n - w_n$
For the case $a_n \in \mathbb{C}$, write $a_n = x_n + iy_n$
Since $|x_n|, |y_n| \leq |a_n| \implies \sum x_n, \sum y_n$ are absolutely convergent.
Then by the previous case $\sum x'_n = \sum x_n$ and $\sum y'_n = \sum y_n$. Since $a'_n = x'_n + iy'_n$, $\sum a_n = \sum a'_n$
□

## 2 Continuity

$E \subseteq \mathbb{C}$ non-empty, $f : E \to \mathbb{C}$ any function, $a \in E$
(includes case in which $f$ is real valued and $E$ is a subset of $\mathbb{R}$)

**Definition.** $f$ is **continuous at a**$\in E$ if for every sequence $z_n \in E$ with $z_n \to a$, we have $f(z_n) \to f(a)$
Equivalently below:

**Definition.** $f$ is **continuous at a**$\in E$, if

$$\text{given } \varepsilon > 0,\ \exists \delta \text{ s.t. if } |z - a| < f,\ \text{then } |f(z) - f(a)| < \varepsilon$$

($\varepsilon$-$f$ definition)

**Claim.** Two definitions equivalent

**Proof.** $2^{\text{nd}} \implies 1^{\text{st}}$:
We know that given $\varepsilon > 0,\ \exists \delta > 0$, s.t. $|z - a| < f,\ z \in E$, then $|f(z) - f(a)| < \varepsilon$.
Let $z_n \to a$.
Then $\exists n_0$ s.t. $\forall n \geq n_0$ we have

$$|z_n - a| < \delta \implies |f(z_n) - f(a)| < \varepsilon$$

$1^{\text{st}} \implies 2^{\text{nd}}$:
Assume $f(z_n) \to f(a)$ whenever $z_n \to a$ ($z_n \in E$). Suppose $f$ is not continuous at $a$, according to $2^{\text{nd}}$ definition.

$$\exists \varepsilon > 0,\ \text{s.t. } |z - a| < \delta \text{ and } |f(z) - f(a)| \geq \varepsilon \qquad (*)$$

Let $\delta = \frac{1}{n}$, from (*) we get $z_n$ s.t. $|z_n - a| < \frac{1}{n}$ and $|f(z_n) - f(a)| \geq \varepsilon$.
Clearly $z_n \to a$, but $f(z_n)$ does not tend to $f(a)$ because $|f(z_n) - f(a)| \geq \varepsilon$※

**Prop 2.1.** $a \in E,\ g, f : E \to \mathbb{C}$ continuous at $a$. Then so are the functions $f(z) + g(z),\ f(z)g(z)$ & $\lambda f(z)$ for any constant. In addition if $f(z) \neq 0\ \forall z \in E$, then $\frac{1}{f}$ is continuous at $a$

**Proof.** Using $1^{\text{st}}$ definition, this is obvious using the analagous results for sequences (Lemma 1.1) e.g.

$$f(z_n) + g(z_n) \to f(a) + g(a) \text{ if } z_n \to a,\ f(z_n) \to f(A) \ \&\ g(z_n) \to g(a) \text{ etc. } \square$$

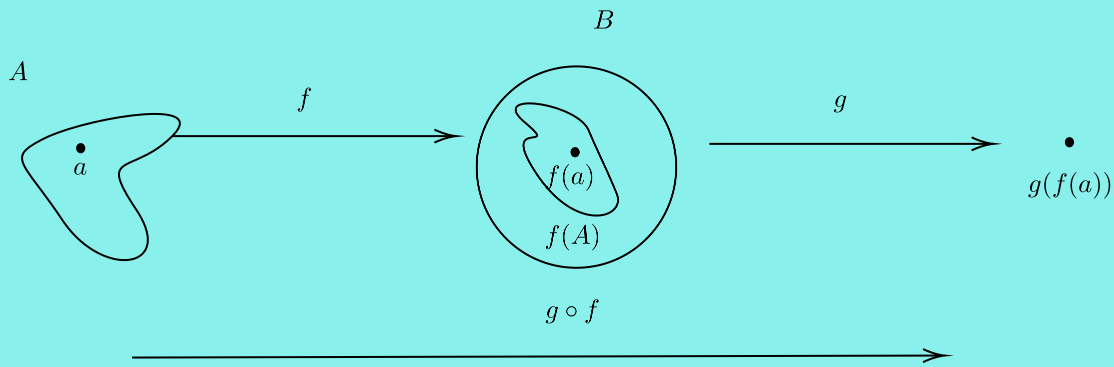
**Example.** The function $f(z) = z$ is continuous, so using the proposition we derive that every polynomial is continuous at every point in $\mathbb{C}$

**Note.** We say $f$ is continuous on $E$ if it is continuous at every $a \in E$.

**Remark.** Still it is instructive to prove above prop directly from the $\varepsilon - \delta$ definition

Next we look at compositions

**Theorem 2.2.** Let $f : A \to \mathbb{C}$ and $g : B \to \mathbb{C}$ be two functions s.t. $f(A) \subseteq B$. Suppose $f$ is continuous at $a \in A$ and $g$ is continuous at $f(a)$. Then $g \circ f : A \to \mathbb{C}$ is continuous at $a$.




**Proof.** Take any sequence $z_n \to a$. By assummpion, $f(z_n) \to f(A)$. Set $w_n = f(z_n)$. then $w_n \in B$ and $w_n \to f(a)$; thus

$$g(w_n) \to g(f(a)) \square$$

**Examples.**

(i)

$$f : \mathbb{R} \to \mathbb{R}$$

$$f(x) = \begin{cases} \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

($\sin(x)$ continuous proved later)
if $x \neq 0$, then 2.1 and 2.2 imply that $f(x)$ is continuous at every $x \neq 0$.
Discontinuous at 0:

$$\frac{1}{x_n} = (2n + \frac{1}{2})\pi$$

$$f(x_n) = 1,\ x_n \to 0 \text{ but } f(0) = 0$$

(ii)

$$f : \mathbb{R} \to \mathbb{R}$$

$$f(x) = \begin{cases} x\sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

$f$ is continuous at 0:
take $x_n \to 0$, then

$$f(x_n)| \leq |x_n| \text{ because } |\sin\left(\frac{1}{x}\right)| \leq 1$$

$$\implies f(x_n) \to 0 = f(0)$$

(iii)

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

Discontinuous at every point:
if $x \in \mathbb{Q}$, take a sequence $x_n \to x$ with $x_n \notin \mathbb{Q}$, then

$$f(x_n) = 0 \not\to f(x) = 1$$

Similarly, if $x \notin \mathbb{Q}$, take a sequence $x_n \to x$ with $x_n \in \mathbb{Q}$, then

$$1 = f(x_n) \not\to f(x) = 0$$

## 2.1 Limit of a function

$$F : E \subseteq \mathbb{C} \to \mathbb{C}$$

We wish to define what is meany by

$$\lim_{z \to a} f(z)$$

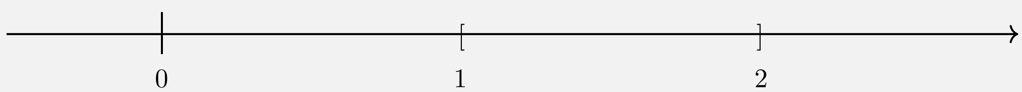
even when $a$ might not be in $E$ e.g.

$$\text{limit at } z \to 0 \frac{\sin z}{z} \ E = \mathbb{C} \backslash \{0\} \ a = 0$$

Also if

$$E \cup [1, 2]$$

it does not make sense to speak about $z \in, \ z \neq 0, z \to 0$




**Definition.** $E \subseteq \mathbb{C}, \ a \in \mathbb{C}$. We say that $a$ is a **limit point** of $E$ if for any $\delta > 0, \exists z \in E$ s.t.

$$0 < |z - a| < \delta$$

**Remark.** $a$ is a limit point iff $\exists$ a sequence $z_n \in E$ s.t. $z_n \to a$ and $z_n \neq a$ for all $n$. (can check equivalence)

**Definition.** $f : E \subseteq \mathbb{C} \to \mathbb{C}$, let $a \in \mathbb{C}$ be a limit point of $E$.
We say that

$$\lim_{z \to a} f(z) = l$$

($f$ tends to $l$ as $z$ tends to $a$)
If given $\varepsilon > 0$, $\exists \delta > 0$ s.t. whenever $0 < |z - a| < \delta$ and $z \in E$, thenn $|f(z) - l| < \varepsilon$
Equivalently: $f(z_n) \to l$ for every sequence $z_n \in E$, $z_n \neq a$ and $z_n \to a$
(proved exactly the same as previously with 2 definitions of continuity).

**Remark.** Straight from the definition, we have if $a \in E$ is a limit point, then

$$\lim_{z \to a} f(z) = f(a) \iff f \text{ is continuous at } a$$

If $a \in E$ is isolated (i.e. $a \in E$ and is not a limit point), continuity of $f$ at $a$ always holds.

The limit of functions has very similar properties to the limit of sequences

(i) it is unique $f(z) \to A,\ f(z) \to B$ as $z \to a$

$$|A - B| \leq |A - f(z)| + |f(z) - B|$$

if $z \in E$ is s.t. $0 < |z - a| < \delta_1, \delta_2$, then

$$|A - B| < 2\varepsilon \implies A = B$$

(the existence of such $z$ is a consequence of the condition that $a$ is s alimit point of $E$)

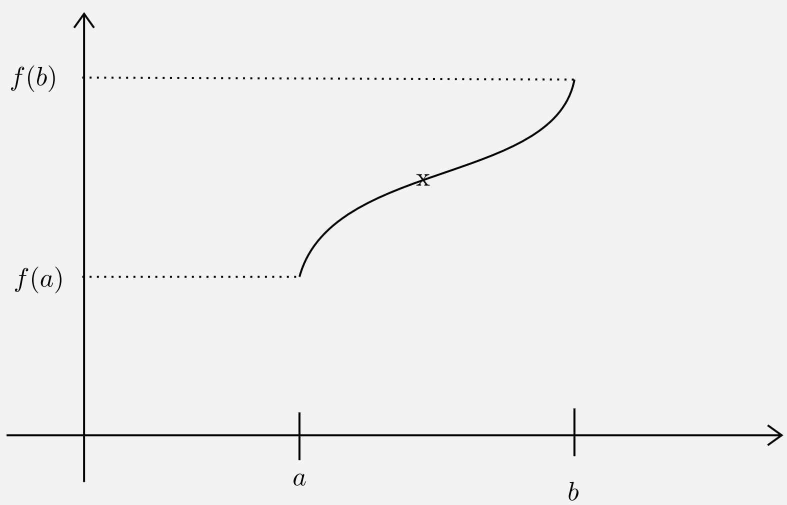
(ii) $f(z) + g(z) \to A + B$ if $f(z) \to A,\ g(z) \to B$ as $z \to a$

(iii) $f(z)g(z) \to AB$

(iv) if $B \neq 0,\ \frac{f(z)}{g(z)} \to \frac{A}{B}$

all proved in the same way as before.

## 2.2 The Intermediate Value Theorem



**Theorem 2.3.** $f : [a, b] \to \mathbb{R}$ continuous and $f(a) \neq f(b)$. Then $f$ takes every value which lies between $f(a)$ and $f(b)$.

**Proof.** Without loss of generality, we may suppose $f(a) < f(b)$.
Take

$$f(a) < \eta < f(b)$$

Let

$$S = \{x \in [a, b] : f(x) < \eta\}$$

$a \in S$, so $S \neq \varnothing$. Clearly $S$ is bounded above by $b$.
Then there is a supremum $C$ where $C \leq b$. By definition of the supremum, given $n$, there exists $x_n \in S$ s.t.

$$C - \frac{1}{n} < x_n \leq C$$

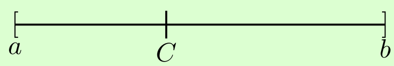
So, $x_n \to C$. Since $x_n \in S$,

$$f(x_n) < \eta$$

By continuity of $f$, $f(x_n) \to f(C)$.
Thus

$$f(C) \leq \eta \tag{*}$$

Now observe that $C \neq b$, for if $C = b$, then $f(b) \leq \eta$ by (*) which is false.




Then for $n$ large

$$C + \frac{1}{n} \in [a, b] \text{ and } C + \frac{1}{n} \to C$$

Again by continuity $f(C + \frac{1}{n}) \to f(C)$. But since

$$C + \frac{1}{n} > C, \; f(C + \frac{1}{n}) \geq \eta$$

Thus

$$f(C) \geq \eta \implies f(C) = \eta \square$$

**Remark.** The theorem is very useful for finding zeros of fixed points.

**Example.** Existence if the $N$-th root of a positive real number

$$f(x) = x^N, \; x \geq 0$$

Let $y$ be a positive number.
$f$ is continuous on $[0, 1 + y]$

$$0 = f(0) < y < (1 + y)^N = f(1 + y)$$

By the IVT, $\exists C \in (0, 1 + y)$ s.t. $f(C) = y$ i.e. $C^N = y$
$C$ is a positive $N$-root of $y$.
Uniqueness: if $d^N = y$ with $d > 0$ and $d \neq C$, wlog suppose $d < c$

$$\implies d^N < c^N \implies y < y ※$$

## 2.3 Bounds of a Continuous Function

**Theorem 2.4.** Let $f : [a, b] \to \mathbb{R}$ be continuous. Then there exists $K$ s.t.

$$|f(x)| \leq K \ \forall x \in [a, b]$$

**Proof.** We argue by contradiction.
Suppose statement is false. Then given any integer $n \geq 1$, there exists $x_n \in [a, b]$ s.t. $|f(x_n)| > n$.
By Bolzano-Weierstrauss, $x_n$ has a convergent subsequence $x_{n_j} \to x$.
Since $a \leq x_{n_j} \leq b$, we must have $x \in [a, b]$. By continuity of $f$,

$$f(x_{n_j}) \to f(x)$$

But

$$|f(x_{n_j}| > n_j \to \infty ※ \square$$

**Theorem 2.5.** $f : [a, b] \to \mathbb{R}$ continuous. Then $\exists x_1, x_2 \in [a, b]$ s.t.

$$f(x_1) \leq f(x) \leq f(x_2) \ \forall x \in [a, b]$$

"A continuous function on a closed, bounded interval is bounded and attains its bounds."

**Proof** (1$^{\text{st}}$)**.** Let

$$A = \{f(x) : c \in [a, b]\} = f([a, b])\}$$

By Theorem 2.4, $A$ is bounded. Since it is clearly non-empty, it has supremum, $M$.
By definition of supremum,

$$\text{given integer } n \geq 1, \ \exists x_n \in [a, b] \text{ s.t. } M - \frac{1}{n} < f(x_n) \leq M \qquad (*)$$

By Bolzano-Weierstrass,

$$\exists x_{n_j} \to x \in [a, b]$$

Since $f(x_{n_j}) \to M$ (because *) and $f$ is continuous, we deduce that $f(x) = M$ so $x_2 = x$. Reason similarly for the minimum $\square$

**Proof** (2$^{\text{nd}}$)**.**

$$A = f([a, b]), \ M = \sup A$$

as before. Suppose $\nexists x_2$ s.t. $f(x_2) = M$.
Let

$$g(x) = \frac{1}{M - f(x)}, \ x \in [a, b]$$

is defined and continuous. By Theorem 2.4 applied to $g$,

$$\exists K > 0 \text{ s.t. } g(x) \leq K \ \forall x \in [a, b]$$

This means that $f(x) \leq M - \frac{1}{K}$ on $[a, b]$. This is absurd since it contradicts that $M$ is the supremum $\square$

**Note.** Theorems 2.4, 2.5 are false if the interval is not closed e.g.

$$x \in (0,1],\ f(x) = \frac{1}{x}$$

## 2.4 Inverse functions

**Definition.** $f$ is **increasing** for $x \in [a,b]$ if $f(x_1) \leq f(x_2)$ for all $x_1, x_2$ s.t. $a \leq x_1 \leq x_2 \leq b$
If $f(x_1) < f(x_2)$ we say that $f$ is **strictly increasing**.
Similarly for **decreasing** and **strictly decreasing**.

**Theorem 2.6.** $f : [a,b] \to \mathbb{R}$ continuous and strictly increasing for $x \in [a,b]$.
Let $c = f(a)$ and $d = f(b)$.
Then $f : [a,b] \to [c,d]$ is bijective and the inverse

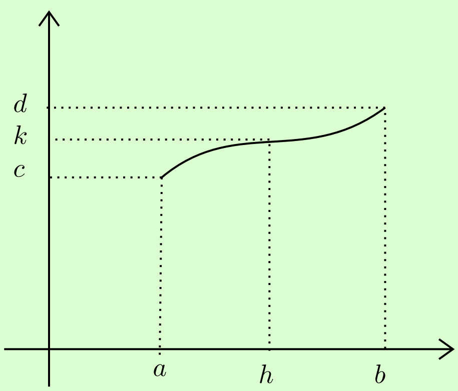
$$g = f^{-1} : [c,d] \to [a,b]$$

is continuous and strictly increasing

**Remark.** A similar theorem holds for strictly decreasing functions.

**Proof.** Take $c < k < d$.
From the intermediate value theorem

$$\exists h \text{ s.t. } f(h) = k$$




Since $f$ is strictly increasing, $h$ is unique.
Define $g(k) = h$ and this gives an inverse $g : [c,d] \to [a,b]$ for $f$. $g$ is strictly increaseing: $y_1 < y_2$

$$y_1 = f(x_1),\ y_2 = f(x_2)$$

If $x_2 \le x_1$, since $f$ is increasing

$$\implies f(x_2) \le f(x_1) \implies y_2 \le y_1 ※$$

$g$ is continuous:
Given $\varepsilon > 0$, let

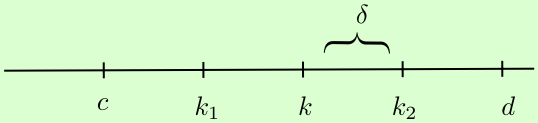
$$k_1 = f(h-\varepsilon),\ k_1 = f(h+\varepsilon)$$

$f$ strictly increasing $\implies$

$$k_1 < k < k_2$$

If $k_1 < y < k_2$ then

$$h - \varepsilon < g(y) < h + \varepsilon$$




$$\delta = \min\{k_2 - k, k - k_1\}$$

(here $k \in (c,d)$ but a similar argument establishes continuity at the end points (can check))

# 3 Differentiability

Let $f : E \subseteq \mathbb{C} \to \mathbb{C}$, ost of the time $E =$ interveral $\subseteq \mathbb{R}$

**Definition.** Let $x \in E$ be a point s.t. $\exists x_n \in E$ with $x_n \neq x$ and $x_n \to x$ (i.e. a limit point) $f$ is said to be **differentiable** at $x$ with derviative $f'(x)$ if

$$\lim_{y \to x} \frac{f(y) - f(x)}{y - x} = f'(x)$$

If $f$ is differentiable at each $x \in E$, we say $f$ is differentiable on $E$

**Note.** Think of $E$ as an interval or disc in the case of $\mathbb{C}$

**Remark.**

(i) Other common notations:

$$\frac{\mathrm{d}y}{\mathrm{d}x}, \ \frac{\mathrm{d}f}{\mathrm{d}x}$$

(ii)

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$(y = x + h)$

(iii) "Another important look at the definition:"
Let

$$\varepsilon(h) = f(x+h) - f(x) - hf'(x)$$

then

$$\lim_{h \to 0} \frac{\varepsilon(h)}{h} = 0$$

$$f(x+h) = f(x) + \underbrace{hf'(x)}_{\text{linear}} + \varepsilon(h)$$

linear as $h \mapsto hf'(x)$

**Definition** (alternative)**.** $f$ is **differentiable** at $x$ if $\exists A$ and $\varepsilon$ s.t.

$$f(x+h) = f(x) + hA + \varepsilon(h)$$

where

$$\lim_{h \to 0} \frac{\varepsilon(h)}{h} = 0$$

If such an $A$ exists, then it is unique, since

$$A = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

**Remark.**

(iv) If $f$ is differentiable at $x$ then $f$ is continuous at $x$ as since $\varepsilon(h) \to 0$,

$$f(x+h) \to f(x) \text{ as } h \to 0$$

(v) Another alternative way of writing things:

$$f(x+h) = f(x) + hf'(x) + h\varepsilon_f(h)$$

with $\varepsilon_f(h) \to 0$ as $h \to 0$
or

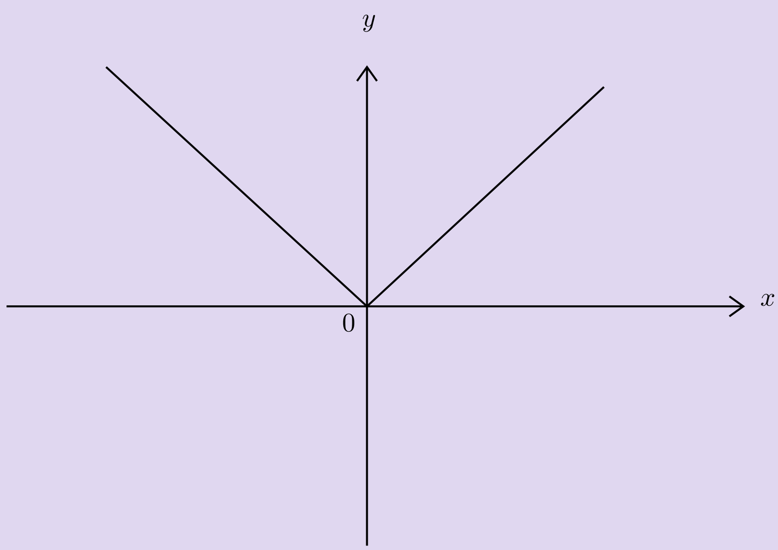
$$f(x) = f(a) + (x-a)f'(a) + (x-a)\varepsilon_f(x)$$

with

$$\lim_{x \to a} \varepsilon_f(x) \to 0$$

**Example.**

$$f(x) = |x|, \ f : \mathbb{R} \to \mathbb{R}$$




$$f'(x) = 1 \text{ if } x > 0$$

$$f'(x) - = 1 \text{ if } x < 0$$

Take $h_n \to 0$ from above:

$$\lim_{n \to \infty} \frac{f(h_n) - f(0)}{h_n} = \lim \frac{h_n}{h_n} = 1$$

Take $h_n \to 0$ from below:

$$\lim_{n \to \infty} \frac{f(h_n) - f(0)}{h_n} = \lim \frac{-h_n}{h_n} = -1$$

So not differentiable at $x = 0$

## 3.1 Differentiation of Sums, Products, etc.

**Prop 3.1.**

(i) IF $f(x) = c \ \forall x \ inE$, then $f$ is differentiable with $f'(x) = 0$

(ii) $f, g$ differentiable at $x$, then so is $f + g$ and

$$(f+g)'(x) = f'(x) + g'(x)$$

(iii) $f, g$ differentiable at $x$, then so is $fg$ and

$$(fg)'(x) = f'(x)g(x) + f(g)g'(x)$$

(iv) If $f$ is differentiable at $x$ and $f(x) \neq 0 \ \forall x \in E$, then $1/f$ is differentiable at $x$ and

$$\left(\frac{1}{f}\right)'(x) = -\frac{f(x)}{[f(x)]^2}$$

**Proof.**

(i)

$$\lim_{h \to 0} \frac{C - C}{h} = 0$$

(ii)

$$\begin{aligned} \lim_{h \to 0} \frac{f(x+h) + g(x+h) - f(x) - g(x)}{h} &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} + \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} \\ &= f'(x) + g'(x) \end{aligned}$$

(iii)

$$\phi(x) = f(x)g(x)$$

$$\begin{aligned} \frac{\phi(x+h) - \phi(x)}{h} &= \frac{f(x+h)g(x+h) - f(x)g(x)}{h} \\ &= f(x+h)\left[\frac{g(x+h) - g(x)}{h}\right] + g(x)\left[\frac{f(x+h) - f(x)}{h}\right] \\ &= f'(x)g(x) + f(x)g'(x) \end{aligned}$$

using standard properties of limits and the fact that $f$ is continuous at $x$

(iv)

$$\phi(x) = 1/f(x)$$

$$\begin{aligned} \frac{\phi(x+h) - \phi(x)}{h} &= \frac{1/f(x+h) - 1/f(x)}{h} \\ &= \frac{f(x) - f(x+h)}{hf(x)f(x+h)} \to -\frac{f'(x)}{[f(x)]^2} \square \end{aligned}$$

**Remark.** From (iii) and (iv) we immediately get

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

**Example.**

$$f(x) = x^n,\ n \in \mathbb{Z},\ n > 0$$

$$n = 1$$

Clearly $f(x) = x,\ f'(x) = 1$

**Claim.**

$$f'(x) = nx^{n-1}$$

**Proof.** Induction:

$$f(x) = x \cdot x^n$$

$$f'(x) = x^n + x(nc^{n-1}) = (n+1)x^n$$

Using prop 3.1

$$f(x) = x^{-n} = \frac{1}{x^n}\ n \in \mathbb{Z},\ n > 0$$

If $x \neq 0$, use prop 3.1 (iv) to derive

$$f'(x) = -\frac{(x^n)'}{x^{2n}} = -\frac{nx^{n-1}}{x^{2n}} = -nx^{-n-1}$$

So can differentiate polynomials, rational functions ✓

**Theorem 3.2** (Chain rule).

$$f : U \to \mathbb{C}$$

is s.t.

$$f(x) \in V \ \forall x \in V$$

If $f$ is differentiable at $a \in U$ and $g : V \to \mathbb{C}$ is differentiable at $f(a)$, then $g \circ f$ is differentiable at $a$ with

$$(g \circ f)'(a) = f'(a)g'(f(a))$$

**Proof.** We know:

$$f(x) = f(a) + (x-a)f'(a) + \varepsilon_f(x)(x-a)$$

where

$$\lim_{x\to a} \varepsilon_f(x) = 0$$

$$g(y) = g(b) + (y-b)g'(b) + \varepsilon_g(y)(y-b)$$

where

$$\lim_{y\to b} \varepsilon_g(y) = 0$$

$$b = f(a)$$

Set

$$\varepsilon_f(a) = 0 \ \& \ \varepsilon_g(b) = 0$$

to make them continuous at $x = a$ and $y = b$.
Now $y = f(x)$ gives

$$\begin{aligned} g(f(x)) &= g(b) + (f(x)-b)g'(b) + \varepsilon_g(f(x))(f(x)-b) \\ &= g(f(a)) + [(x-a)f'(a) + \varepsilon_f(x)(x-a)][g'(b) + \varepsilon_g(f(x))] \\ &= g(f(a)) + (x-a)f'(a)g'(b) + (x-a)\underbrace{[\varepsilon_f(x)g'(b) + \varepsilon_g(f(x))(f'(a)+\varepsilon_f(x))]}_{\sigma(x)} \end{aligned}$$

$$\sigma(x) = \underbrace{\varepsilon_f(x)g'(b)}_{0} + \underbrace{\varepsilon_g(f(x))}_{\text{0 as continuous comp.}} \underbrace{(f'(a)+\varepsilon_f(x))}_{f'(a)}$$

so

$$\lim_{x\to a} \sigma(x) = 0$$

**Examples.**

(i)

$$f(x) = \sin(x^2)$$

$$(\sin x)' = \cos x$$

(to be seen later)

$$f'(x) = 2x\cos(x^2)$$

(ii)

$$f(x) = \begin{cases} \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

(this is continuous at every $x$)
differentiable at every $x \neq 0$ by the previous theorem.
At $x = 0$,

$$\frac{f(x) - f(0)}{x - 0} = \frac{x\sin(1/x)}{x} = \sin(1/x)$$

$$\implies \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0}$$

does not exist $\implies$ $f$ is not differentiable at $x = 0$.

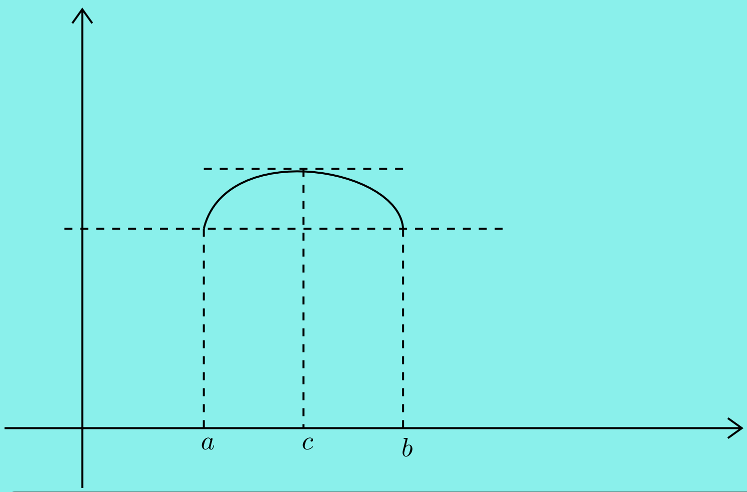
## 3.2 The Mean Value Theorem

**Theorem 3.3** (Rolle's Theorem)**.**

$$f : [a, b] \to \mathbb{R}$$

continuous on $[a, b]$ and differentiable on $[a, b)$. If $f(a) = f(b)$,

$$\exists c \in (a, b) \text{ s.t. } f'(c) = 0$$




**Proof.** Let

$$M = \max_{x\in[a,b]} f(x),\ m = \min_{x\in[a,b]} f(x)$$

Recall (Theorem 2.5) that these values are achieved.
Let $k = f(a)$. If $M = m = k$, then $f$ is constant and $f'(c) = 0\ \forall c \in (a, b)$
Then $M > k$ or $m < k$. Suppose $M > k$
By Theorem 2.5,

$$\exists c \text{ s.t. } f(c) = M$$

If $f'(c) > 0$, then there are values to the right of $c$ for which $f(x) > f(c)$ since

$$f(x + h) - f(x) = h(f'(c) + \varepsilon(h)) > 0$$

Since $\varepsilon(h) \to 0$ as $h \to 0$ and thus

$$f'(x) + \varepsilon(h) > 0 \text{ if } h \text{ small}$$

This contradicts that $M$ is the maximum.
Similarly, if $f'(c) < 0$, $\exists x$ to the left of $c$ for which $f(x) > f(c)$

$$\implies f'(c) = 0\square$$

**Note.** A simple tweak gives below:

**Theorem 3.4** (The Mean Value Theorem)**.** Let $f : [a, b] \to \mathbb{R}$ be a continuous function which is differentiable on $(a, b)$. Then $\exists c \in (a, b)$ st.

$$f(b) - f(a) = f'(c)(b - a)$$

**Proof.** Write

$$\phi(x) = f(x) - kx$$

Choose $k$ s.t. $\phi(a) = \phi(b)$

$$\implies f(b) - bk = f(a) - bk \implies k = \frac{f(b) - f(a)}{b - a}$$

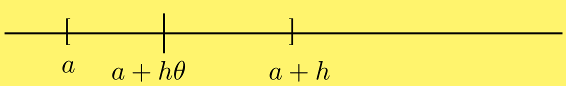
By Rolle's theorem applied to $\phi$

$$\exists c \in (a, b) \text{ s.t. } \phi'(c) = 0$$

i.e. $f'(x) = k$□

**Remark.** We will often write

$$f(a + h) = f(A) + hf'(a + \theta h)$$




$$\theta \in (0, 1)$$

$$(b = a + h$$

**Warning.**

$$\theta = \theta(h)$$

**Corollary 3.5.** $f : [a, b] \to \mathbb{R}$ continuous and differentiable on $(a, b)$. Then we have

(i) If $f'(x) > 0 \ \forall x \in (a, b)$, then $f$ is strictly increasing on $[a, b]$
(i.e. if $b \geq y > x \geq a$, then $f(y) > f(x)$)

(ii) If $f'(x) \geq 0 \ \forall x \in (a, b)$, then $f$ is increasing (i.e. if $b \geq y > x \geq a$, then $f(y) \geq f(x)$)

(iii) If $f'(x) = 0 \ \forall x \in (a, b)$, then $f$ is constant on $[a, b]$

**Proof.**

(i) Have

$$f(y) - f(x) = f'(c)(y - x) \ c \in (x, y)$$

from MVT
so

$$f'(c) > 0 \implies f(y) > f(x)$$

(ii) same: but $f'(c) \geq 0 \implies f(y) \geq f(x)$

(iii) Take $x \in [a, b]$. Then use MVT in $[a, x]$ to get $x \in (a, x)$ s.t.

$$f(x) - f(a) = f'(x)(x - a) = 0$$

$$\implies f(x) = f(a) \implies f \text{ is constant}\square$$

**Remark.** We have similar statements for decreasing functions

## 3.3 Inverse Rule/ Inverse Function Theorem

**Theorem 3.6.** $f : [a, b] \to \mathbb{R}$ continuous and differentiable on $(a, b)$ with

$$f'(x) > 0 \; \forall x \in (a, b)$$

Let $f(a) = c$ and $f(b) = d$. Then the function $f : [a, b] \to [c, d]$ is bijective and $f^{-1}$ is differentiable on $(c, d)$ with

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

**Proof.** By corollary 3.5, $f$ is strictly increasing on $[a, b]$. By Theorem 2.6

$$\exists g : [c, d] \to [a, b]$$

which is continuous, strictly increasing inverse of $f$.
RTP: $g$ is differentiable and $g'(y) = \frac{1}{f'(x)}$ where $y = f(x),\; x \in (a, b)$
If $k \neq 0$ is given, let $h$ be given by

$$y + k = f(x + h)$$

That is, $g(y + k) = x + h,\; h \neq 0$
Then

$$\frac{g(y + k) - g(y)}{k} = \frac{x + h - x}{f(x + h) - f(x)} \to \frac{1}{f'(x)}$$

Let $k \to 0$, then $h \to 0$ ($g$ is continuous)

$$g'(y) = \lim_{h \to 0} \frac{g(y + k) - g(y)}{k} = \frac{1}{f'(x)}$$

**Example.**

$$g(x) = x^{1/q}$$

($x > 0,\; q$ positive integer)

$$f(x) = x^q \; (g(f(x) = x)$$

$$f'(x) = qx^{q-1}$$

Since $f$ is differentiable, so if $g$ and by the inverse rule

$$g'(x) = \frac{1}{q(x^{1/q})^{1-q}} = \frac{1}{q} x^{1/q - 1}$$

Now if $g(x = x^{p/q}$ ($p$ integer, $q$ positive integer)
We can find $g'(x)$ by using the chain rule

$$g(x) = (x^p)^{1/q} = (x^{1/q})^p$$

We find (can check)

$$g'(x) = \frac{p}{q} x^{\frac{p}{q} - 1}$$

So, if $g(x) = x^r \; r \in \mathbb{Q}$
then $g'(x) = rx^{r-1}$

**Remark.** Suppose $f, g : [a, b] \to \mathbb{R}$ are continuous, differentiable on $(a, b)$ and $g(a) \neq g(b)$. Then the MVT gives us $s, t \in (a, b)$ s.t.

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{(b-a)f'(s)}{(b-a)g'(t)} = \frac{f'(s)}{g'(t)}$$

Cauchy showed that one can take $s = t$

**Theorem 3.7** (Cauchy's mean value theorem)**.** Let $f, g : [a, b] \to \mathbb{R}$ be continuous functions and differentiable on $(a, b)$.
Then $\exists t \in (a, b)$ s.t.

$$(f(b) - f(a))g'(t) = f'(t)(g(b) - g(a))$$

**Proof.** Let

$$\phi(x) = \begin{vmatrix} 1 & 1 & 1 \\ f(a) & f(x) & f(b) \\ g(a) & g(x) & g(b) \end{vmatrix}$$

$\phi$ iscontinuous on $[a, b]$ and differentiable on $(a, b)$
Also,

$$\phi(a) = \phi(b) = 0$$

By Rolle's theorem, $\exists t \in (a, b)$ s.t. $\phi'(t) = 0$
If we expand the determinant, we get the desired result:

$$\begin{aligned} \phi'(x) &= f'(x)g(b) - g'(x)f(b) + f(a)g'(x) - g(a)f'(x) \\ &= f'(x)[g(b) - g(a)] + g'(x)[f(a) - f(b)] \end{aligned}$$

$\phi'(t) = 0$ gives the result $\square$

**Note.** We recover the MVT if we take $g(x) = x$

**Example.** "L'Hopital's rule"

$$\lim_{x \to 0} \frac{e^x - 1}{\sin x} = \frac{e^x - e^0}{\sin x - \sin 0} = \frac{e^t}{\cos t}$$

as $x \to 0$, $t \to 0$, so

$$\frac{e^t}{\cos t} \to 1$$

**Note.** We want to entend the MVT to include higher order derivatives

**Theorem 3.8** (Taylor's theorem with Lagrange's remainder)**.** Suppose $f$ and its derivatives up to order $n-1$ are continuous in $[a, a+h]$ and $f^{(n)}$ exist for $x \in (a, a+h)$. Then

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \cdots + \frac{h^{n-1}f^{(n-1)}(a)}{(n-1)!} + \frac{h^n}{n!}f^{(n)}(a+\theta h)$$

Where $\theta \in (0,1)$

**Proof.** Define for $0 \leq t \leq h$

$$\phi(t) = f(a+t) - f(a) - tf'(a) - \cdots - \frac{t^{n-1}}{(n-1)!}f^{(n-1)}(a) - \frac{t^n}{n!}\beta$$

where we choose $\beta$ s.t. $\phi(h) = 0$
(recall in the proof of the MVT we used $f(x) - kx$ and we picked $k$ s.t. we could use Rolle's theorem)
We see that

$$\phi(0) = \phi'(0) = \cdots = \phi^{(n-1)}(0) = 0$$

We use Rolle's Theorem $n$-times:

$$\phi(0) = \phi(h) = 0 \implies \phi'(h_1) = 0 \; 0 < h_1 < h$$

$$\phi'(0) = \phi(h_1) = 0 \implies \phi''(h_2) = 0 \; 0 < h_2 < h_1$$

Finally

$$\phi^{(n-1)}(0) = \phi^{(n-1)}(h_{n-1}) = 0 \implies \phi^{(n)}(h_n) = 0$$

$$0 < h_n < h_{n-1} < \cdots < h$$

So $h_n = \theta h$ for $\theta \in (0,1)$
Now

$$\phi^{(n)}(t) = f^{(n)}(a+t) - \beta$$

$$\implies \beta = f^{(n)}(a+\theta h)$$

Set $t = h$, $\phi(h) = 0$ and put this value of $\beta$ in the second line in the proof □

**Note.**

(i) For $n = 1$,we get back the MVT, so this is a "$n$-th order mean value theorem"

(ii)

$$R_n = \frac{h^n}{n!}f^{(n)}(a+\theta h)$$

is known as Lagrange's form of the remainder

**Theorem 3.9** (Taylor's theorem with Cauchy's form of remainder)**.** With the same hypothesis as in Theorem 3.8 and $a = 0$ (to simplify), we have

$$f(h) = f(0) + hf'(0) + \cdots + \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(0) + R_n$$

where

$$R_n = \frac{h^n (1-\theta)^{n-1} f^{(n)}(\theta h)}{(n-1)!}, \; \theta \in (0,1)$$

**Proof.** Define

$$F(t) = f(h) - f(t) - (h-t)f'(t) - \cdots - \frac{(h-t)^{n-1} f^{(n-1)}(t)}{(n-1)!}$$

with $t \in [0, h]$

$$F'(t) = -f'(t) + f'(t) - (h-t)f''(t) + (h-t)f''(t) - \frac{(h-t)^2}{2} f''(t) + \cdots - \frac{(h-t)^{n-1}}{(n-1)!} f^{(n)}(t)$$

$$\implies F'(t) = -\frac{(h-t)^{n-1}}{(n-1)!} f^{(n)}(t)$$

Set

$$\phi(t) = F(t) - \left[\frac{h-t}{h}\right]^p F(0)$$

where $p \in \mathbb{Z}, 1 \leq p \leq n$
Then $\phi(0) = \phi(h) = 0$ so by Rolle's theorem,

$$\exists \theta \in (0,1) \text{ s.t. } \phi'(\theta h) = 0$$

But

$$\phi'(\theta h) = F'(\theta h) + \frac{p(1-\theta)^{p-1}}{h} F(0) = 0$$

Thus

$$0 = -h^{n-1} \frac{(1-\theta)^{n-1}}{(n-1)!} f^{(n)}(\theta h) + \frac{p(1-\theta)^{p-1}}{h} \left[ f(h) - f(0) - hf'(0) - \cdots - \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(0) \right]$$

$$\implies f(h) = f(0) + hf'(0) + \cdots + \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(0) + \frac{h^n (1-\theta)^{n-1} f^{(n)}(\theta h)}{(n-1)! \cdot p \cdot (1-\theta)^{p-1}}, \; \theta \in (0,1)$$

If $p = n$ we get Lagrange's remainder
If $p = 1$ we get Cauchy's remainder

**Method.** To get a Taylor Series for $f$, one needs to show that $R_n \to 0$ as $n \to \infty$. This requires "estimates" and "effort"

**Remark.** Theorems 3.8 and 3.9 work equally well in n interval $[a+h, a]$ with $h < 0$

**Example** (The Binomial Series).

$$f(x) = (1+x)^r,\ r \in \mathbb{Q}$$

**Claim.** if $|x|| < 1$ then

$$(1+x)^r = 1 + \binom{r}{1}x + \dots + \binom{r}{n}x^n + \dots$$

where

$$\binom{r}{n} = \frac{r(r-1)\dots(r-n+1)}{n!}$$

**Proof.** Clearly

$$f^{(n)}(x) = r(r-1)\dots(r-n+1)(1+x)^{r-n}$$

If $r \in \mathbb{Z},\ r \geq 0$, then $f^{(r+1)} \equiv 0$, we have a polynomial of degree $r$.
In general (Lagrange),

$$\begin{aligned} R_n &= \frac{x^n}{n!} f^{(n)}(\theta x) \\ &= \binom{r}{n} \frac{x^n}{(1+\theta x)^{n-r}} \end{aligned}$$

$\theta \in (0,1)$ so have interval $[0,x]$ Note: in principle, $\theta$ depends on both $x$ and $n$.
For $0 < x < 1$

$$(1+\theta x)^{n-r} > 1 \text{ for } n > r$$

Now observe that the series

$$\sum \binom{r}{n} x^n$$

is absolutely convergent for $|x| < 1$.
Indeed by the ratio test

$$a_n = \binom{r}{n} x^n$$

$$\begin{aligned} \left|\frac{a_{n+1}}{a_n}\right| &= \left|\frac{r(r-1)\dots(r-n+1)(r-n)x^{n+1}}{(n+1)!}\right| \left|\frac{n!}{r(r-1)\dots(r-n+1)x^n}\right| && (1) \\ &= \left|\frac{(r-n)x}{n+1}\right| \to |x| \text{ as } n \to \infty && (2) \end{aligned}$$

In particular, $a_n \to 0$, so $\binom{r}{n}x^n \to 0$for $|x| < 1$
Hence for $n > r$ and $0 < x < 1$, we have

$$|R_n| \leq \left|\binom{r}{n} x^n\right| = |a_n| \to 0 \text{ as } n \to \infty$$

So the claim is proved in the range $0 \leq x < 1$

**Example** (continued)**.**

**Proof** (continued)**.** If $-1 < x < 0$ the argument above breaks down, but Cauchy's form of $R_n$ works:

$$\begin{aligned}
R_n &= \frac{(1-\theta)^{n-1} r(r-1)\dots(r-n+1)(1+\theta x)^{r-n} x^n}{(n-1)!} \\
&= \underbrace{\frac{r(r-1)\dots(r-n+1)}{(n-1)!}}_{r\binom{r-1}{n-1}} \frac{(1-\theta)^{n-1}}{(1+\theta x)^{n-r}} x^r \\
&= r\binom{r-1}{n-1} x^n (1+\theta x)^{r-1} \underbrace{\left(\frac{1-\theta}{1+\theta x}\right)^{n-1}}_{<1 \text{ for } x \in (-1,1)}
\end{aligned}$$

$$|R_n| \leq \left| r\binom{r-1}{n-1} x^n \right| (1+\theta x)^{n-1}$$

Can check:

$$(1+\theta x)^{r-1} < \max\{1, (1+x)^{r-1}\}$$

$$K_r = r\max\{1, (1+x)^{r-1}\}$$

which is independent of $n$

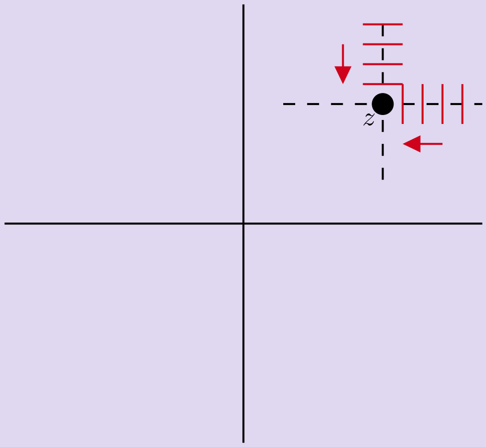
$$|R_n| \leq K_r \left| \binom{r-1}{n-1} x^n \right| \to 0$$

because $a_n \to 0$. Thus $R_n \to 0$ □

## 3.4 Remarks on Complex Differentiation

**Remark.** Formally, we have regarding sums, products, chain rule etc. but it is much more restrictive than differentiability of functions on the real line.

**Example.** $f(z) = \bar{z}$ is no-where $\mathbb{C}$-differentiable




$$z_n = z + \frac{1}{n} \to z$$

$$\frac{f(z_n) - f(z)}{z_n - z} = \frac{\bar{z} + \frac{1}{n} - \bar{z}}{z + \frac{1}{n} - z} = 1$$

$$z_n = z + \frac{i}{n} \to z$$

$$\frac{f(z_n) - f(z)}{z_n - z} = \frac{\bar{z} - \frac{i}{n} - \bar{z}}{z + \frac{i}{n} - z} = -1$$

so

$$\lim_{w \to z} \frac{f(w) - f(z)}{w - z} \text{ does not exist}$$

On the other hand $f(x, y) = (x, -y)$ is differentiable

$$z = x + iy$$

**Note.** IB Complex Analysis explores the consequences of $\mathbb{C}$-differentiability

## 4 Power Series

We want to look at $\sum_{n=0}^{\infty} a_n z^n$ with $z_n \in \mathbb{C},\ a_n \in \mathbb{C}$.
(The case $\sum_{n=0}^{\infty} a_n (z - z_0)^n,\ z_0$ fixed follows this one by translation)

**Lemma 4.1.** If $\sum_0^\infty a_n z_1^n$ converges and $|z| < |z_1|$, then $\sum_0^\infty a_n z^n$ converges absolutely

**Proof.** Since $\sum_0^\infty a_n z_1^n$ converges, $a_n z_1^n \to 0$. Thus $\exists K > 0$ s.t.

$$|a_n z_1^n| < K \ \forall n$$

Then

$$|a_n z^n| \leq K \left| \frac{z}{z_1} \right|^n$$

Since the geometric series $\sum_0^\infty \left| \frac{z}{z_1} \right|^n$ converges, the lemma follows by comparison □

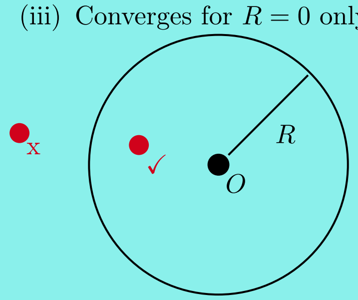
Using this lemma, we will prove that every power series has a radius of convergence

**Theorem 4.2.** A power series either

(i) Converges absolutely for all $z$, or

(ii) Converges absolutely for all $z$ inside a circle $|z| = R$ and diverges for all $z$ outside it, or

(iii) Converges for $R = 0$ only




**Proof.** Let $S = \{x \in \mathbb{R},\ x \geq 0 \text{ and } \sum a_n x^n \text{ converges}\}$ Clearly $0 \in S$. By Lemma 4.1, if $x_1 \in S$, then $[0, x_1] \in S$.

If $S = [0, \infty)$, we have case (i)

If not, there exists a finite supremum $R$ ($R \geq 0$). For $S$, $R = \sup S < \infty$

If $R > 0$, we'll prove that if $|z_1| < R$, then $\sum a_n z_1^n$ converges absolutely:
choose $R_0$ s.t.$|z_1 < |R_0 < R$. Then $R_0 \in S$ and the series converges if $z = R_0$.
By Lemma 4.1, $\sum |a_n z_1^n|$ converges

Finally we show that if $|z_2| > R \geq 0$, then the series does not converge for $z_2$. Now take $R_0$ s.t. $R < R_0 < |z_2|$. If $\sum a_n z_2^n$ converes, by Lemma 4.1, $\sum a_n R_0^n$ would be convergent, which contradicts that $R = \sup S$. □

**Definition.** The circle $|z| = R$ is called the **circle of convergence** and $R$ is the **radius of convergence**.
In (i), we agree that $R = \infty$ and in (iii) $R = 0$

The following lemma is useful for computing $R$

**Lemma 4.3.** If

$$\left|\frac{a_{n+1}}{a_n}\right| \to l$$

as $n \to \infty$, then $R = \frac{1}{l}$

**Proof.** By the ratio test, we have absolute convergence if

$$\lim \left|\frac{a_{n+1}}{a_n}\frac{z^{n+1}}{z^n}\right| < 1$$

so if $|z| < \frac{1}{l}$, we have absolute convergence. If $|z| > \frac{1}{l}$, the series diverges , again by the ratio test □

**Remark.** One can also use the root test to get $|a_n|^{1/n} \to l$ then $R = \frac{1}{l}$

**Examples.**

(i) $\sum_0^\infty \frac{z^n}{n!}$

$$\left|\frac{a_{n+1}}{a_n}\right| - \frac{n!}{(n+1)|} = \frac{1}{n+1} \to 0 = l \implies R = \infty$$

(ii) Geometric series, $\sum_0^\infty z^n$
$R = 1$. Note that at $|z| = 1$, we have divergence

(iii) $\sum_0^\infty n!z^n$, has $R = 0$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)!z^{n+1}}{n!z^n} = (n+1)z \to \infty$$

Only converges at $z = 0$

(iv) $\sum_1^\infty \frac{z^n}{n}$ has $R = 1$, but diverges for $z = 1$ (harmonic series)
What happens for $|z| = 1$ and $z \neq 1$?
Consider

$$\sum_1^\infty \frac{z^n}{n}(1-z)$$

$$\begin{aligned} S_N &= \sum_1^N \frac{z^n - z^{n+1}}{n} = \sum_1^N \frac{z^n}{n} - \sum_1^N \frac{z^{n+1}}{n} \\ &= \sum_1^N \frac{z^n}{n} - \sum_2^{N+!} \frac{z^n}{n-1} \\ &= z - \frac{z^{N+1}}{N} + \sum_2^{N+1} \frac{-z^n}{n(n-1)} \end{aligned}$$

if $|z| = 1$, $\frac{z^{N+1}}{N} \to 0$ as $N \to \infty$ and $\sum_2^\infty \frac{z^n}{n}$ converges for all $z$ with $|z| = 1,\ z \neq 1$

(v) $\sum_1^\infty \frac{z^n}{n^2}$, $R = 1$ and converges for all $z$ with $|z| = 1$

(vi) $\sum_0^\infty nz^n$, $R = 1$ but diverges for all $|z| = 1$

**Remark.** In principle, nothing can be said about $|z| = R$ and each case has to be discussed separately.
Within the radius of convergence 'life is great". Power series will "behave as if they were polynomials"

**Theorem 4.4.** $f(z) = \sum_0^\infty a_n z^n$ has radius of convergence $R$. Then $f$ is differentiable at all points with $|z| < R$ with

$$f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}$$

**Proof.** By Lemma 4.5, we may define

$$f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}, \ |z| < R$$

RTP:

$$\lim_{h \to 0} \frac{f(z+h) - f(z) - hf'(z)}{h} \to 0$$

Let

$$\begin{aligned} I &= \frac{f(z+h) - f(z) - hf'(z)}{h} \\ &= \frac{1}{h} \sum_0^\infty a_n \left( (z+h)^n - z^n - hnz^{n-1} \right) \\ |I| &= \frac{1}{|h|} \left| \lim_{N \to \infty} \sum_0^N a_n \left( (z+h)^n - z^n - hnz^{n-1} \right) \right| \\ &\leq \frac{1}{|h|} \sum_0^\infty |a_n| |(z+h)^n - z^n - nhz^{n-1}| \\ &\leq \frac{1}{|h|} \sum_2^\infty |a_n| n(n-1)(|z| + |h|)^{n-2} |h|^2 \end{aligned}$$

By Lemma 4.5, for $|h|$ small enough,

$$\sum_2^\infty |a_n| n(n-1)(|z| + |h|)^{n-2}$$

converges to $A(h)$, but $A(h) \leq A(r)$ for $h < r$ and $|z| + r < R$

$$\implies |I| \leq |h| A(h) \leq |h| A(r) \text{ as } h \to 0$$

**Lemma 4.5.** If $\sum_0^\infty a_n z^n$ has radius of convergence $R$, then so do

$$\sum_1^\infty n a_n z^{n-1} \text{ and } \sum_2^\infty n(n-1)a_n z^{n-2}$$

**Proof.** Take $z$ and $R_0$ s.t. $0 < |z| < R_0 < R$. Since $a_n R_0^n \to 0$,

$$\exists K \text{ s.t. } |a_n R_0^n| \leq K \ \forall n \geq 0$$

Thus

$$\begin{aligned} |a_n n z^{n-1}| &= \frac{n}{|z|} |a_n R_0^n| \left| \frac{z}{R_0} \right|^n \\ &\leq \frac{Kn}{|z|} \left| \frac{z}{R_0} \right|^n \end{aligned}$$

But $\sum n |\frac{z}{R_0}|$ converges by the ratio test

$$\frac{n+1}{n} \left| \frac{z}{R_0} \right|^{n+1} \left| \frac{R_0}{z} \right|^n = \frac{n+1}{n} \left| \frac{z}{R_0} \right| \to \left| \frac{z}{R_0} \right| < 1$$

if $|z| > R$, the series diverges since $|a_n z^n|$ is unbounded, hence so is $n|a_n z^n|$
Same proof applies to

$$\sum_2^\infty n(n-1)a_n z^{n-2} \ \square$$

**Lemma 4.6.**

(i)

$$\binom{n}{r} \le n(n-1)\binom{n-2}{r-2}$$

for all $2 \le r \le n$

(ii)

$$|(z+h)^n - z^n - nhz^{n-1}| \le n(n-1)(|z|+|h|)^{n-2}|h|^2 \ \forall z \in \mathbb{C},\ h \in \mathbb{C}$$

**Proof.**

(i)

$$\begin{aligned} \frac{\binom{n}{r}}{\binom{n-2}{r-2}} &= \frac{n!}{r!(n-r)!}\frac{(r-2)!(n-r)!}{(n-2)!} \\ &= \frac{n(n-1)}{r(r-1)} \\ &\le n(n-1) \ \checkmark \end{aligned}$$

(ii)

$$\begin{aligned} (z+h)^n - z^n - nhz^{n-1} &= \sum_{r=2}^{n}\binom{n}{r} z^{n-r}h^r \text{ thus} \\ |(z+h)^n - z^n - nhz^{n-1}| &\le \sum_{r=2}^{n}\binom{n}{r}|z|^{n-r}|h|^r \\ &\le n(n-1)\underbrace{\left[\sum_{r=2}^{n}\binom{n-2}{r-2}|z|^{n-r}|h|^{r-2}\right]}_{(|z|+|h|)^{n-2}}|h|^2 \end{aligned}$$

## 4.1 The Standard Functions

We have already seen that

$$\sum_0^\infty \frac{z^n}{n!}$$

has $R = \infty$
Define $e : \mathbb{C} \to \mathbb{C}$

$$e(z) = \sum_0^\infty \frac{z^n}{n!}$$

Straight from Theorem 4.4, $e$ is differentiable and $e'(z) = e(z)$

**Claim.** Observation: If $F : \mathbb{C} \to \mathbb{C}$ has $f'(z) = 0 \; \forall z \in \mathbb{C}$, then $F$ is constant

**Proof.** Consider

$$\begin{aligned} g(t) &= F(tz) \\ &= u(t) + iv(t) \end{aligned}$$

By the chain rule:

$$g'(t) = F'(tz)z = 0 = u'(t) + iv'(t)$$

$$\implies u' = v' = 0$$

Now apply Corollary 3.5 □

Now let $a, b \in \mathbb{C}$ and consider

$$F(z) = e(a + b - z)e(z)$$

$$F'(z) = -e(a + b - z)e(z) + e(a + b - z)^z = 0$$

$$\implies F \text{is constant}$$

$$e(a + b - z)e(z) = F(0) = e(a + b)$$

Set $z = b$

$$e(a)e(b)e(a + b)$$

Now we restrict $e : \mathbb{R} \to \mathbb{R}$

**Theorem 4.7.**

(i) $e : \mathbb{R} \to \mathbb{R}$ is everywhere differentiable and $e'(x) = e(x)$

(ii) $e(x + y) = e(x)e(y)$

(iii) $e(x) > 0 \ \forall x \in \mathbb{R}$

(iv) $e$ is strictly increasing

(v) $e(x) \to \infty$ as $x \to \infty$, and $e(x) \to 0$ as $x \to -\infty$

(vi) $e : \mathbb{R} \to (0, \infty)$ is a bijection

**Proof.**

(i) done ✓

(ii) done ✓

(iii) Clearly $e(x) > 0 \ \forall x \geq 0$ and $e(0) = 1$
Also

$$e(0) = e(x - x) = e(x)e(-x)$$

$$\implies e(-x) > 0 \ \forall x > 0$$

(iv)

$$e'(x) = e(x) > 0 \implies e \text{ is strictly increasing}$$

(v)

$$e(x) > 1 + x \text{ for } x > 0$$

So if $x \to \infty, \ e(x) \to \infty$
For $x > 0$ since

$$e(-x) = \frac{1}{e(x)}, \ e(x) \to 0 \text{ as } x \to -\infty$$

(vi) injectivity: follows right away from being strictly increasing
surjectivity: Take $y \in (0, \infty)$, since $e(x) \to \infty$ as $x \to \infty$ and $e(x) \to 0$ as $x \to -\infty$,

$$\exists a, b \in \mathbb{R} \text{ s.t. } e(a) < y < e(b)$$

By the intermediate value theorem, $\exists x \in \mathbb{R}$ s.t. $e(x) = y$

**Remark.**

$$e : (\mathbb{R}, +) \to ((0, \infty), \times)$$

is a group isomorphism.

Since $e$ is a bijection, consider the inverse function

$$l : (0, \infty) \to \mathbb{R}$$

**Theorem 4.8.**

(i)
$$l : (0, \infty) \to \mathbb{R}$$
is a bijection and
$$l(e(x)) = x \ \forall x \in \mathbb{R}$$
and
$$e(l(t) = t \ \forall t \in (0, \infty)$$

(ii) $l$ is differentiable and
$$l'(t) = \frac{1}{t}$$

(iii)
$$l(xy) = l(x) + l(y) \ \forall x, y \in (0, \infty)$$

**Proof.**

(i) obvious from the definition

(ii) Inverse rule (Theorem 3.6):
$l$ is differenitable and
$$l'(t) = \frac{1}{e(l(t))} = \frac{1}{t}$$

(iii) from IA Groups, if $e$ is an isomorphism, so is its inverse □

Now define for $\alpha \in \mathbb{R}$ and $x > 0$,
$$r_\alpha(x) = e(\alpha l(x))$$

**Theorem 4.9.** Suppose $x, y > 0$ and $\alpha, \beta \in \mathbb{R}$. Then:

(i)
$$r_\alpha l(xy) = r_\alpha(x) r_\alpha(y)$$

(ii)
$$r_{\alpha+\beta}(x) = r_\alpha(x) r_\beta(x)$$

(iii)
$$r_\alpha(r_\beta(x)) = r_{\alpha\beta}(x)$$

(iv)
$$r_1(x) = x, \; r_0(x) = 1$$

**Proof.**

(i)
$$\begin{aligned} r_\alpha(xy) &= e(\alpha l(xy)) \\ &= e(\alpha l(x) + \alpha l(y)) \\ &= e(\alpha l(x)) e(\alpha l(y)) \\ &= r_\alpha(x) r_\alpha(y) \end{aligned}$$

(ii)
$$\begin{aligned} r_{\alpha+\beta}(x) &= e((\alpha+\beta) l(x)) \\ &= e(\alpha l(x)) e(\beta l(x)) \\ &= r_\alpha(x) r_\beta(x) \checkmark \end{aligned}$$

(iii)
$$\begin{aligned} r_{\alpha\beta}(x) &= r_\alpha(e(\beta l(x))) \\ &= e(\alpha l e(\beta l(x))) \\ &= e(\alpha\beta l(x)) \\ r_{\alpha\beta}(x) &\checkmark \end{aligned}$$

(iv)
$$r_1(x) = e(l(x)) = x \checkmark$$
$$r_0(x) = e(0) = 1 \checkmark \; \square$$

**Equation.**

$$r_n(x) = r_{1+\cdots+1}(x) = x \cdot x \ldots x = x^n$$

$$r_1(x) r_{-1}(x) = r_0(x) = 1$$

So

$$r_{-1}(x) = \frac{1}{x}$$

$$\implies r_{-n}(x) = \frac{1}{x^n}$$

$$(r_{1/q}(x))^q = r_1(x) = x \implies r_{1/q}(x) = x^{1/q}$$

$$r_{p/q} = (r_{1/q}(x))^p = x^{p/q}$$

Thus $r_\alpha(x)$ agrees with $\alpha \in \mathbb{Q}$ as previously defined.

Now we do a "baptism ceremony"

$$\exp(x) = e(x) \; x \in \mathbb{R}$$

$$\log x = l(x) \; x \in (0, \infty)$$

$$x^\alpha = r_\alpha(x) \; \alpha \in \mathbb{R}, \; x \in (0, \infty)$$

$$e(x) = e(x \log e) = r_x(e) = e^x$$

where

$$e = \sum_0^\infty \frac{1}{n!} = e(1)$$

so $\exp(x)$ is also a power, which we may as well denote $e^x$
Finally, we compute $(x^\alpha)'$

$$(x^\alpha)' = \left(e^{\alpha \log x}\right)' = e^{\alpha \log x} \frac{\alpha}{x} = \alpha x^{\alpha - 1} \checkmark$$

**Note.** If we let $f(x) = a^x$, $a > 0$ then

$$f'(x) = \left(e^{x \log a}\right)' = e^{x \log a} \log a = a^x \log a$$

**Remark.** "Exponentials beat polynomials"

$$\lim_{x \to \infty} \frac{e^x}{x^k} = \infty \text{ for } k > 0$$

$$e^x = \sum_0^\infty \frac{x^j}{j!} > \frac{x^n}{n!} \text{ for } x > 0$$

and pick $n > k$ so

$$\frac{e^x}{x^k} > \frac{x^{n-k}}{n!} \to \infty \text{ as } x \to \infty$$

## 4.2 Trigonometric Functions

**Definition.**

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots = \sum_0^\infty \frac{(-1)^k z^{2k}}{(2k)!}$$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots = \sum_0^\infty \frac{(-10)^k z^{2k+1}}{(2k+1)!}$$

Both power series have infinite radius of convergence and by theorem 4.4., they are differentiable and

$$(\sin z)' = \cos z$$

$$(\cos z)' = -\sin z$$

**Notation.** Write

$$e^x = e(z)$$

**Equation.**

$$e^{iz} = \sum_0^\infty \frac{(-z)^n}{n!} = \sum_0^\infty \frac{(iz)^{2k}}{(2k)!} + \sum_0^\infty \frac{(iz)^{2k+1}}{(2k+1)!}$$

$$(iz)^{2k} = (-1)^k z^{2k},\ (iz) = i(-1)^k z^{2k+1}$$

$$\implies e^{iz} = \cos z + i \sin z$$

Similarly,

$$e^{-iz} = \cos z - i \sin z$$

which gives:

$$\cos z = \frac{1}{2}\left(e^{iz} + e^{-iz}\right)$$

$$\sin z = \frac{1}{2i}\left(e^{iz} - e^{-iz}\right)$$

From this we get many trigonometric identities:

$$\cos z = \cos(-z),\ \sin(z) = -\sin z$$

$$\cos(0) = 1,\ \sin(0) = 0$$

(i)

$$\sin(z+w) = \sin z \cos w + \cos z \sin w$$

(ii)

$$\cos(z+w) = \cos z \cos w - \sin z \sin w \ \ z, w \in \mathbb{C}$$

Follows from

$$e^{a+b} = e^a \cdot e^b$$

to prove (ii) write:

$$\cos(z+w) = \frac{1}{2}\left\{e^{i(z+w)} + e^{-i(z+w)}\right\}$$
$$= \frac{1}{2}\left\{e^{iz} \cdot e^{iw} + e^{-iz} \cdot e^{-iw}\right\}$$

$$\cos z \cos w - \sin z \sin w = \frac{1}{4}(e^{iz} + e^{-iz})(e^{iw} + e^{-iw}) + \frac{1}{4}(e^{iz} - e^{-iz})(e^{iw} - e^{-iw}) \qquad (*)$$

operate to get same result use (*) to get

$$\sin^2 z + \cos^2 z = 1 \ \forall z \in \mathbb{C}$$

Now if $x \in \mathbb{R}$, then $\sin x, \cos x \in \mathbb{R}$
and (*) gives

$$|\sin x|,\ |\cos x| \leq 1$$

**Warning.**

$$\cos(iy) = \frac{1}{2}(e^{-y} + e^y)\ (y \in \mathbb{R})$$

as $y \to \infty,\ \cos(iy) \to \infty$

### 4.2.1 Periodicity of the Trigonometric Functions

**Prop 4.10.** There is a smallest positive number $\omega$ (where $\sqrt{2} < \frac{\omega}{2} < \sqrt{3}$ s.t.

$$\cos\left(\frac{\omega}{2}\right) = 0$$

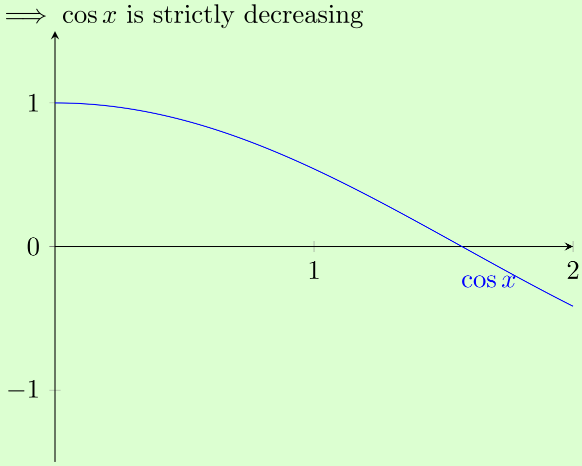
**Proof.** If $0 < x < 2$

$$\sin x = \left(x - \frac{x^3}{3!}\right) + \left(\frac{x^5}{5!} - \frac{x^7}{7!}\right) + \cdots > 0$$

(if $0 < x < 2$ then $\frac{x^{2n-1}}{(2n-1)!} > \frac{x^{2n+1}}{(2n+1)}$)
So for $0 < x < 2$,

$$(\cos x)' = -\sin x < 0$$

$\implies \cos x$ is strictly decreasing




We'll show that $\cos\sqrt{2} > 0$ and $\cos\sqrt{3} < 0$. Then by the intermediate value theorem the existence of $\omega$ follows.

$$\cos\sqrt{2} = \left(\frac{(\sqrt{2})^4}{4!} - \frac{(\sqrt{2})^6}{6!}\right) + \underset{>0}{(\ )} + \underset{>0}{(\ )} + \cdots > 0$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \underbrace{\left(\frac{x^6}{6!} - \frac{x^8}{8!}\right)}_{>0} - \dots$$

$x = \sqrt{3}$:

$$1 - \frac{3}{2} + \frac{9}{4 \times 3 \times 2} = 1 - \frac{3}{2} + \frac{3}{8} = -\frac{1}{8} < 0$$

$$\implies \cos\sqrt{3} < 0 \ \square$$

**Corollary 4.11.**

$$\sin \frac{\omega}{2} - 1$$

**Proof.**

$$\sin^2 \frac{\omega}{2} + \cos \frac{\omega}{2} = 1$$

and

$$\sin \frac{\omega}{2} > 0 \; \square$$

**Notation.** Now define $\pi = \omega$

**Theorem 4.12.**

(i)

$$\sin\left(z + \frac{\pi}{2}\right) = \cos z, \; \cos\left(z + \frac{\pi}{2}\right) - \sin z$$

(ii)

$$\sin(z + \pi) = -\sin z, \; \cos(z + \pi) = -\cos z$$

(iii)

$$\sin(z + 2\pi) = \sin z, \; \cos(z + 2\pi) \cos z$$

**Proof.** immediate from addition formulas and

$$\cos \frac{\pi}{2}, \; \sin \frac{\pi}{2} = 1 \; \square$$

**Note.** This implies

$$\begin{aligned} e^{iz+2\pi i} &= \cos(z + 2\pi) + i \sin(z + 2\pi) \\ &= \cos(z) i \sin z \\ &= e^{iz} \end{aligned}$$

$$\implies e^z \text{ is periodic with period } 2\pi i$$

**Remark.** We can "relate the trig functions with geometry".
Given two vectors $x, y \in \mathbb{R}^2$, define $x \cdot y$ as in vector and matrices

$$x \cdot y = x_1 y_1 + x_2 y_2,\ x = (x_1, x_2) \text{ and } y = (y_1, y_2)$$

By Cauchy-Swarz:

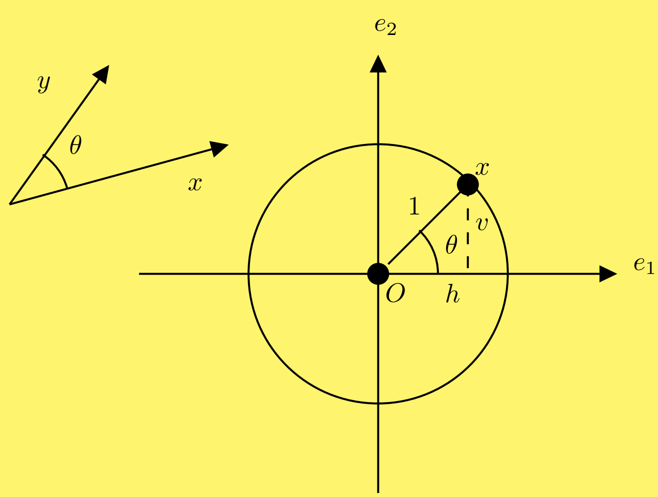
$$|x \cdot y| \leq \|x\| \|y\|$$

Thus if $x \neq 0,\ y \neq 0$

$$-1 \leq \frac{x \cdot y}{\|x\|\|y\|} \leq 1$$

So we define the angle between $x$ and $y$ as the unique $\theta \in [0, \pi]$ s.t.

$$\cos\theta = \frac{x \cdot y}{\|x\|\|y\|}$$




$$x = (h, v)$$

$$\cos\theta = x \cdot e_1 = h$$

## 4.3 Hyperbolic Functions

**Definition.**

$$\cosh z = \frac{1}{2}(e^z + e^{-z})$$

$$\sinh z = \frac{1}{2}(e^z - e^{-z})$$

$$\implies \cosh z = \cos(iz),\ \sinh = -i\sin(iz)$$

**Claim.**

$$(\cosh z)' = \sinh z$$

$$(\sinh z)' = \cosh z$$

$$\cosh^2 z - \sinh^2 z = 1,\ \text{etc.}$$

**Proof.** Exercise

**Note.** The rest of the trigonometric functions $(\tan, \cot, \sec, \mathrm{cosec})$ are defined in the usual way

# 5 Integration

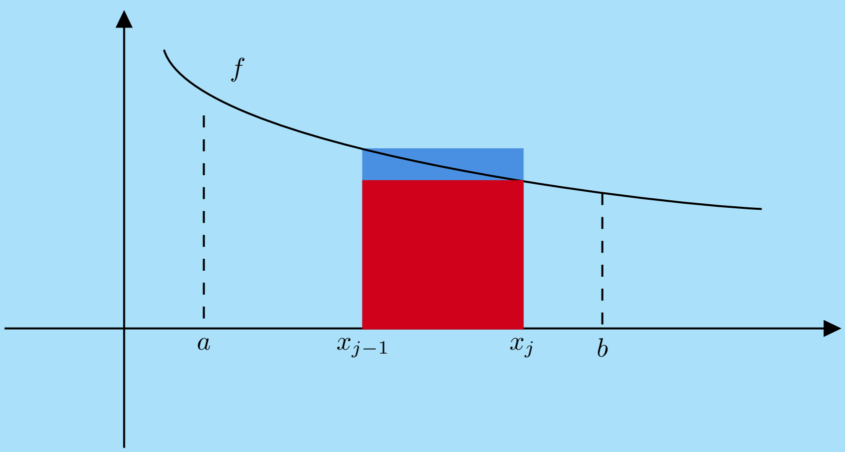
**Note.** $f : [a, b] \to \mathbb{R}$ bounded meand:

$$\exists K \text{ s.t. } |f(X)| \leq K,\ \forall x \in [a, b]$$

**Definition.** A **dissection** (or partition) $\mathcal{D}$ of $[a, b]$ is a finite subset of $[a, b]$ containing the end points of $a$ and $b$.
We write

$$\mathcal{D} = \{x_0, x_1, \ldots, x_n\} \text{ with}$$

$$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$$




**Definition.** We define the **upper sum** and **lower sum** associated with $\mathcal{D}$ by

$$S(f, \mathcal{D}) = \sum_{j=1}^{n} (x_j - x_{j-1}) \sup_{x \in [x_{j-1}, x_j]} f(x) \text{ (upper}$$

$$s(f, \mathcal{D} = \sum_{j=1}^{n} (x_j - x_{j-1}) \inf_{x \in [x_{j-1}, x_j]} f(x) \text{ (lower}$$

Clearly

$$s(d, \mathcal{D}) \leq S(d, \mathcal{D}) \ \forall \mathcal{D}$$

**Lemma 5.1.** If $\mathcal{D}$ and $\mathcal{D}'$ are dissections with $\mathcal{D} \subseteq \mathcal{D}'$, then

$$S(d, \mathcal{D}) \geq S(d, \mathcal{D}') \geq s(f, \mathcal{D}') \geq s(f, \mathcal{D})$$

**Proof.**

$$S(d, \mathcal{D}') \geq s(f, \mathcal{D}')$$

is obvious.
Suppose $\mathcal{D}'$ contains an extra point than $\mathcal{D}$, let's say $y \in (x_{r-1}, x_r)$
clearly:

$$\sup_{x \in [x_{r-1}, y]} f(x), \sup_{x \in [y, x_r]} f(x) \leq \sup_{x \in [x_{r-1}, x_r]} f(x)$$

$$\implies (x_r - x_{r-1}) \sup_{x \in [x_{r-1}, x_r]} f(x) \geq (y - x_{r-1}) \sup_{x \in [x_{r-1}, y]} f(x) + (x_r - y) \sup_{x \in [y, x_r]} f(x)$$

$$S(f, \mathcal{D}) \geq s(f, \mathcal{D}')$$

The same for $s$ and the same if $\mathcal{D}'$ has more extra points than $\mathcal{D}$

**Lemma 5.2.** $\mathcal{D}_1, \mathcal{D}_2$ two arbitrary dissections. Then

$$S(f, \mathcal{D}_1) \geq S(f, \mathcal{D}_1 \cup D_2) \geq s(f, \mathcal{D}_1 \cup \mathcal{D}_2) \geq s(f, \mathcal{D}_2)$$

So

$$S(f, \mathcal{D}_1) \geq s(f, \mathcal{D}_2)$$

**Proof.** Take

$$\mathcal{D}' = \mathcal{D}_1 \cup \mathcal{D}_2 \supseteq \mathcal{D}_1 \mathcal{D}_2$$

ad apply the previous lemma. □

**Definition.** The **upper integral** of $f$ is

$$I^*(f) = \inf_{\mathcal{D}} S(f, \mathcal{D})$$

(this always exists)
The **lower integral** of $f$ is

$$I_*(f) = \sup_{\mathcal{D}} s(f, \mathcal{D})$$

(this always exists)

**Claim.** By lemma 5.2,

$$I^*(f) \geq I_*(f)$$

**Proof.**

$$S(f, \mathcal{D}_1) \geq s(f, \mathcal{D}_2)$$

$$I^*(f) = \inf_{\mathcal{D}_1} S(f, \mathcal{D}_\infty) \geq s(f, \mathcal{D}_2)$$

$$I^*(f) \geq \sup_{\mathcal{D}_2} s(f, \mathcal{D}_\in) \geq s(f, \mathcal{D}_2) = I_*(f)$$

**Definition.** A bounded function $f : [a, b] \to \mathbb{R}$ is said to be **Reimann integrable** (or first integrable) if

$$I^*(f) = I_*(f)$$

and we set

$$\int_a^b f(x)\,\mathrm{d}x = I^*(f) = I_*(f) = \int_a^b f$$

**Example.**

$$f(x) = \begin{cases} 1 & x \in \mathbb{Q} \cap [0, 1] \\ 0 & x \notin \mathbb{Q} \cap [0, 1] \end{cases}$$

$f : [0, 1] \to \mathbb{R}$ is not Reimann integrable

$$\sup_{[x_{j-1}, x_j]} = 1, \quad \inf_{[x_{j-1}, x_j]} = 0 \ \forall \mathcal{D}$$

$$\implies I^*(f) = 1, \text{ but } I_*(f) = 0$$

A useful criterion for integrability:

**Theorem 5.3.** A bounded function

$$f : [a, b] \to \mathbb{R}$$

is Riemann integrable iff given $\varepsilon > 0, \exists \mathcal{D}$ s.t.

$$S(f, \mathcal{D}) - s(f, \mathcal{D}) < \varepsilon$$

**Proof.** For every dissection $\mathcal{D}$, we have

$$0 \leq I^*(f) - I_*(f) \leq S(f, \mathcal{D}) - s(f, \mathcal{D})$$

If the given condition holds, then

$$0 \leq I^*(f) - I_*(f) \leq S(f, \mathcal{D}) - s(f, \mathcal{D}) < \varepsilon \ \forall \varepsilon > 0$$

$$\implies I^*(f) = I_*(f)$$

Conversely, if $f$ is integrable, by definition of sup, inf, there are partitions $\mathcal{D}_1$ and $\mathcal{D}_2$ s.t.

$$\int_a^b f \, \mathrm{d}x - \frac{\varepsilon}{2} = I_*(f) - \frac{\varepsilon}{2} < s(f, \mathcal{D}_1)$$

$$S(f, \mathcal{D}_2) < I^*(f) + \frac{\varepsilon}{2} = \int_a^b f \, \mathrm{d}x + \frac{\varepsilon}{2}$$

By lemma 5.1,

$$(\mathcal{D}_1 \cup \mathcal{D}_2 \supseteq \mathcal{D}_1, \mathcal{D}_2)$$

$$S(f, \mathcal{D}_1 \cup \mathcal{D}_2) - s(f, \mathcal{D}_1 \cup \mathcal{D}_2) \leq S(f, \mathcal{D}_2) - s(f, \mathcal{D}_1) < \int_a^b f \, \mathrm{d}x + \frac{\varepsilon}{2} - \int_a^b f \, \mathrm{d}x + \frac{\varepsilon}{2} = \varepsilon \ \square$$

We now use this condition to show that monotonic and continuous functions (separately) are integrable.

**Remark.** Monotonic and continuous are bounded (thm 2.6 for the case of continuous functions)

**Theorem 5.4.** $f : [a, b] \to \mathbb{R}$ monotonic. Then $f$ is integrable

**Proof.** Suppose $f$ is increasing (same proof for $f$ decreasing)
Then

$$\sup_{x \in [x_{j-1}, x_j]} = f(x_j)$$

$$\inf_{x \in [x_{j-1}, x_j]} = f(x_{j-1})$$

Thus

$$S(f, \mathcal{D}) - s(f, \mathcal{D}) = \sum_{j=1}^{n} (x_j - x_{j-1})[f(x_j) - f(x_{j-1})]$$

Now choose

$$\mathcal{D} = \{a, a + \frac{b-a}{n}, a + \frac{2(b-a)}{n}, \ldots, b\}$$

$$x_j = a + \frac{(b-a)j}{n},\ 0 \leq j \leq n$$

$$S(f, \mathcal{D}) - s(f, \mathcal{D}) = \frac{(b-a)}{n}(f(b) - f(a))$$

Take $n$ large enough s.t.

$$\frac{b-a}{n}(f(b) - f(a)) < \varepsilon$$

and use Theorem 5.3 □

### 5.0.1 Continuous Functions

**Note.** First we need an auxiliary lemma

**Lemma 5.5.** $f : [a, b] \to \mathbb{R}$ continuous. Then

$$\text{given } \varepsilon > 0, exists \delta > 0 \text{ s.t } |x - y| < \delta \implies |f(x) - f(y)| < \varepsilon$$

(uniform continuity)

**Note.** The point is $\delta$ works $\forall x, y$ as long as $|x - y| < \delta$
(in the definition of continuity of $f$ at $x$, $\delta = \delta(x)$)

**Proof.** Suppose the claim is false. Then $\exists \varepsilon > 0$ s.t. $\forall \delta > 0$, we can find $x, y \in [a, b]$ s.t. $|x - y| < \delta$ but $|f(x) - f(y) \geq \varepsilon$
Take $\delta = \frac{1}{n}$, to gen $x_n, y_n$ with

$$|x_n - y_n| < \frac{1}{n}, \text{ but } |f(x_n) - f(y_n)| \geq \varepsilon$$

By Bolzano-Weierstrass, $\exists x_{n_k} > C$

$$|y_{n_k} - C| \leq |y_{n_k} - x_{n_k}| + |x_{n_k} - C| \to 0$$

(both parts of sum converge to 0)
But

$$|f(x_{n_k}) - f(y_{n_k})| \geq \varepsilon$$

$$0 \geq \varepsilon \text{ ※ } \square$$

**Theorem 5.6.** Let $f : [a, b] \to \mathbb{R}$ be continuous. Then $f$ is Riemann integrable.

**Proof.** given $\varepsilon > 0$, $\exists \delta > 0$ s.t. $|x - y| < \delta$

$$\implies |f(x) - f(y)| < \varepsilon$$

Let $\mathcal{D} = \{a + \frac{(b-a)j}{n}, j = 0, 1, \dots, n\}$
Choose $n$ large enough s.t.

$$\frac{b-a}{n} < \delta$$

Then for $x, y \in [x_{j-1}, x_j]$

$$|f(x) - f(y)| < \varepsilon \tag{*}$$

since

$$|x - y| \leq |x_j - x_{j-1}| = \frac{b-a}{n} < \delta$$

This means that

$$\max_{x \in [x_{j-1}, x_j]} f(x) - \min_{x \in [x_{j-1}, x_j]} f(x) = f(p_j) - f(q_j) \; p_j, q_j \in [x_{j-1}, x_j]$$

(max and min exist due to continuity)

$$\begin{aligned}
\implies S(f, \mathcal{D}) - s(f, \mathcal{D}) &= \sum_{j=1}^{n} (x_j - x_{j-1}) \left[ \max_{x \in [x_{j-1}, x_j]} f(x) - \min_{x \in [x_{j-1}, x_j]} f(x) \right] \\
&= \sum_{j=1}^{n} \frac{(b-a)}{n} (\underbrace{f(p_j) - f(q_j)}_{< \varepsilon \text{ by } (*)}) \\
&< \varepsilon (b - a)
\end{aligned}$$

Now use Theorem 5.3 □

**Remark.** More complicated functions can be Riemann integrable

**Example.** $f : [0,1] \to \mathbb{R}$

$$f(x) = \begin{cases} 1/q, & x = p/q \in (0,1] \text{in its lowest form} \\ 0, & \text{otherwise} \end{cases}$$

Clearly $s(f, \mathcal{D}) = 0 \ \forall \mathcal{D}$.
We will show that given $\varepsilon > 0, \exists \mathcal{D}$ s.t.

$$S(f, \mathcal{D}) < \varepsilon$$

This implies $f$ is integrable with

$$\int_0^1 f = 0$$

Take $N \in \mathbb{N}$ s.t.

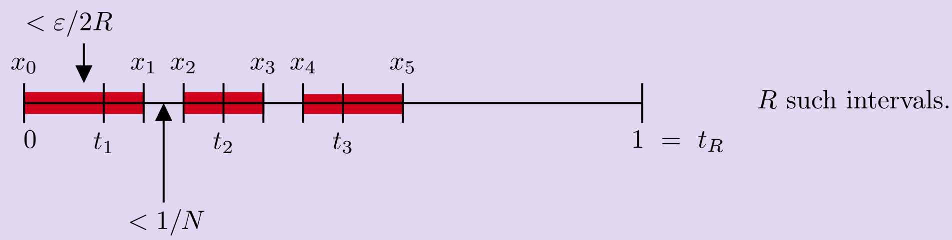
$$\frac{1}{N} < \frac{\varepsilon}{2}$$

Consider the set

$$\{x \in [0,1] : f(x) \geq 1/N\} = \{p/q : 1 \leq q \leq N \text{ and } 1 \leq p \leq q$$

This is a finite set $0 < t_1 << t_2 < \cdots < t_R = 1$
Consider a dissection of $[a, b]$ s.t.

(i) Each $t_k, 1 \leq k \leq R$ is in some $[x_{j-1}, x_j]$
(ii) $\forall k$, the unique interval containing $t_R$ has length at most $\varepsilon/2R$




Not: $f \leq 1$ everywhere

$$S(f, \mathcal{D}) \leq \frac{1}{N} + \frac{\varepsilon}{2} < \varepsilon$$

## 5.1 Elementary Properties of the Integral

**Claim.** For $f, g$ bounded and integrable on $[a,b]$:

(i) If $f \leq g$ on $[a,b]$, then

$$\int_a^b f \leq \int_a^b g$$

(ii) $f+g$ is integrable on $9a,b]$ and

$$\int_a^b f+g = \int_a^b f + \int_a^b g$$

(iii) For any constant $k$, $kf$ is integrable and

$$\int_a^b kg = k\int_a^b f$$

(iv) $|f|$ is integrable and

$$\left|\int_a^b f\right| \leq \int_a^b |f|$$

(v) The product $fg$ is integrable

**Proof.**

(i) if $f \leq g$, then

$$\int_a^b f = I^*(f) \leq S(f,\mathcal{D}) \leq S(g,\mathcal{D})$$

hence

$$\int_a^b f = I^*(f) \leq I^*(g) = \int_a^b g$$

(ii)

$$\sup_{[x_{j-1},x_j]}(f+g) \leq \sup_{[x_{j-1},x_j]} f + \sup_{[x_{j-1},x_j]} g$$

$$\implies S(f+g,\mathcal{D}) \leq S(f,\mathcal{D}) + S(g,\mathcal{D})$$

Now take two dissections $\mathcal{D}_1$ and $\mathcal{D}_2$

$$\begin{aligned} I^*(f+g) &\leq S(f+g,\mathcal{D}_1\cup\mathcal{D}_2) \leq S(f,\mathcal{D}_1\cup\mathcal{D}_2) + S(g,\mathcal{D}_1\cup\mathcal{D}_2) \\ &\leq S(f,\mathcal{D}_1) + S(g,\mathcal{D}_2) \end{aligned}$$

last from lemma 5.1. Fix $\mathcal{D}_1$ and inf over $\mathcal{D}_2$ to get

$$I^*(f+g) \leq I^*(f) + I^*(g) = \int_a^b f + \int_a^b g$$

Similarly

$$\int_a^b f + \int_a^b g \leq I_*(f+g)$$

$\implies$ $f+g$ is integrable with integral equal to the sum of the integrals.

(iii) Exercise!

**Claim** (cont.).

**Proof** (cont.).

(iv) Consider

$$f_+(x) = \max(f(x), 0)$$

$$\sup_{[x_{j-1},x_j]} f_+ - \inf_{[x_{j-1},x_j]} f_+ \leq \sup_{[x_{j-1},x_j]} f - \inf_{[x_{j-1},x_j]} f$$

(can check)
and we know that given $\varepsilon > 0$, $\exists \mathcal{D}$ s.t.

$$S(f, \mathcal{D}) - s(f, \mathcal{D}) < \varepsilon$$

$$\implies S(f_+, \mathcal{D}) - s(f_+, \mathcal{D}) < \varepsilon$$

$\implies f_+$ is integrable
But $|f| = 2f_+ - f$ By (ii) and (iii), $|f|$ is integrable.
Since $-|f| \leq f \leq |f|$, we use property (i) to see

$$\left| \int_a^b f \right| \leq \int_a^b |f|$$

(v) Take $f$ integrable and $\geq 0$
Then

$$\sup_{[x_{j-1},x_j]} f^2 = \Bigg( \overbrace{\sup_{[x_{j-1},x_j]} f}^{M_j} \Bigg)^2$$

$$\inf_{[x_{j-1},x_j]} f^2 = \Bigg( \underbrace{\inf_{[x_{j-1},x_j]} f}_{m_j} \Bigg)^2$$

Thus

$$\begin{aligned} S(f^2, \mathcal{D}) - s(f^2, \mathcal{D}) &= \sum_{j=1}^{n} (x_j - x_{j-1})(M_j^2 - m_j^2) \\ &= \sum_{j=1}^{n} (x_j - x_{j-1}(M_j + m_j)(M_j - m_j) \\ &\leq 2K(S(f, \mathcal{D}) - s(f, \mathcal{D})) \end{aligned}$$

using $|f(x)| \leq K \ \forall x \in [a, b]$
Using the criterion in Theorem 5.3, we deduce that $f^2$ is integrable.
Now take any $f$, then $|f| \geq 0$ and is integrable. Since $f^2 = |f|^2$.
We deduce that $f^2$ is integrable for any $f$
Finally for $fg$, note:

$$4fg = (f + g)^2 - (f - g)^2$$

$$\implies fg \text{ is integrable given what we proved}$$

**Claim** (6)**.** $f$ is integrable on $[ab]$. If $a < c < b$, then $f$ is integrable over $[a, c]$ and $[c, b]$ and

$$\int_a^b f = \int_a^c f + \int_c^b f$$

Conversely if $f$ is integrable over $[a, c]$ and $[c, b]$, then $f$ is integrable over $[a, b]$ and

$$\int_a^b f = \int_a^c f + \int_c^b f$$

**Proof.** We first make two observations:
if $\mathcal{D}_1$ is a dissection of $[a, c]$ and $\mathcal{D}_2$ is a dissection of $[b, c]$, then

$$\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2$$

is a dissection of $[a, b]$ and

$$S(f, \mathcal{D}_1 \cup \mathcal{D}_2) = S(f|_{[a,c]}, \mathcal{D}_1) + S(f|_{[c,b]}, \mathcal{D}_2) \qquad (*_1)$$

Also if $\mathcal{D}$ is a dissection of $[a, b]$, then

$$\begin{aligned} S(f, \mathcal{D}) &\geq S(f, \mathcal{D} \cup \{c\}) \\ &= S(f|_{[a,c]}, \mathcal{D}_1) + S(f|_{[c,b]}, \mathcal{D}_2) \qquad (*_2) \end{aligned}$$

where $\mathcal{D}_1$ dissects $[a, c]$ and $\mathcal{D}_2$ dissects $[a, b]$

$$(*_1) \implies I^*(f) \leq I^*(f|_{[a,c]}) + I^*(f|_{[c,b]})$$

$$(*_2) \implies I^*(f) \geq I^*(f|_{[a,c]}) + I^*(f|_{[c,b]})$$

Similarly

$$I_*(f) = I_*(f|_{[a,c]}) + I_*(f|_{[c,b]})$$

Thus

$$0 \leq I^*(f) - I_*(f) = \underbrace{I^*(f|_{[a,c]}) - I_*(f|_{[a,c]})}_{\geq 0} + \underbrace{I^*(f|_{[c,b]}) - I_*(f|_{[c,b]})}_{\geq 0}$$

From this, claim follows right away. □

**Notation.** We have a convention that is if $a > b$, then

$$\int_a^b f = -\int_b^a f$$

if $a = b$, we agree that its value is zero.
With this convention, if $|f| \leq K$, then

$$\left| \int_c^b f \right| \leq K|b - a|$$

(from property (4) and convention)

## 5.2 The Fundamental Theorem of Calculus (FTC)

$f : [a, b] \to \mathbb{R}$ bounded and integrable. Write

$$F(x) = \int_a^x f(t)\,\mathrm{d}t,\ x \in [a, b]$$

**Theorem 5.7.** $F$ is continuous

**Proof.**

$$F(x+h) - F(x) = \int_x^{x+h} f(t)\,\mathrm{d}t$$

$$|F(x+h) - F(x)| = \left|\int_x^{x+h} f(t)\,\mathrm{d}t\right| \leq K|h|$$

if $|f(t)| \leq K,\ \forall t \in [a, b]$. Now let $h \to 0$ and we are done. □

**Theorem 5.8** (FTC)**.** If in addition $f$ is continuous at $x$, then $F$ is differentiable at $x$ and

$$F'(x) = f(x)$$

**Proof.** We need to consider $(x + h \in [a, b]$ & $h \neq 0$

$$\begin{aligned} \left|\frac{F(x+h) - F(x)}{h} - f(x)\right| &= \frac{1}{|h|}\left|\int_x^{x+h} f(t)\,\mathrm{d}t - hf(x)\right| \\ &= \frac{1}{|h|}\left|\int_x^{x+h} [f(t) - f(x)]\,\mathrm{d}t\right| \end{aligned}$$

$f$ continuous at $x$, means that given $\varepsilon > 0,\ \exists\delta > 0$ s.t. if $|t - x| < \delta$ then

$$|f(t) - f(x)| < \varepsilon$$

IF $|h| < \delta$, we can write

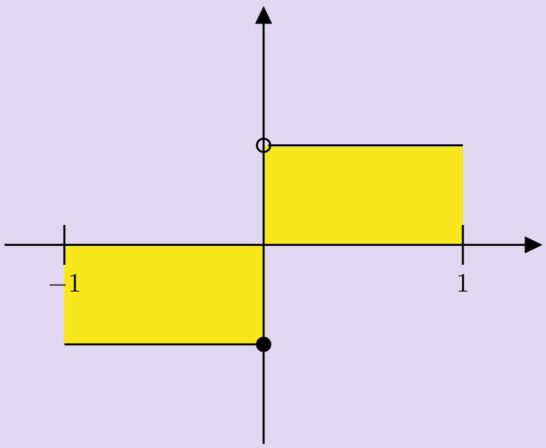
$$\begin{aligned} \frac{1}{|h|}\left|\int_x^{x+h} [f(t) - f(x)]\,\mathrm{d}t\right| &\leq \frac{1}{|h|}\varepsilon|h| \\ &= \varepsilon \end{aligned}$$

This means

$$\lim_{h \to 0} \frac{F(x+h) - F(x)}{h} = f(x)\ \square$$

**Example.**

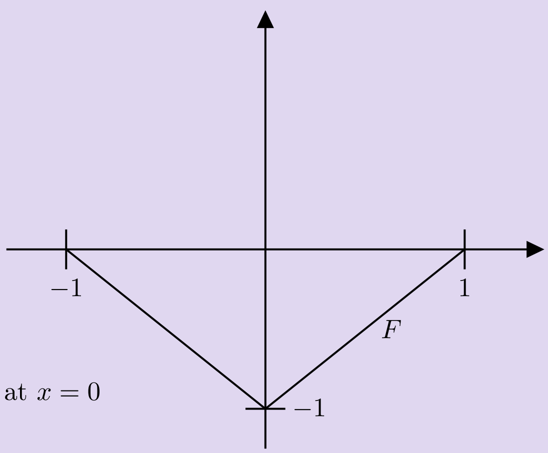
$$f(x) = \begin{cases} -1 & x \in [-1, 0) \\ 1 & x \in (0, 1] \end{cases}$$




$$\text{monotonic} \implies \text{integrable}$$

$$f(x) = \begin{cases} -x - 1 & x \leq 0 \\ x - 1 & x \geq 0 \end{cases}$$

$$F(x) = -1 + |x|$$




$F$ not differentiable at $x = 0$

**Corollary 5.9** (integration is the inverse of differentiation)**.** If $f = g'$ is continuous on $[a, b]$, then

$$\int_a^x f(t)\,\mathrm{d}t = g(x) - g(a)\ \forall x \in [a, b]$$

**Proof.** From Theorem 5.8, $F - g$ has zero derivative in $[a, b] \implies F - g$ is constant and since $F(a) = 0$,

$$F(x) = g(x) - g(a)\ \square$$

**Notation.** Every continuous function has an indefinite integral or anti-derivative written

$$\int f(x)\,\mathrm{d}x$$

which is determined up to a constant.

**Remark.** We have solved the ODE:

$$\begin{cases} y'(x) = f(x) \\ y(a) = y_0 \end{cases}$$

**Corollary 5.10** (integration by parts)**.** Suppose $f'$ and $g'$ exist and are ontinuous on $[a, b]$. Then

$$\int_a^b f'g = f(b)g(b) - f(a)g(a) - \int_a^b fg'$$

**Proof.** By the product rule,

$$(fg)' = f'g + fg'$$

By 5.9,

$$f(b)g(b) - f(a)g(a) = \int_a^b f'g + \int_a^b fg' \ \square$$

**Corollary 5.11** (integration by substitution)**.** Let $g : [\alpha, \beta] \to [a, b]$ with $g(\alpha) = a$ and $g(\beta) = b$, $g'$ exists and is continuous on $[\alpha, \beta]$. Let $f : [a, b] \to \mathbb{R}$ be continuous. Then

$$\int_a^b f(x)\,\mathrm{d}x = \int_\alpha^\beta f(g(t))g'(t)\,\mathrm{d}t$$

**Proof.** Set

$$F(x) = \int_a^x f(t)\,\mathrm{d}t$$

as before. Let $h(t) = F(g(t))$ defined since $g$ takes values in $[a, b]$). Then

$$\begin{aligned} \int_\alpha^\beta f(g(t))g'(t)\,\mathrm{d}t &\underset{\text{FTC}}{=} \int_\alpha^\beta F'(g(t))g'(t)\,\mathrm{d}t \\ &\underset{\text{chain rule}}{=} \int_\alpha^\beta h'(t)\,\mathrm{d}t \\ &= h(\beta) - h(\alpha) \\ &= F(b) - F(a) \\ &= \int_a^b f(x)\,\mathrm{d}x \ \square \end{aligned}$$

**Theorem 5.12** (Taylor's theorem with remainder an integral)**.** Let $f^{(n)}(x)$ be continuous for $x \in [0, h]$. Then

$$f(h) = f(0) + \cdots + \frac{h^{n-1} f^{(n-1)}}{(n-1)!} + R_n$$

where

$$R_n = \frac{h^n}{(n-1)!} \int_0^1 (1-t)^{n-1} f^{(n)}(th)\,\mathrm{d}t$$

**Proof.** By substituting $u = th$

$$R_n = \frac{1}{(n-1)!} \int_0^h (h-u)^{n-1} f^{(n)}(u)\,\mathrm{d}u$$

Integrating by parts now, we get:

$$R_n = -\frac{h^{n-1} f^{(n-1)}(0)}{(n-1)!} + \underbrace{\frac{1}{(n-2)!} \int_0^h (h-u)^{n-2} f^{(n-1)}(u)\,\mathrm{d}u}_{R_{n-1}}$$

If we integrate by parts $n-1$ times, we arrive at:

$$R_n = -\frac{h^{n-1} f^{(n-1)}(0)}{(n-1)!} - \cdots - hf'(0) + \underbrace{\int_0^h f'(u)\,\mathrm{d}u}_{f(h)-f(0)} \quad \square$$

**Remark.** Now we can get the Cauchy & Lagrange form of the remainder.
However, note that the proof above uses continuity of $f^{(n)}$ not just mere existence as in section 3.
But first need to prove:

**Theorem 5.13.** $f, g : [a,b] \to \mathbb{R}$ continuous with $g(x) \neq 0\ \forall x \in (a,b)$. Then $\exists c \in (a,b)$ s.t.

$$\int_a^b f(x)g(x)\,\mathrm{d}x = f(x)\int_a^b g(x)\,\mathrm{d}x$$

**Proof.** We're going to use Cauchy's MVT (Theorem 3.7)

$$F(x) = \int_a^x fg,\ G(x) = \int_a^x g$$

$$\text{Theorem 3.7} \implies \exists c \in (a,b) \text{ s.t.}$$

$$(F(b) - F(a))G'(c) = F'(c)(G(b) - G(a))$$

$$\left(\int_a^b fg\right) g(c) = f(c)g(c)\int_a^b g$$

if $g(c) \neq 0$, we simplify ans we're done $\square$

**Note.** if we take $g(x) \equiv 1$, we get

$$\int_a^b f(x)\,\mathrm{d}x = f(c)(b-a)$$

**Claim.** We can get the Cauchy & Lagrange form of the remainder from Taylor's theorem with remainder (given continuity of $f^{(n)}$)

**Proof.** Now we want to apply this to

$$R_n = \frac{h^n}{(n-1)!}\int_0^1 (1-t)^{n-1} f^{(n)}(th)\,\mathrm{d}t$$

First we use Theorem 5.13 with $g \equiv 1$, to get

$$R_n = \frac{h^n}{(n-1)!}(1-\theta)^{n-1} f^{(n)}(\theta h) m\theta \in (0,1)$$

Which is Cauchy's form of the remainder!
To get Lagrange, we use Theorem 5.13 with $g(t) = (1-t)^{n-1}$ which is $> 0$ for $t \in (0,1)$

$$\implies \exists \theta \in (0,1) \text{ s.t. } R_n = \frac{h^n}{(n-1)!} f^{(n)}(\theta h)\underbrace{\left[\int_0^1 (1-t)^{n-1}\,\mathrm{d}t\right]}_{=1/n}$$

$$\int_0^1 (1-t)^{n-1}\,\mathrm{d}t = -\frac{(1-t)^n}{n}\Bigg]_0^1 = \frac{1}{n}$$

$$\implies R_n = \frac{h^n}{n!} f^{(n)}(\theta h),\ \theta \in (0,1)$$

which is Lagrange's form of the remainder!

## 5.3 Improper Integrals

**Definition.** Suppose $f : [a, \infty] \to \mathbb{R}$ integrable (and bounded) on every interval $[a, R]$ and that as $R \to \infty$

$$\int_a^R f(x)\,\mathrm{d}x \to l$$

Then we say that $\int_a^\infty f(x)\,\mathrm{d}x$ exists or converges and that its value is $l$. If $\int_a^R f(x)\,\mathrm{d}x$ does not ten to a limit, we say that $\int_a^\infty f(x)\,\mathrm{d}x$ diverges.
A similar definition applies to $\int_{-\infty}^a f(x)\,\mathrm{d}x$. If

$$\int_a^\infty f(x)\,\mathrm{d}x = l_1$$

and

$$\int_{-\infty}^a f(x)\,\mathrm{d}x = l_2$$

we write

$$\int_{-\infty}^\infty = l_1 + l_2$$

(independent of the particular value of $a$)

**Warning.** This is not the same as saying that

$$\lim_{R\to\infty} \int_{-R}^R f(x)\,\mathrm{d}x$$

exists. It is stronger: e.g.

$$\int_{-R}^R x\,\mathrm{d}x = 0$$

**Example.**

$$\int_1^\infty \frac{\mathrm{d}x}{x^k} \text{ converges iff } k > 1$$

Indeed, if $k \neq 1$,

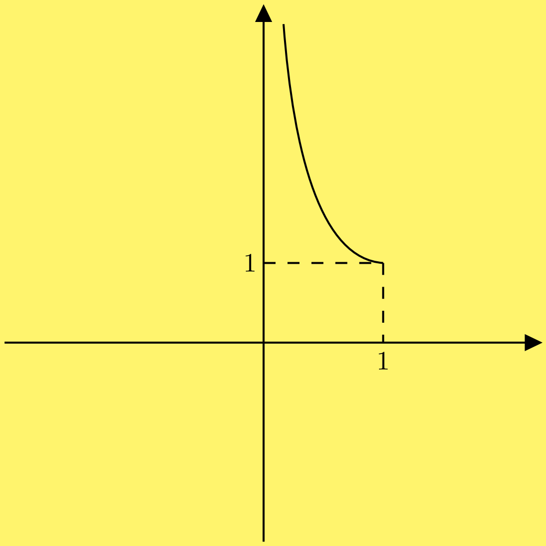
$$\int_1^R \frac{\mathrm{d}x}{x^k} = \frac{x^{1-k}}{1-k}\bigg]_1^R = \frac{R^{1-k}}{1-k}$$

and as $R \to \infty$, this limit is finite iff $k > 1$ (and equals $-\frac{1}{1-k}$)
if $k = 1$,

$$\int_1^R \frac{\mathrm{d}x}{x} = \log R \to \infty$$

**Remark.** $1/\sqrt{x}$ continuous on $[\delta, 1]$, for any $\delta > 0$. and

$$\int_\delta^1 \frac{\mathrm{d}x}{\sqrt{x}} = 2\sqrt{x}\Big]_\delta^1 = 2 - 2\sqrt{\delta} \to 2 \text{ as } \delta \to 0$$




$1/\sqrt{x}$ is unbounded on $[0, 1]$

$$\int_0^1 \frac{\mathrm{d}x}{\sqrt{x}} = \lim_{\delta \to 0} \int_\delta^1 \frac{\mathrm{d}x}{\sqrt{x}} = 2$$

Exercise: give a general definition

$$\int_0^1 \frac{\mathrm{d}x}{x} = \lim_{\delta \to 0} \int_\delta^1 \frac{\mathrm{d}x}{x} = \lim_{\delta \to 0} \log x\Big]_\delta^1 = \log 1 - \log \delta$$

limit does not exist as $\delta \to 0$

**Remark.** If $f \geq 0$ and $g \geq 0$ for $x \geq a$ and $f(x) \leq Kg(x)$, $K$ constant $x \geq a$, then

$$\int_a^\infty g \text{ converges } \implies \int_a^\infty f \text{ converges}$$

and

$$\int_a^\infty f \leq K \int_a^\infty g$$

Just note that

$$\int_a^R f \leq K \int_a^R g$$

The function $R \to \int_a^R f$ is increasing ($f \geq 0$) and bounded above ($\int_a^\infty g$ converges)
Take

$$l = \sup_{R \geq a} \int_a^R f < \infty$$

Now check that

$$\lim_{R \to \infty} \int_a^R f = l$$

given $\varepsilon > 0, \exists R_0$ s.t.

$$\int_a^{R_0} f \geq l - \varepsilon$$

Thus

$$\forall R \geq R_0, \int_a^R f \geq \int_a^{R_0} \geq l - \varepsilon$$

$$\implies 0 \leq l - \int_a^R f \leq \varepsilon \checkmark$$

**Example.**

$$\int_0^\infty e^{-x^2/2} \, \mathrm{d}x$$

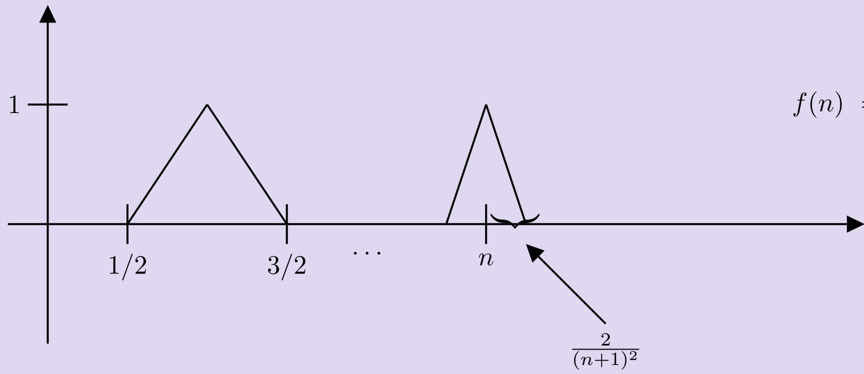
$$e^{-x^2/2} \leq e^{-x/2}, x \geq 1$$

$$\int_1^R e^{-x/2} \, \mathrm{d}x = \frac{1}{2}[e^{-1/2} - e^{-R/2}] \to \frac{e^{-1/2}}{2}$$

$$\implies \int_0^\infty e^{-x^2/2} \, \mathrm{d}x \text{ converges}$$

**Remark.** We know that if $\sum a_n$ converges, then $a_n \to 0$. We have to be careful with improper integrals.
$\int_a^\infty f$ converges may not imply that $f \to 0$

**Example.**




$$f(n) = 1$$

$$\text{area}(\Delta) = \frac{2}{(n+1)^2}$$

## 5.4 The Integral Test

**Theorem 5.14** (integral test)**.** Let $f(x)$ be a positive decreasing function for $x \geq 1$. Then

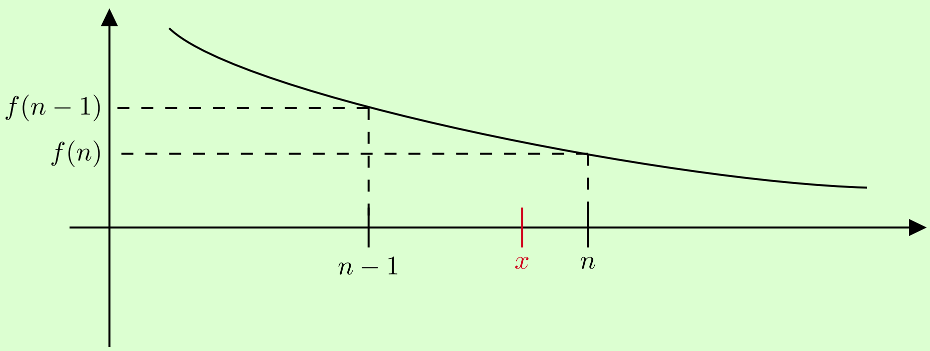
(i) Th integral $\int_1^\infty f(x)\,\mathrm{d}x$ and the series $\sum_1^\infty f(n)$ both converge or diverge.

(ii) As $n \to \infty$,

$$\sum_{r=1}^{n} f(r) - \int_1^n f(x)\,\mathrm{d}x$$

tends to a limit $l$ s.t. $0 \leq l \leq f(1)$

**Proof.**




($f$ decreasing $\implies$ $f$ integrable on every bounded subinterval by Theorem 5.4)
If $n-1 \leq x \leq n$, then

$$f(n-1) \geq f(x) \geq f(n)$$

$$\implies f(n-1) \geq \int_{n-1}^{n} f(x)\,\mathrm{d}x \geq f(n) \qquad (*)$$

Adding:

$$\sum_{r=1}^{n-1} f(r) \geq \int_1^n f(x)\,\mathrm{d}x \geq \sum_2^n f(r) \qquad (**)$$

From this claim (i) is obvious.
For the proof of (ii) set

$$\phi(n) = \sum_1^n f(r) - \int_1^n f(x)\,\mathrm{d}x$$

Then

$$\phi(n) - \phi(n-1) = f(n) - \int_{n-1}^{n} f(x)\,\mathrm{d}x \leq 0$$

using (*).
Also from (**),

$$0 \leq \phi(n) \leq g(1)$$

thus $\phi(n)$ is decreasing and tends ot a limit $l$ s.t.

$$0 \leq l \leq f(1) \;\square$$

**Examples.**

(i)

$$\sum_{1}^{\infty} \frac{1}{n^k} \text{ converges iff } k > 1$$

We saw that

$$\int_1^{\infty} \frac{1}{x^k} \text{ converges iff } k > 1$$

so we just apply the integral test.

(ii)

$$\sum_{1}^{\infty} \frac{1}{n \log n}, \ f(x) = \frac{1}{x \log x}, \ x \geq 2$$

$$\int_2^R \frac{\mathrm{d}x}{x \log x} = \log(\log x)]_2^R$$

$$\log(\log R) - \log(\log 2) \to \infty \text{ as } R \to \infty$$

then by the integral test

$$\sum_{2}^{\infty} \frac{1}{n \log n} \text{ diverges}$$

**Corollary 5.15** (Euler's constant)**.** As $n \to \infty$,

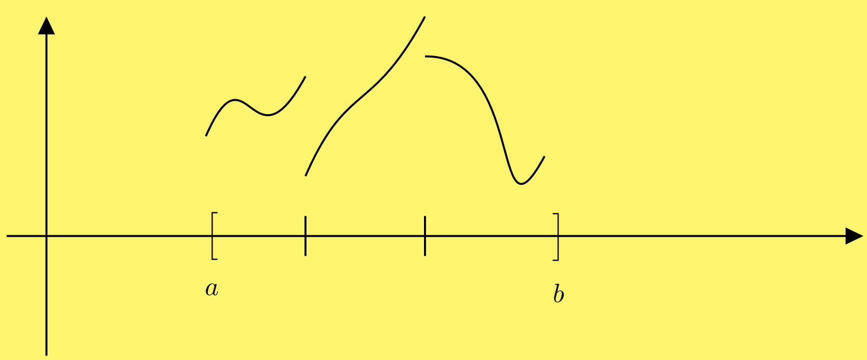
$$1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log n \to \gamma$$

with $0 \leq \gamma \leq 1$

**Proof.** Set $f(x) = 1/x$ and use Theorem 5.14 □

**Remark.** We have an open problem: is $\gamma$ irrational?
($\gamma \sim 0.577$)

**Remark.** We have seen: monotone functions and continuous functions are integrable
We can generalise this a bit and say that piece-wise continuous functions are integrable



**Definition.** A function $f : [a, b] \to \mathbb{R}$ is said to be **piece-wise continuous** if there is a dissection $\mathcal{D} = \{x_0 = a, x_1, \ldots, x_n = b\}$ s.t.
(i) $f$ is continuous on $(x_{j-1}, x_j)$ $\forall j$
(ii) the one-sided limits

$$\lim_{x \to x_{j-1}^+} f(x), \quad \lim_{x \to x_{j-1}^-} f(x) \text{ exist}$$

## 5.5 Characterization for Riemann integrability (Non-Examinable)

**Note.** It is now an exercise to check that $f$ is Riemann integrable:
first check that $f|_{[x_{j-1}, x_j]}$ is integrable for each $j$ (the values of $f$ at the end points won't really matter) and use additivity of domain (property (6))

**Note.** Q: How large can the discontinuity of $f$ be while $f$ is still Riemann integrable?
Recall the example

$$f(x) = \begin{cases} 1/q & x = p/q \\ 0 & \text{otherwise} \end{cases}$$

The question has been answered by Henri Lebesgue:
Characterization for Riemann integrability:
$f : [a, b] \to \mathbb{R}$ bounded. Then $f$ is Riemann integrable iff the set of discontinuity points has measure zero.

**Definition.** Let $l(I)$ be the length of an interval $I$.
A subset $A \subseteq \mathbb{R}$ is said to have **measure zero** if for each $\varepsilon > 0$ $\exists$ a countable family of intervals st.

$$A \subseteq \bigcup_{j=1}^{\infty} I_j$$

and

$$\sum_j l(I_j) < \varepsilon$$

**Lemma 5.16.**

(i) Every countable set has measure zero.

(ii) if $B$ has measure zero and $A \subseteq B$, the $A$ has measure zero.

(iii) if $A_k$ has measure zero $\forall k \in \mathbb{N}$ then $\bigcup_{k\in\mathbb{N}} A_k$ also has measure zero.

**Note.** The proof of Lebesgue's criterion uses the concept of oscillation of $f$:
$I$ interval:

$$\omega_f(I) = \sup_I f - \inf_I f$$

Oscillation at a point

$$\omega_f(x) = \lim_{\varepsilon\to 0} \omega_f(x-\varepsilon, x+\varepsilon)$$

**Proof** (Sketch)**.**

$$\begin{aligned} D &= \{x \in [a,b] : f \text{ discontinuous at } x\} \\ &= \{x : \omega_f(x) > 0\} \end{aligned}$$

$\implies$ RTP: $D$ has measure zero.

$$N(\alpha) = \{x : \omega_f(x) \geq \alpha\}$$

$$D = \bigcup_1^\infty N(1/k)$$

We'll show that for fixed $\alpha$, $N(\alpha)$ has measure zero.
Let $\varepsilon > 0, \exists \mathcal{D}$ s.t.

$$S(f,\mathcal{D}) - s(f,\mathcal{D}) < \frac{\varepsilon\alpha}{2}$$

$$S(f,\mathcal{D}) - s(f,\mathcal{D}) = \sum_{j=1}^n \omega_f([x_{j-1},x_j])(x_j - x_{j-1})$$

$$F = \{j : (x_{j-1},x_j) \cap N(\alpha) \neq \varnothing\}$$

then for each $j \in F$,

$$\omega_f([x_{j-1},x_j]) \geq \alpha$$

$$\alpha \sum_{j\in F}(x_j - x_{j-1}) \leq \sum_{j\in F} \omega_f([x_{j-1},x_j])(x_j - x_{j-1}) < \frac{\varepsilon\alpha}{2}$$

$$\implies \sum_{j\in F}(x_j - x_{j-1}) < \frac{\varepsilon}{2}$$

These cover $N(\alpha)$ except perhaps for $\{x_0, x_1, x_n\}$. But these can be covered by intervals of total length $< \frac{\varepsilon}{2}$
$\implies$ $N(\alpha)$ can be covered by intervals of total length $< \varepsilon$✓

**Lemma 5.17** (cont.)**.**

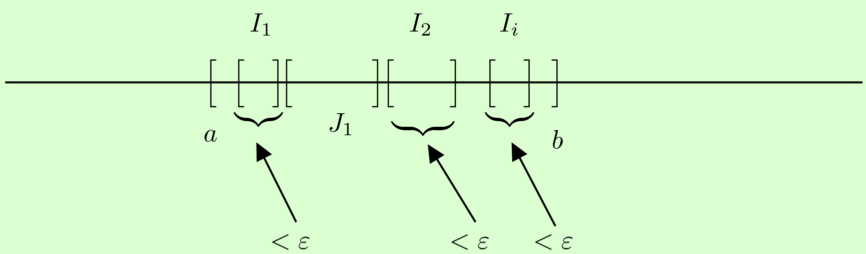
**Proof** (cont.)**.** $\Longleftarrow$ : let $\varepsilon > 0$ be given

$$N(\varepsilon) \subseteq D$$

so $N(\varepsilon)$ has measure zero. It is closed and bounded, $\implies$ it can be covered with finitely many open sets of total length $< \varepsilon$

$$N(\varepsilon) \subseteq \bigcup_{i=1}^{m} U_i$$

let $I_i = \overline{U_i}$ (closure = adding end points)
wlog, $I_i$ do not overlap




The complement

$$K = [a,b] \setminus \bigcup_{i=1}^{m} U_i$$

is compact so it can be covered by finitely many disjoint closed intervals $J_i$ s.t.

$$\omega_f(J_j) < \varepsilon$$

Now the $I_i$'s and $J_j$'s give a dissection for $[a,b]$ s.t.

$$\begin{aligned}\sum_1^n \omega_f([x_{j-1}, x_j])(x_j - x_{j-1}) &= \sum_{i=1}^{m} \underbrace{\omega_f(I_i)}_{\leq 2K} l(I_i) + \sum_{j=1}^{k} \underbrace{\omega_f(J_j)}_{<\varepsilon} l(J_j) \\ &\leq 2K \sum_1^m l(I_i) + \varepsilon(b-a) \\ &\leq 2K\varepsilon + \varepsilon(b-a) \ \square\end{aligned}$$

(using $|f| \leq K$)

**Lemma 5.18.** $f$ is continuous at $x$ iff $\omega_f(x) = 0$

**Proof.** Exercise.