Analysis Summary

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Contents

1 Limits and Convergence

1.1 Review of Numbers and Sets

Notation. Write sequences as: a_n, $(a_n)_{n=1}^{\infty}$, $a_n \in \mathbb{R}$

Definition. We say that $a_n \to a$ as $n \to \infty$ if given $\varepsilon > 0$, $\exists N$ s.t. $|a_n - a| < \varepsilon$ for all $n \geq N$

Note. $N = N(\varepsilon)$

Note. Say monotone if stays increasing or stays decreasing

1.2 Fundamental Axiom of the real numbers

Axiom. If $a_n \in \mathbb{R}, \forall n \geq 1, A \in \mathbb{R}$ and $a_1 \leq a_2 \leq a_3 \leq \dots$ with $a_n \leq A$ for all n, there exists $a \in \mathbb{R}$ s.t. $a_n \to a$ as $n \to \infty$
i.e. an increasing sequence of real numbers bounded above converges.

Definition (supremum)**.** For$S \subseteq \mathbb{R}$, $S \neq \varnothing$, $\sup S = K$ if

(i) $x \leq K$, $\forall x \in S$

(ii) given $\varepsilon > 0, \exists x \in S$, s.t. $x > K - \varepsilon$

Note. Supremum is unique (see N&S notes), infinimum defined similarly.

Lemma 1.1.

(i) The limit is unique. That is, if $a_n \to a$, and $a_n \to b$, then $a = b$

(ii) If $a_n \to a$ as $n \to \infty$ and $n_1 < n_2 < n_3 < \dots$, then $a_{n_j} \to a$ as $j \to \infty$ (subsequences converge to the same limit)

(iii) If $a_n = C \ \forall n$, then $a_n \to C$ as $n \to \infty$

(iv) If $a_n \to a$ & $b_n \to b$, then

$$a_n + b_n \to a + b$$

(v) If $a_n \to a$ & $b_n \to b$, then

$$a_n b_n \to ab$$

(vi) If $a_n \to a$, $a_n \neq 0 \ \forall n \,\&\, a \neq 0$ then

$$\frac{1}{a_n} \to \frac{1}{a}$$

(vii) If $a_n \leq A \ \forall n$ and $a_n \to a$, then $a \leq A$

Proof.

(i) Suppose we have 2 limits, use triangle inequality on $|(a_n - a) - (a_n - b)|$

(ii) For given ε, we can find N and show $n_N \geq N$

(iii) $N = 1$ for any given ε works

(iv) Use triangle inequality

(v) $|a_n b_n - ab| \leq |a_n b_n - a_n b| + |a_n b - ab|$

(vi) Combine into same fraction then use N so $|a_n| > |l|/2$ by triangle inequality

(vii) If not then we can get within $|A - a|$, which is contradiction using triangle inequality

Lemma 1.2.

$$\frac{1}{n} \to 0 \text{ as } n \to \infty$$

Proof. $1/n$ is a decreasing sequence bounded below so by the fundamental Axiom it has limit a. Show $a = 0$ by using fact that $1/(2n)$ tends to same limit

Remark. The definition of limit of a sequence makes perfect sence for $a_n \in \mathbb{C}$

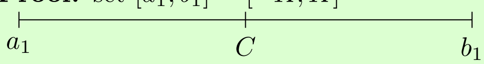
Definition. $a_n \to a$ if given $\varepsilon > 0$, $\exists N$ s.t. $\forall n \geq N$, $|a_n - a| < \varepsilon$.
First six parts of Lemma 1.1 are the same over $\mathbb{C}$.
The last one does not makes sense (over $\mathbb{C}$) since it uses the order of $\mathbb{R}$.

1.3 Bolzano-Weierstass Theorem

Theorem 1.3 (Bolzano-Weierstass)**.** If $x_n \in \mathbb{R}$ and there exists K s.t. $|x_n| \leq K \ \forall n$, then we can find $n_1 < n_2 < n_3 < \dots$ and $x \in \mathbb{R}$ s.t. $x_{n_j} \to x$ as $j \to \infty$
In other words: every bounded sequence has a convergent subsequence.

Remark. We say nothing about uniqueness of limit, $x_n = (-1)^n$, $x_{2n+1} \to -1$, $x_{2n} \to 1$

Proof. set $[a_1, b_1] = [-K, K]$



$C =$ mid point
We show either $[a_1, C]$ or $[c, b_1]$ contains infinitely many values in sequence. Define a_2, b_2 based on the case. Have

$$b_n - a_n = \frac{b_{n-1} - a_{n-1}}{2}$$

Then use algebra of limits to show $b = a$ and we choose our x_{n_j} from $[a_j, b_j]$

Proof (Faster)**.** Define x_m is a peak if $x_m \geq x_n$ for all $n \geq m$. If (x_n) has infinitely many peaks then we must have a decreasing sequence: the subsequence of peaks. If we only have finitely many peaks, we must have a final peak, say x_N, and so $\forall n_i \geq N+1$, we can find $n_{i+1} > n_i$ for which $x_{n_{i+1}} < x_{n_i}$ thus we have an increasing subsequence: $x_{n_1} = x_{N+1}$ and (x_{n_i}) defined inductively as shown.

1.4 Cauchy Sequences

Definition. $a_n \in \mathbb{R}$ is called a **Cauchy sequence** if given $\varepsilon > 0$, $\exists N > 0$ s.t. $|a_n - a_m| < \varepsilon \ \forall n, m \geq N$

Lemma 1.4. A convergent sequence is a Cauchy sequence.

Proof. Triangle inequality on $|(a_n - a) - (a_m - a)|$

Theorem 1.5. Every Cauchy sequence is convergent.

Proof. Show a_n bounded by taking $N = N(1)$ in Cauchy property, triangle inequality on $|a_m - a_N|$, then take $\max\{a_1, \dots, a_{N-1}, 1 + |a_N|\}$.
Then we have a limit, a, of a subsequence by Bolzano-Weierstass. We use the fact this subsequence convergent and the Cauchy property of a_n to get $|a_n - a| \leq |a_n - a_{n_j}| + |a_{n_j} - a| < 2$

Remark. Thus on $\mathbb{R}$ a sequence is convergent iff it is Cauchy

Note. This is a useful property since we do not need to know what the limit is.

1.5 Series

Definition. $a_n \in \mathbb{R}, \mathbb{C}$. We say that $\sum_{j=1}^{\infty} a_j$ **converges to** s if the sequence of partial sums

$$S_N = \sum_{j=1}^{N} a_j \to s$$

as $N \to \infty$
We write $\sum_{j=1}^{\infty} a_j = s$

If S_N does not converge, we say that $\sum_{j=1}^{\infty} a_j$ diverges.

Remark. Any problem on series can be turned into a problem on sequences just by considering the sequence of partial sums.

Lemma 1.6.

(i) If $\sum_{j=1}^{\infty} a_j$ & $\sum_{j=1}^{\infty} b_j$ converge, then so does $\sum_{j=1}^{\infty} (\lambda a_j + \mu b_j)$ where $\lambda, \mu \in \mathbb{C}$

(ii) Suppose $\exists N$ s.t. $a_j = b_j \, \forall j \geq N$, then either $\sum_{j=1}^{\infty} a_j$ & $\sum_{j=1}^{\infty} b_j$ both converge or both diverge (initial terms do not matter)

Proof.

(i)

$$S_N = \lambda \sum_{j=1}^{N} a_j + \mu \sum_{j=1}^{N} b_j = \lambda c_N + \mu d_N$$

$c_N \to c \& d_N \to d$ so by lemma 1.1 (version $\mathbb{C}$), $s_N \to \lambda c + \mu d$

(ii) $n \geq N$,

$$s_n = \sum_{1}^{n} a_j$$

$$d_n = \sum_{1}^{n} b_j$$

consider $s_n - d_n$ to show s_n converges iff d_n does

1.5.1 The Geometric Series

Claim. The geometric series converges iff $|x| < 1$

Proof. Evaluate partial sum to n explicitly for $x \neq 1$ and consider relevant cases

Note. Say $S_n \to \infty$ if given A, $\exists N$ s.t. $S_n > A$, $\forall n \geq N$
$S_n \to -\infty$, if given A, $\exists N$ s.t. $S_n < -A$ for all $n \geq N$
If S_n does not converge or tend to $\pm\infty$, we say that S_n oscillates.

Claim. $x^n \to 0$ if $|x| < 1$

Proof. Write $1/|x| = 1 + \delta$ and use binomial expansion to bound by $1/(1+\delta n)$

Lemma 1.7. If $\sum_{j=1}^{\infty} a_j$ converges, then:

$$\lim_{j\to\infty} a_j = 0$$

Proof.

$$a_n = S_n - S_{n-1}$$

algebra of limits on RHS shows $a_n \to 0$

Remark. The converse of 1.7 is false! Shown by example below:

Claim. $\sum_{1}^{\infty} \frac{1}{n}$ diverges (harmonic series)

Proof.

$$S_n = \sum_{1}^{\infty} \frac{1}{j}$$

$$S_{2n} = S_n + \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n} > S_n + \frac{1}{2}$$

Since $\frac{1}{n+k} \geq \frac{1}{2n}$ for $k = 1, 2, \ldots, n$
So if $S_n \to a$, then $S_{2n} \to a$ also and thus

$$a \geq a + \frac{1}{2} ※$$

1.5.2 Series of Positive/ Non-negative terms

Theorem 1.8 (The Comparison Test)**.** Suppose $0 \leq b_n \leq a_n \forall n$
Then if $\sum_{1}^{\infty} a_n$ converges, so does $\sum_{1}^{\infty} b_n$

Proof. Show partial sums bounded above by $\sum_{1}^{\infty} a_n$, increasing.

An example using this below:

Claim. $\sum_{1}^{n} \frac{1}{n^2}$ converges

Proof.

$$\frac{1}{n^2} < \frac{1}{n(n-1)} = \frac{1}{n-1} - \frac{1}{n} = a_n \text{ for } n \geq 2$$

By comparison, $\sum_{1}^{n} \frac{1}{n^2}$ converges

In fact, we get $\sum_{1}^{n} \frac{1}{n^2} \leq 1 + 1 = 2$

Theorem 1.9 (Root test/ Cauchy's test for convergence)**.** Assume $a_n \geq 0$ and $a_n^{1/n} \to a$ as $n \to \infty$. Then if $a < 1$, $\sum a_n$ converges; if $a > 1$, $\sum a_n$ diverges

Proof. If $a < 1$, choose $a < r < 1$.
Eventually have:

$$a_n^{1/n} < r \implies a_n < r^n$$

Show divergence in other case as terms do not tend to 0

Remark. Nothing can be said if $a = 1$.

Theorem 1.10 (Ratio test/ D'Alanbert's test)**.** Suppose $a_n > 0$ and $\frac{a_{n+1}}{a_n} \to l$
If $l < 1$, $\sum a_n$ converges.
If $l > 1$, $\sum a_n$ diverges

Proof. Suppose $l < 1$ and choose r with $l < r < 1$
Then $\exists N$ s.t. $\forall n \geq N$,

$$\frac{a_{n+1}}{a_n} < r$$

Therefore

$$a_n = \frac{a_n}{a_{n-1}} \frac{a_{n-1}}{a_{n-2}} \cdots \frac{a_{N+1}}{a_N} a_N < a_N r^{n-N}, \; n > N$$

$$\implies a_n < Kr^n$$

with K independent of n
Since $\sum r^n$ converges, so does $\sum a_n$
If $l > 1$, choose $1 < r < l$
Then $\frac{a_{n+1}}{a_n} > r \, \forall n \geq N$
And as before:

$$a_n = \frac{a_n}{a_{n-1}} \frac{a_{n-1}}{a_{n-2}} \cdots \frac{a_{N+1}}{a_N} a_N > a_N r^{n-N}, \; n > N$$

$$a_N r^{n-N} \to \infty \text{ as } n \to \infty$$

So $\sum a_n$ diverges. □

Remark. Nothing can be said if $a = 1$.

Remark. Can use root test when we have things like $\sum (f(n))^n$

Theorem 1.11 (Cauchy's Condensation Test)**.** Let a_n be a decreasing sequence of positive terms. Then $\sum_1^\infty a_n$ converges iff
$\sum_1^\infty 2^n a_{2^n}$ converges.

Proof. Just bound as one does in the $\sum 1/n$ case

1.5.3 Alternating Series

Theorem 1.12 (The alternating series test)**.** If a_n decreases and tends to zero as $n \to \infty$, then the series $\sum_1^\infty (-1)^{n+1} a_n$ converges

Proof. Show $S_{2n} \leq a_1$ and increasing by bracketing. $S_{2n+1} = S_{2n} + a_{n+1}$.
This implies convergence as we consider $\max\{N_1, N_2\}$ defined appropriately for odd and even n

1.5.4 Absolute Convergence

Definition. Take $a_n \in \mathbb{C}$. If $\sum_{n=1}^{\infty} |a_n|$ is convergent, then the series is **absolutely convergent**

Note. Since $|a_N| \geq 0$ we can use th previous tests to check absolute convergence; this is particularly useful for $a_n \in \mathbb{C}$.

Theorem 1.13. If Σa_n is absolutely convergent, then it is convergent.

Proof. Suppose first that $a_n \in \mathbb{R}$
Let $v_n = a_n$ when a_n positive, 0 otherwise. $w_n = -a_n$ when a_n negative, 0 otherwise. $|a_n| = v_n + w_n$. $\sum v_n$ converges by comparison. As does $\sum w_n$. $a_n = v_n - w_n$.
In $\mathbb{C}$ case, write $a_n = x_n + iy_n$ and use absolute convergence of x_n, y_n by comparison with $|a_n|$

Definition. If $\sum a_n$ converges but $\sum |a_n|$ does not, it is said sometimes that $\sum a_n$ is **conditionally convergent**.

Note. "conditional": because the sum to which the series converges is conditional on the order in which the elements of the sequence are taken.
If rearranged, the sum is altered.

Definition. Let σ be a bijection of the positive integers,

$$a'_n = a_{\sigma(n)}$$

is a **rearrangement**.

Theorem 1.14. If $\sum_1^{\infty} a_n$ is absolutely convergent, every series consisting of the same terms in any order (i.e. a rearrangement) has the same sum.

Proof. We do the proof first for $a_n \in \mathbb{R}$.
Consider partial sums of series and permutation. Can sum to 'far enough' in original series so as to include all terms in partial sum of permutation. Thus can bound above permutation with limit of original series. By symmetry, can show $\leq$ and $\geq$ so have equality.
If a_n has any sign, use v_n and w_n from Theorem 1.13.
We use above to show any rearrangement of the v_n has same sum, similarly for the w_n. Then $a_n = v_n - w_n$. Permuting the a_n is just applying same permutation to v_n and w_n, which in turn gives us same sum.
For the case $a_n \in \mathbb{C}$, write $a_n = x_n + iy_n$
Since $|x_n|, |y_n| \leq |a_n|$, we have $\sum x_n, \sum y_n$ are absolutely convergent.
We use previous case with $a'_n = x'_n + iy'_n$ to show sums equal $\square$

2 Continuity

Let $E \subseteq \mathbb{C}$ be non-empty. Let $f : E \to \mathbb{C}$ be any function. Have $a \in E$
(includes case in which f is real valued and E is a subset of $\mathbb{R}$)

Definition. f is **continuous at a**$\in E$ if for every sequence $z_n \in E$ with $z_n \to a$, we have $f(z_n) \to f(a)$
Equivalently below:

Definition. f is **continuous at a**$\in E$, if

$$\text{given } \varepsilon > 0,\ \exists \delta \text{ s.t. if } |z - a| < \delta,\ \text{then } |f(z) - f(a)| < \varepsilon$$

(ε-δ definition)

Claim. Two definitions equivalent

Proof. $2^{\text{nd}} \implies 1^{\text{st}}$:
Eventually, the z_n get within δ of a
$1^{\text{st}} \implies 2^{\text{nd}}$:
Suppose f not continuous at a according to 2^{nd} definition. Then we can generate, using $\delta = 1/n$, a sequence $z_n \to a$ but $|f(z_n) - f(a)| > \varepsilon$ for some $\varepsilon > 0$ so do not satisfy 1^{st} definition

Prop 2.1. $a \in E,\ g, f : E \to \mathbb{C}$ continuous at a.
Then so are the functions $f(z) + g(z),\ f(z)g(z)$ & $\lambda f(z)$ for any constant. In addition if $f(z) \neq 0\ \forall z \in E$, then $1/f$ is continuous at a

Proof. Using 1^{st} definition, this is obvious using the analagous results for sequences (Lemma 1.1) e.g.

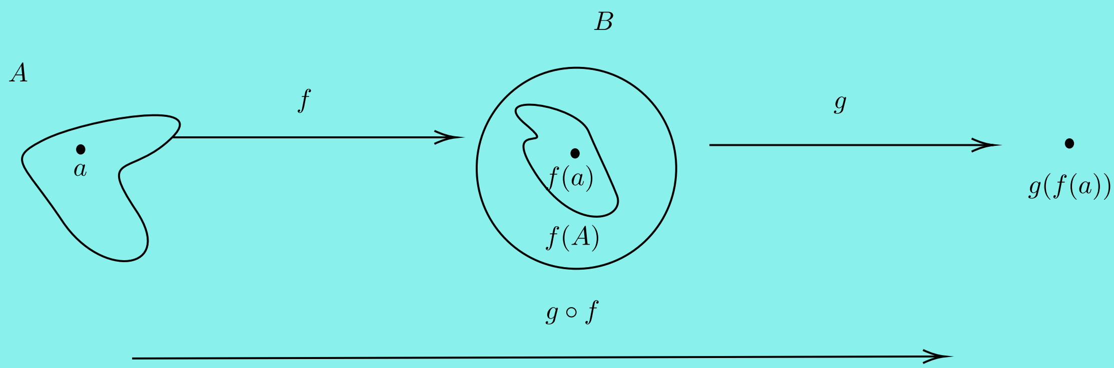
$$f(z_n) + g(z_n) \to f(a) + g(a) \text{ if } z_n \to a,\ f(z_n) \to f(a) \ \&\ g(z_n) \to g(a) \text{ etc. } \square$$

Example. The function $f(z) = z$ is continuous, so using the proposition we derive that every polynomial is continuous at every point in $\mathbb{C}$

Note. We say f is continuous on E if it is continuous at every $a \in E$.

Next we look at compositions

Theorem 2.2. Let $f : A \to \mathbb{C}$ and $g : B \to \mathbb{C}$ be two functions s.t. $f(A) \subseteq B$. Suppose f is continuous at $a \in A$ and g is continuous at $f(a)$. Then $g \circ f : A \to \mathbb{C}$ is continuous at a.



Proof. Take any sequence $z_n \to a$. By assummpion, $f(z_n) \to f(A)$. Set $w_n = f(z_n)$. then $w_n \in B$ and $w_n \to f(a)$; thus

$$g(w_n) \to g(f(a)) \square$$

2.1 Limit of a function

$$f : E \subseteq \mathbb{C} \to \mathbb{C}$$

We wish to define what is meany by

$$\lim_{z \to a} f(z)$$

even when a might not be in E e.g.

$$\text{limit as } z \to 0 \text{ of } \frac{\sin z}{z} \; E = \mathbb{C} \backslash \{0\} \; a = 0$$

Also if

$$E = \{0\} \cup [1, 2]$$

it does not make sense to speak about $z \in E, \; z \neq 0, z \to 0$

Definition. $E \subseteq \mathbb{C}, \; a \in \mathbb{C}$. We say that a is a limit point of E if for any $\delta > 0, \exists z \in E$ s.t.

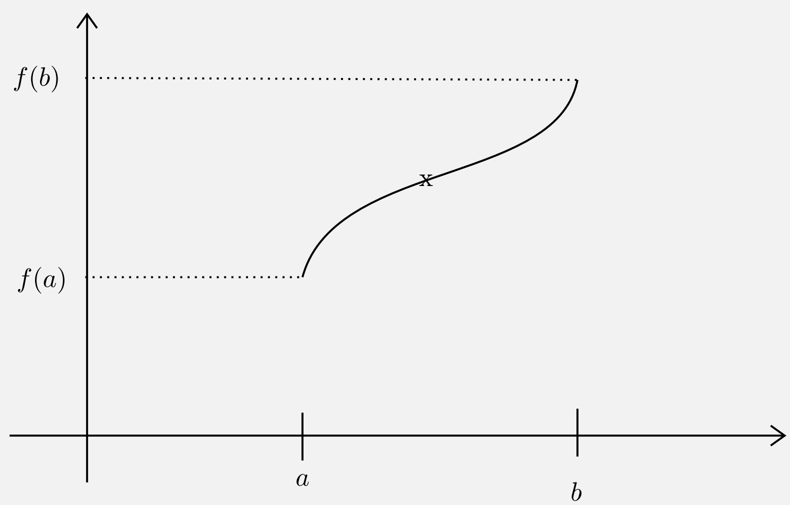
$$0 < |z - a| < \delta$$

Remark. a is a limit point iff $\exists$ a sequence $z_n \in E$ s.t. $z_n \to a$ and $z_n \neq a$ for all n. (can check equivalence)

Definition. $f : E \subseteq \mathbb{C} \to \mathbb{C}$, let $a \in \mathbb{C}$ be a limit point of E.
We say that
$$\lim_{z \to a} f(z) = l$$
If given $\varepsilon > 0$, $\exists \delta > 0$ s.t. whenever $0 < |z - a| < \delta$ and $z \in E$, then $|f(z) - l| < \varepsilon$
Equivalently: $f(z_n) \to l$ for every sequence $z_n \in E$, $z_n \neq a$ and $z_n \to a$
(proved exactly the same as previously with 2 definitions of continuity).

Remark. Straight from the definition, we have if $a \in E$ is a limit point, then
$$\lim_{z \to a} f(z) = f(a) \iff f \text{ is continuous at } a$$
If $a \in E$ is isolated (i.e. $a \in E$ and is not a limit point), continuity of f at a always holds.

2.2 The Intermediate Value Theorem



Theorem 2.3. $f : [a, b] \to \mathbb{R}$ continuous and $f(a) \neq f(b)$. Then f takes every value which lies between $f(a)$ and $f(b)$.

Proof. Without loss of generality, we may suppose $f(a) < f(b)$.
Take

$$f(a) < \eta < f(b)$$

Let

$$S = \{x \in [a, b] : f(x) < \eta\}$$

$a \in S$, so $S \neq \varnothing$. Clearly S is bounded above by b.
Then there is a supremum C where $C \leq b$. We construct a sequence $x_n = C - 1/n \to C$ using the fact it is a supremum.
Have $f(x_n) \to C$ and as $f(x_n) < \eta$, $f(C) \leq \eta$.
Also have $f(C + 1/n) > \eta$ as C supremum, and $C \neq b$ since $f(C) \leq \eta < f(b)$. Taking this limit to C shows $f(C) \geq \eta$
Hence, $f(C) = \eta$

Remark. The theorem is very useful for finding zeros of fixed points.

2.3 Bounds of a Continuous Function

Theorem 2.4. Let $f : [a, b] \to \mathbb{R}$ be continuous. Then there exists K s.t.

$$|f(x)| \leq K \; \forall x \in [a, b]$$

Proof. We argue by contradiction.
Suppose statement is false. Then given any integer $n \geq 1$, there exists $x_n \in [a, b]$ s.t. $|f(x_n)| > n$.
By Bolzano-Weierstass, x_n has a convergent subsequence $x_{n_j} \to x$.
Since $a \leq x_{n_j} \leq b$, we must have $x \in [a, b]$. By continuity of f,

$$f(x_{n_j}) \to f(x)$$

But

$$|f(x_{n_j})| > n_j \to \infty ※ \square$$

Theorem 2.5. $f : [a,b] \to \mathbb{R}$ continuous. Then $\exists x_1, x_2 \in [a,b]$ s.t.

$$f(x_1) \le f(x) \le f(x_2) \ \forall x \in [a,b]$$

"A continuous function on a closed, bounded interval is bounded and attains its bounds."

Proof (1^{st})**.** Let

$$A = \{f(x) : x \in [a,b]\} = f([a,b])$$

By Theorem 2.4, A is bounded. Since it is clearly non-empty, it has supremum, M.
By definition of supremum,

$$\text{given integer } n \ge 1,\ \exists x_n \in [a,b] \text{ s.t. } M - \frac{1}{n} < f(x_n) \le M \qquad (*)$$

By Bolzano-Weierstass,

$$\exists x_{n_j} \to x \in [a,b]$$

Since $f(x_{n_j}) \to M$ (because (*)) and f is continuous, we deduce that $f(x) = M$ so $x_2 = x$.
Reason similarly for the minimum □

Proof (2^{nd})**.**

$$A = f([a,b]),\ M = \sup A$$

as before. Suppose $\nexists x_2$ s.t. $f(x_2) = M$.
Let

$$g(x) = \frac{1}{M - f(x)},\ x \in [a,b]$$

is defined and continuous. By Theorem 2.4 applied to g,

$$\exists K > 0 \text{ s.t. } g(x) \le K \ \forall x \in [a,b]$$

This means that $f(x) \le M - \frac{1}{K}$ on $[a,b]$. This is absurd since it contradicts that M is the supremum □

Note. Theorems 2.4, 2.5 are false if the interval is not closed e.g.

$$x \in (0,1],\ f(x) = \frac{1}{x}$$

2.4 Inverse functions

Definition. f is **increasing** for $x \in [a,b]$ if $f(x_1) \le f(x_2)$ for all x_1, x_2 s.t. $a \le x_1 \le x_2 \le b$
If $f(x_1) < f(x_2)$ we say that f is **strictly increasing**.
Similarly for **decreasing** and **strictly decreasing**.

Theorem 2.6. $f : [a, b] \to \mathbb{R}$ continuous and strictly increasing for $x \in [a, b]$.
Let $c = f(a)$ and $d = f(b)$.
Then $f : [a, b] \to [c, d]$ is bijective and the inverse

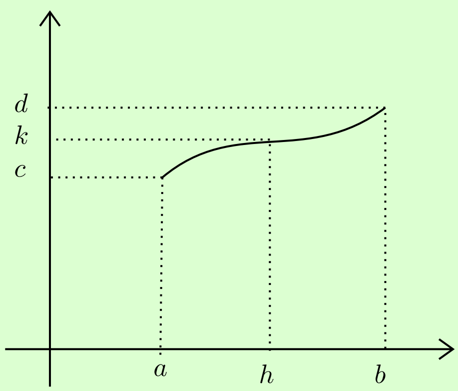
$$g = f^{-1} : [c, d] \to [a, b]$$

is continuous and strictly increasing

Remark. A similar theorem holds for strictly decreasing functions.

Proof. Take $c < k < d$.
From the intermediate value theorem

$$\exists h \text{ s.t. } f(h) = k$$



Since f is strictly increasing, h is unique.
Define $g(k) = h$ and this gives an inverse $g : [c, d] \to [a, b]$ for f. g is strictly increaseing: $y_1 < y_2$

$$y_1 = f(x_1),\ y_2 = f(x_2)$$

If $x_2 \leq x_1$, since f is increasing

$$\implies f(x_2) \leq f(x_1) \implies y_2 \leq y_1 ※$$

g is continuous:
Given $\varepsilon > 0$, let

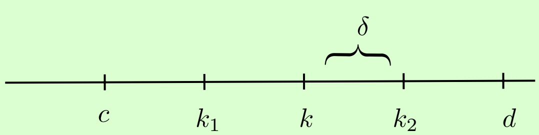
$$k_1 = f(h - \varepsilon),\ k_1 = f(h + \varepsilon)$$

f strictly increasing $\implies$

$$k_1 < k < k_2$$

If $k_1 < y < k_2$ then

$$h - \varepsilon < g(y) < h + \varepsilon$$



$$\delta = \min\{k_2 - k, k - k_1\}$$

(here $k \in (c, d)$ but a similar argument establishes continuity at the end points (can check))

3 Differentiability

Let $f : E \subseteq \mathbb{C} \to \mathbb{C}$. Most of the time, $E \subseteq \mathbb{R}$ is an interval.

Definition. Let $x \in E$ be a point s.t. $\exists x_n \in E$ with $x_n \neq x$ and $x_n \to x$ (i.e. a limit point) f is said to be **differentiable** at x with derviative $f'(x)$ if

$$\lim_{y \to x} \frac{f(y) - f(x)}{y - x} = f'(x)$$

If f is differentiable at each $x \in E$, we say f is differentiable on E

Note. Think of E as an interval or disc in the case of $\mathbb{C}$

Remark.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Definition (alternative)**.** f is **differentiable** at x if $\exists A$ and ε s.t.

$$f(x+h) = f(x) + hA + \varepsilon(h)$$

where

$$\lim_{h \to 0} \frac{\varepsilon(h)}{h} = 0$$

If such an A exists, then it is unique, since

$$A = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Remark.

(i) Differentiable $\implies$ continuous

(ii) Another alternative way of writing things:

$$f(x+h) = f(x) + hf'(x) + h\varepsilon_f(h)$$

with $\varepsilon_f(h) \to 0$ as $h \to 0$
or

$$f(x) = f(a) + (x-a)f'(a) + (x-a)\varepsilon_f(x)$$

with

$$\lim_{x \to a} \varepsilon_f(x) \to 0$$

3.1 Differentiation of Sums, Products, etc.

Prop 3.1.

(i) If $f(x) = C\ \forall x \in E$, then f is differentiable with $f'(x) = 0$

(ii) f, g differentiable at x, then so is $f + g$ and

$$(f+g)'(x) = f'(x) + g'(x)$$

(iii) f, g differentiable at x, then so is fg and

$$(fg)'(x) = f'(x)g(x) + f(g)g'(x)$$

(iv) If f is differentiable at x and $f(x) \neq 0\ \forall x \in E$, then $1/f$ is differentiable at x and

$$\left(\frac{1}{f}\right)'(x) = -\frac{f(x)}{[f(x)]^2}$$

Proof.

(i) trivial

(ii) linearity of lim

(iii)

$$\phi(x) = f(x)g(x)$$

$$\begin{aligned}\frac{\phi(x+h) - \phi(x)}{h} &= \frac{f(x+h)g(x+h) - f(x)g(x)}{h} \\ &= f(x+h)\left[\frac{g(x+h) - g(x)}{h}\right] + g(x)\left[\frac{f(x+h) - f(x)}{h}\right] \\ &= f'(x)g(x) + f(x)g'(x)\end{aligned}$$

using standard properties of limits and the fact that f is continuous at x

(iv)

$$\phi(x) = 1/f(x)$$

$$\begin{aligned}\frac{\phi(x+h) - \phi(x)}{h} &= \frac{1/f(x+h) - 1/f(x)}{h} \\ &= \frac{f(x) - f(x+h)}{hf(x)f(x+h)} \to -\frac{f'(x)}{[f(x)]^2}\end{aligned}$$ □

Remark. From (iii) and (iv) we immediately get

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

Theorem 3.2 (Chain rule)**.**

$$f : U \to \mathbb{C}$$

is s.t.

$$f(x) \in V \ \forall x \in U$$

If f is differentiable at $a \in U$ and $g : V \to \mathbb{C}$ is differentiable at $f(a)$, then $g \circ f$ is differentiable at a with

$$(g \circ f)'(a) = f'(a)g'(f(a))$$

Proof. We know:

$$f(x) = f(a) + (x - a)f'(a) + \varepsilon_f(x)(x - a)$$

where

$$\lim_{x \to a} \varepsilon_f(x) = 0$$

$$g(y) = g(b) + (y - b)g'(b) + \varepsilon_g(y)(y - b)$$

where

$$\lim_{y \to b} \varepsilon_g(y) = 0$$

$$b = f(a)$$

Set

$$\varepsilon_f(a) = 0 \ \& \ \varepsilon_g(b) = 0$$

to make them continuous at $x = a$ and $y = b$.
Set $y = f(x)$, and work through the algebra, using the fact $\varepsilon_f, \ \varepsilon_g$ continuous

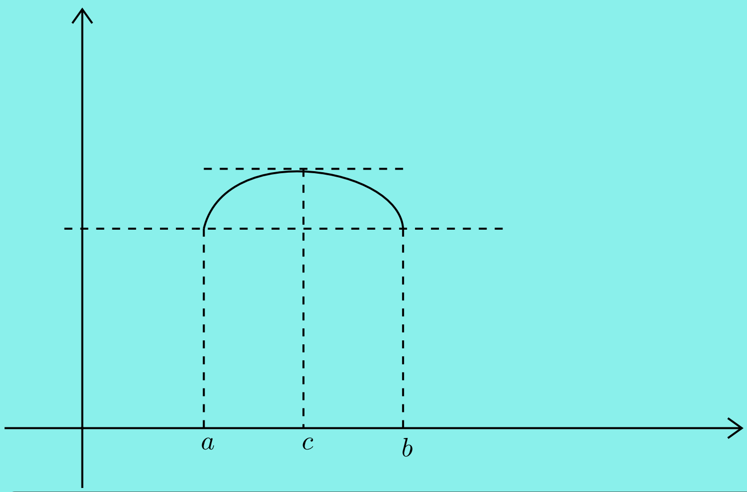
3.2 The Mean Value Theorem

Theorem 3.3 (Rolle's Theorem)**.**

$$f : [a, b] \to \mathbb{R}$$

continuous on $[a, b]$ and differentiable on (a, b). If $f(a) = f(b)$,

$$\exists c \in (a, b) \text{ s.t. } f'(c) = 0$$



Proof. Have function attains maximum, M, in interval. If $M = f(a)$ then f constant implying result. Otherwise, suppose $f'(M) \neq 0$ and get contradiction that M is not maximum

Theorem 3.4 (The Mean Value Theorem)**.** Let $f : [a, b] \to \mathbb{R}$ be a continuous function which is differentiable on (a, b). Then $\exists c \in (a, b)$ st.

$$f(b) - f(a) = f'(c)(b - a)$$

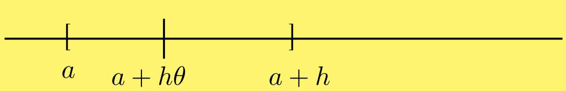
Proof. Apply Rolle's to:

$$\phi(x) = f(x) - kx$$

$$k = \frac{f(b) - f(a)}{b - a}$$

Remark. We will often write

$$f(a + h) = f(a) + hf'(a + \theta h)$$



$$\theta \in (0, 1)$$

$$(b = a + h)$$

Warning.

$$\theta = \theta(h)$$

Corollary 3.5. $f : [a, b] \to \mathbb{R}$ continuous and differentiable on (a, b). Then we have

(i) If $f'(x) > 0\ \forall x \in (a, b)$, then f is strictly increasing on $[a, b]$
(i.e. if $b \geq y > x \geq a$, then $f(y) > f(x)$)

(ii) If $f'(x) \geq 0\ \forall x \in (a, b)$, then f is increasing (i.e. if $b \geq y > x \geq a$, then $f(y) \geq f(x)$)

(iii) If $f'(x) = 0\ \forall x \in (a, b)$, then f is constant on $[a, b]$

Proof.

(i) Use MVT in $[x, y]$

(ii) Use MVT in $[x, y]$

(iii) Use MVT in $[a, x]$

3.3 Inverse Rule/ Inverse Function Theorem

Theorem 3.6. $f : [a, b] \to \mathbb{R}$ continuous and differentiable on (a, b) with

$$f'(x) > 0\ \forall x \in (a, b)$$

Let $f(a) = c$ and $f(b) = d$. Then the function $f : [a, b] \to [c, d]$ is bijective and f^{-1} is differentiable on (c, d) with

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

Proof. By corollary 3.5, f is strictly increasing on $[a, b]$. By Theorem 2.6

$$\exists g : [c, d] \to [a, b]$$

which is continuous, strictly increasing inverse of f.
Set $y + k = f(x + h)$ with $y = f(x)$. Use continuity of g and consider

$$\lim_{h \to 0} \frac{g(y + k) - g(y)}{k}$$

Theorem 3.7 (Cauchy's mean value theorem)**.** Let $f, g : [a, b] \to \mathbb{R}$ be continuous functions and differentiable on (a, b).
Then $\exists t \in (a, b)$ s.t.

$$(f(b) - f(a))g'(t) = f'(t)(g(b) - g(a))$$

Proof. Consider

$$\phi(x) = \begin{vmatrix} 1 & 1 & 1 \\ f(a) & f(x) & f(b) \\ g(a) & g(x) & g(b) \end{vmatrix}$$

ϕ is continuous on $[a, b]$ and differentiable on (a, b)
We can apply Rolle's to ϕ as $\phi(a) = \phi(b) = 0$

Theorem 3.8 (Taylor's theorem with Lagrange's remainder)**.** Suppose f and its derivatives up to order $n-1$ are continuous in $[a, a+h]$ and $f^{(n)}$ exist for $x \in (a, a+h)$. Then

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \cdots + \frac{h^{n-1}f^{(n-1)}(a)}{(n-1)!} + \frac{h^n}{n!}f^{(n)}(a+\theta h)$$

Where $\theta \in (0,1)$

Proof. Define for $0 \leq t \leq h$

$$\phi(t) = f(a+t) - f(a) - tf'(a) - \cdots - \frac{t^{n-1}}{(n-1)!}f^{(n-1)}(a) - \frac{t^n}{n!}\beta$$

where we choose β s.t. $\phi(h) = 0$
(recall in the proof of the MVT we used $f(x) - kx$ and we picked k s.t. we could use Rolle's theorem)
Use Rolle's Theorem on the first $n-1$ derivatives to get

$$0 < h_n < \cdots < h_1 < h$$

where h_i is a point where $\phi^{(i)}(h_i) = 0$

Theorem 3.9 (Taylor's theorem with Cauchy's form of remainder)**.** With the same hypothesis as in Theorem 3.8 and $a = 0$ (to simplify), we have

$$f(h) = f(0) + hf'(0) + \cdots + \frac{h^{n-1}}{(n-1)!}f^{(n-1)}(0) + R_n$$

where

$$R_n = \frac{h^n(1-\theta)^{n-1}f^{(n)}(\theta h)}{(n-1)!},\ \theta \in (0,1)$$

Proof. Define

$$F(t) = f(h) - f(t) - (h-t)f'(t) - \cdots - \frac{(h-t)^{n-1}f^{(n-1)}(t)}{(n-1)!}$$

with $t \in [0, h]$

$$F'(t) = -f'(t) + f'(t) - (h-t)f''(t) + (h-t)f''(t) - \frac{(h-t)^2}{2}f''(t) + \cdots - \frac{(h-t)^{n-1}}{(n-1)!}f^{(n)}(t)$$

$$\implies F'(t) = -\frac{(h-t)^{n-1}}{(n-1)!}f^{(n)}(t)$$

Set

$$\phi(t) = F(t) - \left[\frac{h-t}{h}\right]^p F(0)$$

where $p \in \mathbb{Z}, 1 \leq p \leq n$
Then $\phi(0) = \phi(h) = 0$ so by Rolle's theorem,

$$\exists \theta \in (0,1) \text{ s.t. } \phi'(\theta h) = 0$$

But

$$\phi'(\theta h) = F'(\theta h) + \frac{p(1-\theta)^{p-1}}{h}F(0) = 0$$

Subbing in for $F'(\theta h)$ and $F(0)$ followed by rearrangement gives result.
If $p = n$ we get Lagrange's remainder
If $p = 1$ we get Cauchy's remainder

Method. To get a Taylor Series for f, one needs to show that $R_n \to 0$ as $n \to \infty$. This requires "estimates" and "effort"

4 Power Series

Lemma 4.1. If $\sum_0^\infty a_n z_1^n$ converges and $|z| < |z_1|$, then $\sum_0^\infty a_n z^n$ converges absolutely

Proof. Since $\sum_0^\infty a_n z_1^n$ converges, $a_n z_1^n \to 0$. Thus $\exists K > 0$ s.t.

$$|a_n z_1^n| < K \ \forall n$$

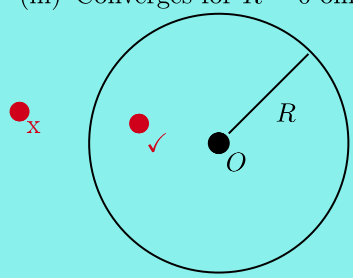
Use this to bound $|a_n z^n|$

Theorem 4.2. A power series either

(i) Converges absolutely for all z, or

(ii) Converges absolutely for all z inside a circle $|z| = R$ and diverges for all z outside it, or

(iii) Converges for $R = 0$ only



Proof. Let $S = \{x \in \mathbb{R},\ x \geq 0 \text{ and } \sum a_n x^n \text{ converges}\}$ Clearly $0 \in S$. By Lemma 4.1, if $x_1 \in S$, then $[0, x_1] \in S$.
Then consider cases $S = [0, \infty)$, $\sup S \neq 0$ and $\sup S = 0$.

Definition. The circle $|z| = R$ is called the **circle of convergence** and R is the **radius of convergence**.
In (i), we agree that $R = \infty$ and in (iii) $R = 0$

Lemma 4.3. If

$$\left|\frac{a_{n+1}}{a_n}\right| \to l$$

as $n \to \infty$, then $R = \frac{1}{l}$

Proof. Apply ratio test

Theorem 4.4. $f(z) = \sum_0^\infty a_n z^n$ has radius of convergence R. Then f is differentiable at all points with $|z| < R$ with

$$f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}$$

Proof. By Lemma 4.5, we may define

$$f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}, \ |z| < R$$

Show

$$\lim_{h \to 0} \left| \frac{f(z+h) - f(z) - hf'(z)}{h} \right| \to 0$$

using

$$|(z+h)^n - z^n - nhz^{n-1}| \leq n(n-1)(|z| + |h|)^{n-2}|h|^2$$

Lemma 4.5. If $\sum_0^\infty a_n z^n$ has radius of convergence R, then so do

$$\sum_1^\infty n a_n z^{n-1} \text{ and } \sum_2^\infty n(n-1) a_n z^{n-2}$$

Proof. Take z and R_0 s.t. $0 < |z| < R_0 < R$. Since $a_n R_0^n \to 0$,

$$\exists K \text{ s.t. } |a_n R_0^n| \leq K \ \forall n \geq 0$$

Then bound above with series of form $\sum K_1 n |z/R_0|^n$ and show convergence of this using ratio test
In case $|z| > R$, have $|n a_n z|^n \not\to 0$ Same proof applies to

$$\sum_2^\infty n(n-1) a_n z^{n-2} \ \square$$

Lemma 4.6.
(i)

$$\binom{n}{r} \leq n(n-1) \binom{n-2}{r-2}$$

for all $2 \leq r \leq n$
(ii)

$$|(z+h)^n - z^n - nhz^{n-1}| \leq n(n-1)(|z| + |h|)^{n-2}|h|^2 \ \forall z \in \mathbb{C}, \ h \in \mathbb{C}$$

Proof.
(i) trivial
(ii) trivially follows

4.1 The Standard Functions

Define $e : \mathbb{C} \to \mathbb{C}$

$$e(z) = \sum_0^\infty \frac{z^n}{n!}$$

Now let $a, b \in \mathbb{C}$ and consider

$$F(z) = e(a + b - z)e(z)$$

$$F'(z) = -e(a + b - z)e(z) + e(a + b - z)e(z) = 0$$

$$\implies F \text{is constant}$$

$$e(a + b - z)e(z) = F(0) = e(a + b)$$

Set $z = b$

$$e(a)e(b)e(a + b)$$

Now we restrict $e : \mathbb{R} \to \mathbb{R}$

Theorem 4.7.

(i) $e : \mathbb{R} \to \mathbb{R}$ is everywhere differentiable and $e'(x) = e(x)$
(ii) $e(x + y) = e(x)e(y)$
(iii) $e(x) > 0 \; \forall x \in \mathbb{R}$
(iv) e is strictly increasing
(v) $e(x) \to \infty$ as $x \to \infty$, and $e(x) \to 0$ as $x \to -\infty$
(vi) $e : \mathbb{R} \to (0, \infty)$ is a bijection

Proof.

(i) done ✓
(ii) done ✓
(iii) Clearly $e(x) > 0 \; \forall x \geq 0$ and $e(0) = 1$
Consider $e(x - x)$
(iv) $e' > 0$
(v) $e(x) > 1 + x$ and $e(-x) = 1/e(x)$
(vi) injectivity: follows right away from being strictly increasing
surjectivity: IVT

Since e is a bijection, consider the inverse function

$$l : (0, \infty) \to \mathbb{R}$$

Theorem 4.8.

(i)
$$l : (0, \infty) \to \mathbb{R}$$
is a bijection and
$$l(e(x)) = x \ \forall x \in \mathbb{R}$$
and
$$e(l(t)) = t \ \forall t \in (0, \infty)$$

(ii) l is differentiable and
$$l'(t) = \frac{1}{t}$$

(iii)
$$l(xy) = l(x) + l(y) \ \forall x, y \in (0, \infty)$$

Proof.

(i) obvious from the definition
(ii) Inverse rule
(iii) from IA Groups, since e is an isomorphism, so is its inverse □

Now define for $\alpha \in \mathbb{R}$ and $x > 0$,
$$r_\alpha(x) = e(\alpha l(x))$$

Theorem 4.9. Suppose $x, y > 0$ and $\alpha, \beta \in \mathbb{R}$. Then:

(i)
$$r_\alpha l(xy) = r_\alpha(x) r_\alpha(y)$$

(ii)
$$r_{\alpha+\beta}(x) = r_\alpha(x) r_\beta(x)$$

(iii)
$$r_\alpha(r_\beta(x)) = r_{\alpha\beta}(x)$$

(iv)
$$r_1(x) = x, \ r_0(x) = 1$$

Proof.

(i) trivial algebra
(ii) trivial algebra
(iii) trivial algebra
(iv) trivial

Now we do a "baptism ceremony"

$$\exp(x) = e(x) \ x \in \mathbb{R}$$

$$\log x = l(x) \ x \in (0, \infty)$$

$$x^\alpha = r_\alpha(x) \ \alpha \in \mathbb{R}, \ x \in (0, \infty)$$

$$e(x) = e(x \log e) = r_x(e) = e^x$$

where

$$e = \sum_0^\infty \frac{1}{n!} = e(1)$$

so $\exp(x)$ is also a power, which we may as well denote e^x
Finally, we compute $(x^\alpha)'$

$$(x^\alpha)' = \left(e^{\alpha \log x}\right)' = e^{\alpha \log x} \frac{\alpha}{x} = \alpha x^{\alpha - 1} \ \checkmark$$

4.2 Trigonometric Functions

Definition.

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots = \sum_0^\infty \frac{(-1)^k z^{2k}}{(2k)!}$$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots = \sum_0^\infty \frac{(-10)^k z^{2k+1}}{(2k+1)!}$$

Both power series have infinite radius of convergence and by theorem 4.4., they are differentiable and

$$(\sin z)' = \cos z$$

$$(\cos z)' = -\sin z$$

Method. Use

$$\cos z = \frac{1}{2}\left(e^{iz} + e^{-iz}\right)$$

$$\sin z = \frac{1}{2i}\left(e^{iz} - e^{-iz}\right)$$

to derive trig identities

4.2.1 Periodicity of the Trigonometric Functions

Prop 4.10. There is a smallest positive number ω (where $\sqrt{2} < \frac{\omega}{2} < \sqrt{3}$) s.t.

$$\cos\left(\frac{\omega}{2}\right) = 0$$

Proof. If $0 < x < 2$

$$\sin x = \left(x - \frac{x^3}{3!}\right) + \left(\frac{x^5}{5!} - \frac{x^7}{7!}\right) + \cdots > 0$$

(if $0 < x < 2$ then $\frac{x^{2n-1}}{(2n-1)!} > \frac{x^{2n+1}}{(2n+1)}$)
So for $0 < x < 2$,

$$(\cos x)' = -\sin x < 0$$

$\implies \cos x$ is strictly decreasing
Show $\cos\sqrt{2} > 0$ and $\cos\sqrt{3} < 0$ so we can use IVT. Show these by bracketing series cleverly

Corollary 4.11.

$$\sin\frac{\omega}{2} = 1$$

Proof.

$$\sin^2\frac{\omega}{2} + \cos\frac{\omega}{2} = 1$$

and

$$\sin\frac{\omega}{2} > 0 \ \square$$

Notation. Now define $\pi = \omega$

Theorem 4.12.

(i)

$$\sin\left(z + \frac{\pi}{2}\right) = \cos z,\ \cos\left(z + \frac{\pi}{2}\right) - \sin z$$

(ii)

$$\sin(z + \pi) = -\sin z,\ \cos(z + \pi) = -\cos z$$

(iii)

$$\sin(z + 2\pi) = \sin z,\ \cos(z + 2\pi)\cos z$$

Proof. immediate from addition formulas and

$$\cos\frac{\pi}{2},\ \sin\frac{\pi}{2} = 1 \ \square$$

Remark. Relate trig functions to geometry using dot product

4.3 Hyperbolic Functions

Definition.

$$\cosh z = \frac{1}{2}(e^z + e^{-z})$$

$$\sinh z = \frac{1}{2}(e^z - e^{-z})$$

$$\implies \cosh z = \cos(iz),\ \sinh = -i\sin(iz)$$

Claim.

$$(\cosh z)' = \sinh z$$

$$(\sinh z)' = \cosh z$$

$$\cosh^2 z - \sinh^2 z = 1,\ \text{etc.}$$

Proof. Trivial

Note. The rest of the trigonometric functions $(\tan, \cot, \sec, \text{cosec})$ are defined in the usual way

5 Integration

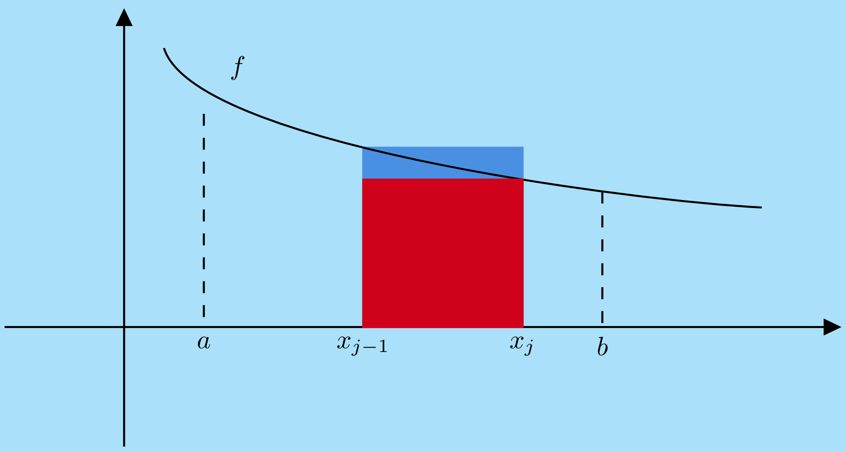
Note. $f : [a, b] \to \mathbb{R}$ bounded means:

$$\exists K \text{ s.t. } |f(x)| \leq K, \ \forall x \in [a, b]$$

Definition. A **dissection** (or partition) $\mathcal{D}$ of $[a, b]$ is a finite subset of $[a, b]$ containing the end points of a and b.
We write

$$\mathcal{D} = \{x_0, x_1, \dots, x_n\} \text{ with}$$

$$a = x_0 < x_1 < \dots < x_{n-1} < x_n = b$$



Definition. We define the **upper sum** and **lower sum** associated with $\mathcal{D}$ by

$$S(f, \mathcal{D}) = \sum_{j=1}^{n} (x_j - x_{j-1}) \sup_{x \in [x_{j-1}, x_j]} f(x)$$

is the **upper sum**.

$$s(f, \mathcal{D}) = \sum_{j=1}^{n} (x_j - x_{j-1}) \inf_{x \in [x_{j-1}, x_j]} f(x)$$

is the **lower sum**. Clearly

$$s(d, \mathcal{D}) \leq S(d, \mathcal{D}) \ \forall \mathcal{D}$$

Lemma 5.1. If $\mathcal{D}$ and $\mathcal{D}'$ are dissections with $\mathcal{D} \subseteq \mathcal{D}'$, then

$$S(d, \mathcal{D}) \geq S(d, \mathcal{D}') \geq s(f, \mathcal{D}') \geq s(f, \mathcal{D})$$

Proof.

$$S(d, \mathcal{D}') \geq s(f, \mathcal{D}')$$

is obvious.
Suppose $\mathcal{D}'$ contains an extra point than $\mathcal{D}$, let's say $y \in (x_{r-1}, x_r)$
Consider sups cleverly

Lemma 5.2. $\mathcal{D}_1, \mathcal{D}_2$ two arbitrary dissections. Then

$$S(f, \mathcal{D}_1) \geq S(f, \mathcal{D}_1 \cup D_2) \geq s(f, \mathcal{D}_1 \cup \mathcal{D}_2) \geq s(f, \mathcal{D}_2)$$

So

$$S(f, \mathcal{D}_1) \geq s(f, \mathcal{D}_2)$$

Proof. Take

$$\mathcal{D}' = \mathcal{D}_1 \cup \mathcal{D}_2 \supseteq \mathcal{D}_1, \mathcal{D}_2$$

ad apply the previous lemma. □

Definition. The **upper integral** of f is

$$I^*(f) = \inf_{\mathcal{D}} S(f, \mathcal{D})$$

(this always exists)
The **lower integral** of f is

$$I_*(f) = \sup_{\mathcal{D}} s(f, \mathcal{D})$$

(this always exists)

Claim. By lemma 5.2,

$$I^*(f) \geq I_*(f)$$

Proof.

$$S(f, \mathcal{D}_1) \geq s(f, \mathcal{D}_2)$$

$$I^*(f) = \inf_{\mathcal{D}_1} S(f, \mathcal{D}_1) \geq s(f, \mathcal{D}_2)$$

$$I^*(f) \geq \sup_{\mathcal{D}_2} s(f, \mathcal{D}_2) = I_*(f)$$

Definition. A bounded function $f : [a, b] \to \mathbb{R}$ is said to be **Reimann integrable** (or first integrable) if

$$I^*(f) = I_*(f)$$

and we set

$$\int_a^b f(x)\,\mathrm{d}x = I^*(f) = I_*(f) = \int_a^b f$$

A useful criterion for integrability:

Theorem 5.3. A bounded function

$$f : [a, b] \to \mathbb{R}$$

is Riemann integrable iff given $\varepsilon > 0, \exists \mathcal{D}$ s.t.

$$S(f, \mathcal{D}) - s(f, \mathcal{D}) < \varepsilon$$

Proof.

$$0 \leq I^*(f) - I_*(f) \leq S(f, \mathcal{D}) - s(f, \mathcal{D})$$

Gives one way

$$\int_a^b f \, \mathrm{d}x - \frac{\varepsilon}{2} = I_*(f) - \frac{\varepsilon}{2} < s(f, \mathcal{D}_1)$$

$$S(f, \mathcal{D}_2) < I^*(f) + \frac{\varepsilon}{2} = \int_a^b f \, \mathrm{d}x + \frac{\varepsilon}{2}$$

Gives reverse

Theorem 5.4. $f : [a, b] \to \mathbb{R}$ monotonic. Then f is integrable

Proof. Set

$$\mathcal{D} = \{a, a + \frac{b-a}{n}, a + \frac{2(b-a)}{n}, \ldots, b\}$$

$$S(f, \mathcal{D}) - s(f, \mathcal{D}) = \frac{(b-a)}{n}(f(b) - f(a))$$

$$\frac{b-a}{n}(f(b) - f(a)) < \varepsilon$$

Lemma 5.5. $f : [a, b] \to \mathbb{R}$ continuous. Then

$$\text{given } \varepsilon > 0, \exists \delta > 0 \text{ s.t } |x - y| < \delta \implies |f(x) - f(y)| < \varepsilon$$

(uniform continuity)

Note. The point is δ works $\forall x, y$ as long as $|x - y| < \delta$
(in the definition of continuity of f at x, $\delta = \delta(x)$)

Proof. Suppose the claim is false. Then $\exists \varepsilon > 0$ s.t. $\forall \delta > 0$, we can find $x, y \in [a, b]$ s.t. $|x - y| < \delta$ but $|f(x) - f(y)| \geq \varepsilon$
Take $\delta = \frac{1}{n}$, to get x_n, y_n with

$$|x_n - y_n| < \frac{1}{n}, \text{ but } |f(x_n) - f(y_n)| \geq \varepsilon$$

Apply Bolzano-Weierstass to get $x_{n_k} \to C$ and show y_{n_k} tend to same limit, use continuity to show $|f(C) - f(C)| \geq \varepsilon$, which is a contradiction

Theorem 5.6. Let $f : [a, b] \to \mathbb{R}$ be continuous. Then f is Riemann integrable.

Proof. given $\varepsilon > 0$, $\exists \delta > 0$ s.t. $|x - y| < \delta$

$$\implies |f(x) - f(y)| < \varepsilon$$

Let $\mathcal{D} = \{a + \frac{(b-a)j}{n} : j = 0, 1, \ldots, n\}$
Choose n large enough s.t.

$$\frac{b-a}{n} < \delta$$

Then for $x, y \in [x_{j-1}, x_j]$

$$|f(x) - f(y)| < \varepsilon \tag{*}$$

since

$$|x - y| \leq |x_j - x_{j-1}| = \frac{b-a}{n} < \delta$$

Use this to get

$$S(f, \mathcal{D}) - s(f, \mathcal{D}) < \varepsilon(b - a)$$

5.1 Elementary Properties of the Integral

Claim. For f, g bounded and integrable on $[a,b]$:

(i) If $f \leq g$ on $[a,b]$, then

$$\int_a^b f \leq \int_a^b g$$

(ii) $f+g$ is integrable on $[a,b]$ and

$$\int_a^b f+g = \int_a^b f + \int_a^b g$$

(iii) For any constant k, kf is integrable and

$$\int_a^b kf = k\int_a^b f$$

(iv) $|f|$ is integrable and

$$\left|\int_a^b f\right| \leq \int_a^b |f|$$

(v) The product fg is integrable

Proof.

(i)

$$\int_a^b f = I^*(f) \leq S(f,\mathcal{D}) \leq S(g,\mathcal{D})$$

$$\implies \int_a^b f = I^*(f) \leq I^*(g) = \int_a^b g$$

(ii)

$$\sup_{[x_{j-1},x_j]}(f+g) \leq \sup_{[x_{j-1},x_j]} f + \sup_{[x_{j-1},x_j]} g$$

$$\implies S(f+g,\mathcal{D}) \leq S(f,\mathcal{D}) + S(g,\mathcal{D})$$

so

$$I^*(f+g) \leq S(f,\mathcal{D}_1) + S(g,\mathcal{D}_2)$$

which then leads us to

$$I^*(f+g) \leq I^*(f) + I^*(g) = \int_a^b f + \int_a^b g$$

Similarly

$$\int_a^b f + \int_a^b g \leq I_*(f+g)$$

(iii) Bound I^*, I_* in natural way

Claim (cont.).

Proof (cont.).

(iv) Consider

$$f_+(x) = \max(f(x), 0)$$

$$\sup_{[x_{j-1},x_j]} f_+ - \inf_{[x_{j-1},x_j]} f_+ \leq \sup_{[x_{j-1},x_j]} f - \inf_{[x_{j-1},x_j]} f$$

(can check)
use this to show f_+ is integrable
But $|f| = 2f_+ - f$ By (ii) and (iii), $|f|$ is integrable.
Since $-|f| \leq f \leq |f|$, we use property (i) to see

$$\left| \int_a^b f \right| \leq \int_a^b |f|$$

(v) Take f integrable and ≥ 0
Then

$$\sup_{[x_{j-1},x_j]} f^2 = \Bigg(\overbrace{\sup_{[x_{j-1},x_j]} f}^{M_j} \Bigg)^2$$

$$\inf_{[x_{j-1},x_j]} f^2 = \Bigg(\underbrace{\inf_{[x_{j-1},x_j]} f}_{m_j} \Bigg)^2$$

Thus

$$\begin{aligned} S(f^2, \mathcal{D}) - s(f^2, \mathcal{D}) &= \sum_{j=1}^n (x_j - x_{j-1})(M_j^2 - m_j^2) \\ &= \sum_{j=1}^n (x_j - x_{j-1})(M_j + m_j)(M_j - m_j) \\ &\leq 2K(S(f, \mathcal{D}) - s(f, \mathcal{D})) \end{aligned}$$

using $|f(x)| \leq K \ \forall x \in [a, b]$
Using the criterion in Theorem 5.3, we deduce that f^2 is integrable.
Now take any f, then $|f| \geq 0$ and is integrable. Since $f^2 = |f|^2$.
We deduce that f^2 is integrable for any f
Finally for fg, note:

$$4fg = (f+g)^2 - (f-g)^2$$

$$\implies fg \text{ is integrable given what we proved}$$

Note. Method was: show f^2 integrable for any f given f is bounded & integrable by showing difference between upper and lower estimates $< \varepsilon$. Then use the fact that $4fg = (f+g)^2 - (f-g)^2$ to get RHS integrable using previous results, implying LHS integrable, implying fg integrable.

Claim (6)**.** f is integrable on $[a,b]$. If $a<c<b$, then f is integrable over $[a,c]$ and $[c,b]$ and

$$\int_a^b f = \int_a^c f + \int_c^b f$$

Conversely if f is integrable over $[a,c]$ and $[c,b]$, then f is integrable over $[a,b]$ and

$$\int_a^b f = \int_a^c f + \int_c^b f$$

Proof. We first make two observations:
if $\mathcal{D}_1$ is a dissection of $[a,c]$ and $\mathcal{D}_2$ is a dissection of $[b,c]$, then

$$\mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2$$

is a dissection of $[a,b]$ and

$$S(f, \mathcal{D}_1 \cup \mathcal{D}_2) = S(f|_{[a,c]}, \mathcal{D}_1) + S(f|_{[c,b]}, \mathcal{D}_2) \qquad (*_1)$$

Also if $\mathcal{D}$ is a dissection of $[a,b]$, then

$$\begin{aligned} S(f,\mathcal{D}) &\geq S(f, \mathcal{D} \cup \{c\}) \\ &= S(f|_{[a,c]}, \mathcal{D}_1) + S(f|_{[c,b]}, \mathcal{D}_2) \end{aligned} \qquad (*_2)$$

where $\mathcal{D}_1$ dissects $[a,c]$ and $\mathcal{D}_2$ dissects $[a,b]$

$$(*_1) \implies I^*(f) \leq I^*(f|_{[a,c]}) + I^*(f|_{[c,b]})$$

$$(*_2) \implies I^*(f) \geq I^*(f|_{[a,c]}) + I^*(f|_{[c,b]})$$

Similarly

$$I_*(f) = I_*(f|_{[a,c]}) + I_*(f|_{[c,b]})$$

Thus

$$0 \leq I^*(f) - I_*(f) = \underbrace{I^*(f|_{[a,c]}) - I_*(f|_{[a,c]})}_{\geq 0} + \underbrace{I^*(f|_{[c,b]}) - I_*(f|_{[c,b]})}_{\geq 0}$$

From this, claim follows right away. ☐

5.2 The Fundamental Theorem of Calculus (FTC)

$f : [a,b] \to \mathbb{R}$ bounded and integrable. Write

$$F(x) = \int_a^x f(t)\,\mathrm{d}t, \;\; x \in [a,b]$$

Theorem 5.7. F is continuous

Proof. Show $|F(x+h) - F(x)| \leq K|h|$ where K is upper bound of $|f(t)|$

Theorem 5.8 (FTC)**.** If in addition f is continuous at x, then F is differentiable at x and

$$F'(x) = f(x)$$

Proof. Show for $x + h \in [a, b],\ h \neq 0$:

$$\left|\frac{F(x+h) - F(x)}{h} - f(x)\right| = \frac{1}{|h|}\left|\int_x^{x+h} [f(t) - f(x)]\,\mathrm{d}t\right|$$

Use continuity at x to bound integral by ε so eventutally get:

$$\lim_{h \to 0} \frac{F(x+h) - F(x)}{h} = f(x) \ \square$$

Corollary 5.9 (integration is the inverse of differentiation)**.** If $f = g'$ is continuous on $[a, b]$, then

$$\int_a^x f(t)\,\mathrm{d}t = g(x) - g(a)\ \forall x \in [a, b]$$

Proof. From Theorem 5.8, $F - g$ has zero derivative in $[a, b] \implies F - g$ is constant and since $F(a) = 0$,

$$F(x) = g(x) - g(a) \ \square$$

Corollary 5.10 (integration by parts)**.** Suppose f' and g' exist and are continuous on $[a, b]$. Then

$$\int_a^b f'g = f(b)g(b) - f(a)g(a) - \int_a^b fg'$$

Proof. By the product rule,

$$(fg)' = f'g + fg'$$

By 5.9,

$$f(b)g(b) - f(a)g(a) = \int_a^b f'g + \int_a^b fg' \ \square$$

Corollary 5.11 (integration by substitution)**.** Let $g : [\alpha, \beta] \to [a, b]$ with $g(\alpha) = a$ and $g(\beta) = b$, g' exists and is continuous on $[\alpha, \beta]$. Let $f : [a, b] \to \mathbb{R}$ be continuous. Then

$$\int_a^b f(x)\,\mathrm{d}x = \int_\alpha^\beta f(g(t))g'(t)\,\mathrm{d}t$$

Proof. Set

$$F(x) = \int_a^x f(t)\,\mathrm{d}t$$

as before. Let $h(t) = F(g(t))$ defined since g takes values in $[a, b]$). Then

$$\begin{aligned}\int_\alpha^\beta f(g(t))g'(t)\,\mathrm{d}t &\underset{\text{FTC}}{=} \int_\alpha^\beta F'(g(t))g'(t)\,\mathrm{d}t \\ &= \int_a^b f(x)\,\mathrm{d}x \;\square\end{aligned}$$

By recognising chain rule. (Let $h(t) = F(g(t))$)

Theorem 5.12 (Taylor's theorem with remainder an integral)**.** Let $f^{(n)}(x)$ be continuous for $x \in [0, h]$. Then

$$f(h) = f(0) + \cdots + \frac{h^{n-1}f^{(n-1)}(0)}{(n-1)!} + R_n$$

where

$$R_n = \frac{h^n}{(n-1)!}\int_0^1 (1-t)^{n-1} f^{(n)}(th)\,\mathrm{d}t$$

Proof. By substituting $u = th$

$$R_n = \frac{1}{(n-1)!}\int_0^h (h-u)^{n-1} f^{(n)}(u)\,\mathrm{d}u$$

Integrating by parts $n - 1$ times

Theorem 5.13. $f, g : [a, b] \to \mathbb{R}$ continuous with $g(x) \neq 0$ $\forall x \in (a, b)$. Then $\exists c \in (a, b)$ s.t.

$$\int_a^b f(x)g(x)\,\mathrm{d}x = f(c)\int_a^b g(x)\,\mathrm{d}x$$

Proof. Use Cauchy's MVT (Theorem 3.7) on:

$$F(x) = \int_a^x fg$$

$$G(x) = \int_a^x g$$

Claim. We can get the Cauchy & Lagrange form of the remainder from Taylor's theorem with remainder (given continuity of $f^{(n)}$)

Proof. Now we want to apply this to

$$R_n = \frac{h^n}{(n-1)!}\int_0^1 (1-t)^{n-1} f^{(n)}(th)\,\mathrm{d}t$$

First we use Theorem 5.13 with $g \equiv 1$, to get

$$R_n = \frac{h^n}{(n-1)!}(1-\theta)^{n-1} f^{(n)}(\theta h),\ \theta \in (0,1)$$

Which is Cauchy's form of the remainder!
To get Lagrange, we use Theorem 5.13 with $g(t) = (1-t)^{n-1}$ which is > 0 for $t \in (0,1)$

$$\implies \exists \theta \in (0,1) \text{ s.t. } R_n = \frac{h^n}{(n-1)!} f^{(n)}(\theta h) \underbrace{\left[\int_0^1 (1-t)^{n-1}\,\mathrm{d}t\right]}_{=1/n}$$

$$\int_0^1 (1-t)^{n-1}\,\mathrm{d}t = -\frac{(1-t)^n}{n}\Bigg]_0^1 = \frac{1}{n}$$

$$\implies R_n = \frac{h^n}{n!} f^{(n)}(\theta h),\ \theta \in (0,1)$$

which is Lagrange's form of the remainder!

5.3 Improper Integrals

Definition. Suppose $f : [a,\infty] \to \mathbb{R}$ integrable (and bounded) on every interval $[a,R]$ and that as $R \to \infty$

$$\int_a^R f(x)\,\mathrm{d}x \to l$$

Then we say that $\int_a^\infty f(x)\,\mathrm{d}x$ exists or converges and that its value is l. If $\int_a^R f(x)\,\mathrm{d}x$ does not tend to a limit, we say that $\int_a^\infty f(x)\,\mathrm{d}x$ diverges.
A similar definition applies to $\int_{-\infty}^a f(x)\,\mathrm{d}x$. If

$$\int_a^\infty f(x)\,\mathrm{d}x = l_1$$

and

$$\int_{-\infty}^a f(x)\,\mathrm{d}x = l_2$$

we write

$$\int_{-\infty}^\infty (x)\,\mathrm{d}x = l_1 + l_2$$

(independent of the particular value of a)

5.4 The Integral Test

Theorem 5.14 (integral test)**.** Let $f(x)$ be a positive decreasing function for $x \geq 1$. Then

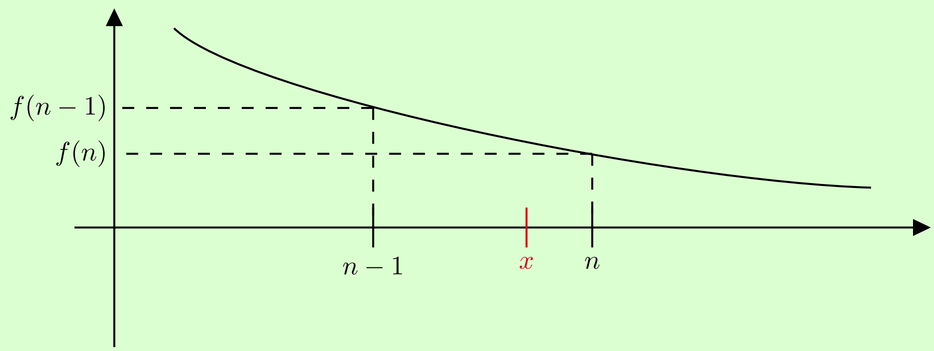
(i) The integral $\int_1^\infty f(x)\,\mathrm{d}x$ and the series $\sum_1^\infty f(n)$ both converge or both diverge.

(ii) As $n \to \infty$,

$$\sum_{r=1}^{n} f(r) - \int_1^n f(x)\,\mathrm{d}x \to l$$

where $0 \leq l \leq f(1)$

Proof.



Use diagram to get to

$$\sum_{r=1}^{n-1} f(r) \geq \int_1^n f(x)\,\mathrm{d}x \geq \sum_{2}^{n} f(r) \qquad (**)$$

Set

$$\phi(n) = \sum_{1}^{n} f(r) - \int_1^n f(x)\,\mathrm{d}x$$

Then

$$\phi(n) - \phi(n-1) = f(n) - \int_{n-1}^{n} f(x)\,\mathrm{d}x \leq 0$$

Right inequality of (**) gives:

$$0 \leq \phi(n) \leq g(1)$$

thus $\phi(n)$ is decreasing and tends ot a limit l s.t.

$$0 \leq l \leq f(1) \ \square$$

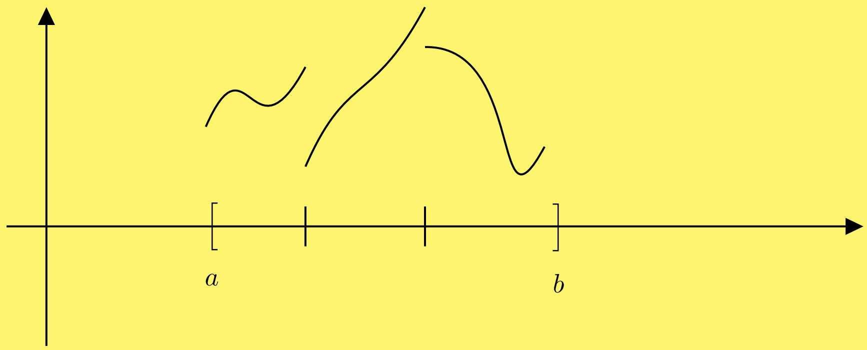
Corollary 5.15 (Euler's constant)**.** As $n \to \infty$,

$$1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log n \to \gamma$$

with $0 \leq \gamma \leq 1$

Proof. Set $f(x) = 1/x$ and use Theorem 5.14 $\square$

Remark. We have seen: monotone functions and continuous functions are integrable
We can generalise this a bit and say that piece-wise continuous functions are integrable



Definition. A function $f : [a, b] \to \mathbb{R}$ is said to be **piece-wise continuous** if there is a dissection $\mathcal{D} = \{x_0 = a, x_1, \ldots, x_n = b\}$ s.t.

(i) f is continuous on (x_{j-1}, x_j) $\forall j$

(ii) the one-sided limits

$$\lim_{x \to x_{j-1}^+} f(x), \quad \lim_{x \to x_{j-1}^-} f(x) \text{ exist}$$