Complex Analysis

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Contents

0 Overview

Recommended books

- "Complex Analysis" by Ahlfors
- "Real and Complex Analysis" by Rudin

1 Basic Notions

Notation. $\mathbb{C}$: complex plane.
$\overline{z}$: complex conjugate of $z \in \mathbb{C}$.
$|z|$: modulus of $z \in \mathbb{C}$.
$d(z,w) = |z-w|$ defines a metric on $\mathbb{C}$ (usual or standard metric).
For $a \in \mathbb{C}$ and $r > 0$:

$$D(a,r) = \{z \in \mathbb{C} : |z-a| < r\}$$

is the open disk (or open ball) with centre a and radius r.

Definition. A subset $U \subset \mathbb{C}$ is **open** if it is open w.r.t. the above metric, i.e. if for every $a \in U$, there exists $r > 0$ such that $D(a,r) \subset U$

- It is easy to check that if we identify $\mathbb{C}$ with $\mathbb{R}^w$ via $x + iy \sim (x,y)$, then $U \subset \mathbb{C}$ is open $\iff$ $U \subset \mathbb{R}^2$ is open w.r.t. the Euclidean metric on $\mathbb{R}^2$
- This course ia about complex valued functions of a (single) complex variable. I.e. functions $f : A \to \mathbb{C}$ where $A \subset \mathbb{C}$

Note. Identifying $\mathbb{C}$ with $\mathbb{R}^2$ in the usual way, we can write $f(z) = u(x,y) + iv(x,y)$ for $z = x+iy$ and a pair of real functions $u, v : A \to \mathbb{R}$. We write $u = \text{Re}(f)$, the real part of f, and $v = \text{Im}(f)$, the imaginary part of f.
Almost exclusively, we will focus on differentiable functions f

Definition. The function f (as above) is **continuous at a point** $w \in A$ if $\forall \varepsilon > 0 \exists \delta > 0$ such that

$$z \in A, |z-w| < \delta \implies |f(z) - f(w)| < \varepsilon$$

This is the same as saying that $\lim_{z \to w} f(z) = f(w)$

- It is easy to check that if we identify $\mathbb{C}$ with $\mathbb{R}^w$, if we write $f(z) = u(x,y) + iv(x,y)$ as above then f is continuous at $w = c + id \in A$ $\iff$ u, v are continuous at $(c,d) \in A$ w.r.t. the (induced) Euclidean metric on $A \subset \mathbb{R}^2$ and the Euclidean metric on the target $\mathbb{R}$

1.1 Complex Differentation

Notation. Let $f : U \to \mathbb{C}$, where $U \subset \mathbb{C}$ is open

Definition. (i) f is **differentiable** at $w \in U$ if the limit

$$f'(w) = \lim_{z \to w} \frac{f(z) - f(w)}{z - w}$$

exists as a complex number. $f'(w)$ is called the derivative of f at w

(ii) f is **holomorphic** at $w \in U$ if there is $\varepsilon > 0$ such that $D(w, \varepsilon) \subset U$ and f is differentiable at every point in $D(w, \varepsilon)$

(iii) f is **holomorphic** in U if f is holomorphic at every point in U, or equivalently, f is differentiable at every point in U

Remark. Sometimes, we use "analytic" to mean holomorphic

Usual rules of differentiation of real functions of a real variable hold for complex functions. Derivatives of sums, products, quotients of functions are obtained in the same way (as can easily be checked).

Equation. The chain rule for homposite functions also holds:

$$f : U \to \mathbb{C}, g : V \to \mathbb{C}, f(U) \subset V, h = g \circ f : U \to \mathbb{C}$$

If f is differentiable at $w \in U, g$ is differentiable at $f(w)$, then h is differentiable at w and

$$h'(w) = g'(f(w))f'(w)$$

(complex multiplication on the RHS)

Examples. (i) Polynomials $p(z) = \sum_{j=0}^{n}$, $a_1, \ldots, a_n \in \mathbb{C}$ are holomorphic on all of $\mathbb{C}$

(ii) If p, q are polynomials, then p/q is holomorphic on $\mathbb{C} \backslash \{z : q(z) = 0\}$

Write $f(z) = u + iv$. Is differentiability of f at a point $w = c + id \in U$ the same as differentiability of u and v at (c, d)?
Recall that $u : U \to \mathbb{R}$ is differentiable at $(c, d) \in U$ if there is a linear transformation $L : \mathbb{R}^2 \to \mathbb{R}$ such that

$$\lim_{(x,y) \to (c,d)} \frac{u(x, y) - (u(c, d) + L(x - c, y - d))}{\sqrt{(x - c)^2 + (y - d)^2}} = 0$$

If u is differentiable at (c, d), then L is uniquely defined and we write $L = Du(c, d)$; moreover, L is given by the partial derivatives of u, i.e.

$$L(x, y) = \left(\frac{\partial u}{\partial x}(c, d)\right) + \left(\frac{\partial u}{\partial y}(c, d)\right) y$$

The answer to the above question is NO! The theorem below characterises differentiability of f in terms of u and v

Theorem 1.1 (Cauchy-Riemann equations)**.** $f = u + iv : U \to \mathbb{C}$ is differentiable at $w = c + id \in U \iff u, v : U \to \mathbb{R}$ are differentiable at $(c, d) \in U$ and u, v satisfy the Cauchy-Riemann equations at (c, d), i.e.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$
$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \text{ at } (c, d)$$

If f is differentiable at $w = c + id$, then

$$f'(w) = \frac{\partial u}{\partial x}(c, d) + i\frac{\partial v}{\partial x}(c, d)$$

(and three other expressions following from the C-R equations)

Proof. f is differentiable at w with derivative $f'(w) = p + iq$

$$\iff \lim_{z \to w} \frac{f(z) - f(w)}{z - w} = p + iq$$
$$\iff \frac{f(z) - f(w) - (z - w)(p + iq)}{|z - w|} = 0$$

Writing $f = u + iv$ and separating real and imaginary parts, the above holds $\iff$

$$\lim_{(x,y)\to(c,d)} \frac{u(x, y) - u(c, d) - p(x - c) + q(y - d)}{\sqrt{(x - c)^2 + (y - d)^2}} = 0 \text{ and}$$

$$\lim_{(x,y)\to(c,d)} \frac{v(x, y) - v(c, d) - q(x - c) - p(y - d)}{\sqrt{(x - c)^2 + (y - d)^2}} = 0$$

$\iff u$ is differentiable at (c, d) with $Du(c, d)(x, y) = px - qy$, and v is differentiable at (c, d) with $Dv(c, d)(x, y) = qx + py$.
$\iff u, v$ is differentiable at (c, d) and $u_x(c, d) = p = v_y(c, d)$, and $u_y(c, d) = iq = -v_x(c, d)$ i.e. C-R equations hold at (c, d). (Here $u_x = \frac{\partial u}{\partial x}$ etc.)
We also get from the above that if f is differentiable at w, then

$$f'(w) = p + iq = u_x(c, d) + iv_x(c, d)$$

Warning. u, v satisfying the C-R equations at a point does NOT guarantee differentiability of f (see example sheet 1)

Remark. If we just want to show: differentiability of f at $w = c + id \implies$ the partial derivatives u_x, u_y, v_c, v_y exist and satisfy Cauchy-Riemann equations, then we can proceed more simply as follows: start with the definition

$$f'(w) = \lim_{z \to w} \frac{f(z) - f(w)}{z - w} = \lim_{h \to 0} \frac{f(w + h) - f(w)}{h}$$

Taking $h = t \in \mathbb{R}$ in this, we get

$$f'(w) = \lim_{t \to 0} \left(\frac{u(c + t, d) - u(c, d)}{t} + i \frac{v(c + t, d) - v(c, d)}{t} \right)$$

This says that $\lim_{t \to 0}(u(c+t, d) - u(c, d))/t$ and $\lim_{t \to 0}(v(c+t, d) - v(c, d))/t$ both exist, i.e. $u_x(c, d)$ and $v_x(c, d)$ exist, and $f'(w) = u_x(c, d) + iv_x(c, d)$. Similarly, taking $h = it$, $t \in \mathbb{R}$, we get that $f'(w) = v_y(c, d) - iu_y(c, d)$. So $u_x = v_y$ and $u_y = -v_x$ at (c, d)

Example. $f(z) = \overline{z} = x - iy$. For this, $u(x, y) = x$, $v(x, y) = -y$, so $u_x = 1, u_y = 0, v_x = 0, v_y = -1$. So C-R equations do not hold anywhere, and f is NOT differentiable at any point

Corollary 1.2. Let $f = u + iv : U \to \mathbb{C}$. If u, v have continuous partial derivatives at a point $(c, d) \in U$ and satisfy the C-R equations at (c, d) then f is differentiable at $w = c + id$. In particular, if u, v are C^1 functions on U (i.e. have continuous partial derivatives in U) satisfying the C-R equations in U, then f is holomorphic in U

Proof. Continuity of partial derivatives of u implies that u is differentiable, and similarly for v. So the corollary follows from Theorem 1.1

We can relax the requirement of continuity of partial derivatives of u, v in U to just continuity of u, v in U. Thus, if $f = u + iv$ is defined on an open set U and is continuous in U, and if u, v satisfy the C-R equations in U, then f is holomorphic in U. This is called the Looman-Menchoff theorem. It is quite non-trivial to prove

Remark. Complex differentiability is much more restrictive than (real) differentiability of real and imaginary parts (because of the additional requirement that C-R equations must hold). This leads to some surprising theorems compared to the real case. For instance:

(i) If $f : \mathbb{C} \to \mathbb{C}$ is holomorphic and bounded (i.e. $|f(z)| \leq K$ for some constant K and all z), then f is constant! (Liouville theorem)

(ii) If $f : U \to \mathbb{C}$ is holomorphic, then f is automoatically infinitely differentiable on U

We will prove these (and much more!) later on

Note. (ii) $\implies$ partial derivatives of u, v of all orders exist. So we can differentiable C-R equations to get:

$$(u_x)_x = (v_y)_x \implies u_{xx} = v_{yx}$$

$$(u_y)_y = (-v_x)_y \implies u_{yy} = -v_{xy}$$

Since $v_{xy} = x_{xy}$, this gives

$$\Delta u = u_{xx} + u_{yy} = 0 \text{ in } U$$

Similarly, $\Delta v = 0$ in U. I.e. real and imaginary parts of a holomorphic function are harmonic

Definition.

- A **curve** is a continuous map $\gamma : [a, b] \to \mathbb{C}$, where $[a, b] \subset \mathbb{R}$ is a closed interval. We say γ is a C^1 curve if γ' exists and is continuous on $[a, b]$. (If $\gamma(t) = x(t) + iy(t)$ then $\gamma'(t) = x'(t) = iy'(t)$; at the end points, γ' is the one-sided derivative)
- An open set $U \subset \mathbb{C}$ is **path-connected** if for any two points $z, w \in U$, there is a curve $\gamma : [0, 1] \to U$ such that $\gamma(0) = z$ and $\gamma(1) = w$
- A **domain** is a non-empty, open, path connected subset of $\mathbb{C}$

Corollary 1.3. If $U \subset \mathbb{C}$ is a domain, $f : U \to \mathbb{C}$ is holomorphic with $f'(z) = 0$ for every $z \in U$, then f is constant

Proof. Write $f == u + iv$. By the C-R equations, $f' = 0 \implies Du = 0$ and $Dv = 0$ in U. Since U is a domain, this means (by a theorem from "Analysis and Topology") that $u = \textit{constant}$ and $v = \textit{constant}$, i.e. f is constant.

So far, we've only seen a few explicity holomorphic functions (namely, polynomials on $\mathbb{C}$ and rational functions on their domains). We'd like to generate more. We do this by looking at the following:

Theorem 1.4 (radius of convergence)**.** If $(c_n)_{n=0}^{\infty}$ is a sequence of complex numbers, then there is a unique number $R \in [0, \infty]$ such that the ower series

$$\sum_{n=0}^{\infty} c_n (z - a)^n, \quad z, a \in \mathbb{C}$$

converges absolutely if $|z - a| < R$ and diverges if $|z - a| > R$. If $0 < r < R$, then the series converges uniformly (with respect to the variable z) on the compact disk $\overline{D(a, r)} = \{z \in \mathbb{C} : |z - a| \leq r\}$

Equation. R is the radius of convergence of the power series. Note that there is no claim about convergence when $|z - a| = r$, $R > 0$. There are various expressions for R. e.g.

$$R = \sup\{r \geq 0 : |c_n| r^n \to 0 \text{ as } n \to \infty\}$$

and

$$\frac{1}{R} = \limsup_{n \to \infty} |c_n|^{1/n}$$

Theorem 1.5. Let $\sum_{n=0}^{\infty}$ be a power series with r. o. c. equal to $R > 0$. Fix $a \in \mathbb{C}$, and define $f : R(a, R) \to \mathbb{C}$ by $f(z) = \sum_{n=0}^{\infty} c_n(z-a)^n$. Then:

(i) f is holomorphic on $D(a, R)$

(ii) the derived series $\sum_{n=1}^{\infty} nc_n(z-a)^{n-1}$ also has r. o. c. equal to R and $f'(z) = \sum_{n=1}^{\infty} nc_n(z-a)^{n-1}$ $\forall z \in D(a, R)$

(iii) f has deriatives of all orders on $D(a, R)$ and $c_n = f^{(n)}(a)/n!$

(iv) if f vanishes on $D(a, \varepsilon)$ for some $\varepsilon > 0$, then $f \equiv 0$ on $D(a, R)$

Proof. (i) By considering $g(z) = f(z+a)$, we assume w.l.o.g. that $a = 0$. So have $f(z) = \sum_{n=0}^{\infty} c_n z^n$ for $z \in D(0, R)$, with r.o.c. $R > 0$.

The derived series $\sum_{n=1}^{\infty} nc_n z^{n-1}$ will have some r.o.c. $R_1 \in [0, \infty]$. To see $R_1 \geq R$, let $z \in D(0, R)$ be arbitrary, and choose ρ such that $|z| < \rho < R$. Then $n|c_n||z|^{n-1} = n|c_n||z/\rho|^{n-1}$. $\rho^{n-1} \leq |c_n|\rho^{n-1}$ for sufficiently large n (as $n|z/\rho|^{n-1} \to 0$ as $n \to \infty$). Since $\sum |c_n|\rho^n$ converges, it follows that $\sum_{n=1}^{\infty} n|c_n||z|^{n-1}$ converges. Thus $D(0, R) \subset D(0, R_1)$ i.e. $R_1 \geq R$.

Since $|c_n||z|^n \leq n|c_n||z^n| = |z|(n|c_n||z|^{n-1})$, if $\sum n|c_n||z|^{n-1}$ converges, then so does $|c_n||z|^n$, so $R_1 \leq R$. So $R_1 = R$.

To prove that f is differentiable with $f'(z) = \sum_{n=1}^{\infty} nc_n z^{n-1}$, fix $z \in D(0, R)$. Key idea: this assertion is equivalent to continuity at z of the function

$$g : D(0, R) \to \mathbb{C}, \quad g(w) = \begin{cases} \frac{f(w)-f(z)}{w-z} & w \neq z \\ \sum_{n=1}^{\infty} nc_n z^{n-1} & w = z \end{cases}$$

By substituting for f, $g(w) = \sum_{n=1}^{\infty} h_n(w),\ w \in D(0, R)$ where

$$h_n(w) = \begin{cases} \frac{c_n(w^n - z^n)}{w-z} & w \neq z \\ nc_n z^{n-1} & w = z \end{cases}$$

Now h_n is continuous on $D(0, R)$ (since $w \mapsto w^n$ is differentiable with derivative nw^{n-1}). Using $(w^n - z^n)/(w-z) = \sum_{j=0}^{n-1} z^j w^{n-1-j}$, we get that for any r with $|z| < r < R$ and any $w \in D(0, r)$,

$$|h_n(w)| \leq n|c_n|r^{n-1} \equiv M_n$$

Since $\sum M_n < \infty$, we have by the Weierstrass M-test $\sum h_n$ converges uniformly on $D(0, r)$. But a uniform of limit of continuous functions is continuous, so $g = \sum h_n$ is continuous in $D(0, r)$ and in particular at z

(ii) proved in (i)

(iii) Repeatedly apply (ii). The formula $c_n = f^{(n)}(a)/n!$ follows by differentiating the series n times and setting $z = a$

(iv) If $f \equiv 0$ in $D(a, \varepsilon)$, then $f^{(n)}(a) = 0$ for all n, so $c_n = 0$ for all n and hence $f \equiv 0$ in $D(a, R)$

Remarks.

(i) This theorem provides a way to generate a large class of holomorphic functions on a disk

(ii) Later we will show that every holomorphic function is locally given by a power series. Once we have that, part (iii) above gives that holomorphic functions are automatically infinitely differentiable in their domain

Definition. If $f : \mathbb{C} \to \mathbb{C}$ is holomorphic on all of $\mathbb{C}$, we say f is **entire**

Prop 1.6. (i) e^z is entire, with $(e^z)' = e^z$

(ii) $e^z \neq 0$ and $e^{z+w} = e^z \cdot e^w$ for all $z, w \in \mathbb{C}$

(iii) $e^{x+iy} = e^x(\cos y + i \sin y)$ for $x, y \in \mathbb{R}$

(iv) $e^z = 1$ if and only if $z = 2n\pi i$ for some integer n

(v) Let $z \in \mathbb{C}$. $\exists w \in \mathbb{C}$ s.t. $e^w = z$ if and only if $z \neq 0$

Proof. (i) r.o.c. of the series is ∞. To see $(e^z)' = z$, differentiate the series term-by-term, using Theorem 1.5

(ii) Fix any $w \in \mathbb{C}$ and set $F(z) = e^{z+w \cdot e^{-z}}/$ Then

$$F'(z) = -e^{z+w} \cdot e^{-z} + e^{z+w} \cdot e^{-z} = 0$$

so

$$F(z) = constant = F(0) = e^w$$

($e^0 = 1$ by direct calculation. Thus

$$e^{z+w} \cdot e^{-z} = e^w \quad \forall z, w \in \mathbb{C} \tag{*}$$

Taking $w = 0$, $e^z \cdot e^{-z} = 1$. So $e^z \neq 0$. Multiplying (*) by e^z, get

$$e^{z+w} = e^z \cdot e^w$$

(iii) $e^{x+iy} = e^x \cdot e^{iy}$ by (ii). Now use the definition of e^{iy}, and the series for $\sin y, \cos y$ for $y \in \mathbb{R}$

(iv) Follows from (iii)

(v) Follows from (iii)

Definition. Given $z \in \mathbb{C}$, we say a complex number $w \in \mathbb{C}$ is a **logarithm** of z if $e^w = z$

Remark. By Proposition 1.6(v), z has a logarithm iff $z \neq 0$.
By Proposition 1.6(ii) and (iv), if $z \neq 0$, then z has infinitely many logarithms, with any two differeing from each other by $2n\pi i$ for some integer n.
If w is a logarithm of z, then $e^{\text{Re}(w)} = |z|$, so $\text{Re}(z) = \ln |z|$ (the real logarithm of the positive number $|z|$); in particular, $\text{Re}(w)$ is uniquely determined by z

Definition. Let $U \subset \mathbb{C}\backslash\{0\}$ be open. Then a **branch of logarithm** on U is a continuous function $\lambda : U \to \mathbb{C}$ such that $e^{\lambda(z)} = z$ for each $z \in U$

Remark. If λ is a branch of log on U, then λ is automatically holomorphic in U, with $\lambda'(z) = 1/z$

Proof. If $w \in U$ then

$$\begin{aligned}\lim_{z\to w}\frac{\lambda(z)-\lambda(w)}{z-w} &= \lim_{z\to w}\frac{\lambda(z)-\lambda(w)}{e^{\lambda(z)}-e^{\lambda(w)}} \\ &= \lim_{z\to w}\frac{1}{\left(\frac{e^{\lambda(z)}-e^{\lambda(w)}}{\lambda(z)-\lambda(w)}\right)} \quad (z\neq w \implies \lambda(z)\neq\lambda(w)) \\ &= \frac{1}{e^{\lambda(w)}}\lim_{z\to w}\frac{1}{\left(\frac{e^{\lambda(z)-\lambda(w)}-1}{\lambda(z)-\lambda(w)}\right)} \\ &= \frac{1}{e^{\lambda(w)}}\lim_{h\to 0}\frac{1}{\left(\frac{e^{h}-1}{h}\right)} \text{ as } \lambda \text{ continuous} \\ &= \frac{1}{e^{\lambda(w)}} = \frac{1}{w}\end{aligned}$$

Definition. The **principal branch of logarithm** is the function

$$\mathrm{Log} : U_1 = \mathbb{C}\backslash\{x\in\mathbb{R} : x\le 0\}\to\mathbb{C}$$

defined by

$$\mathrm{Log}(z) = \ln|z| + i\arg(z)$$

where $\arg(z)$ is the unique argument of $z \in U_w$ in $(-\pi, \pi)$

Log is a branch of logarithm in U_1: to check continuity of Log, note that $z\mapsto\log|z|$ is continuous on $\mathbb{C}\backslash\{0\}$ (by continuity of $z\mapsto|z|$ and continuity of $r\mapsto\log r$ for $r>0$); also $z\mapsto\arg(z)$ is continuous, since $\theta\mapsto e^{i\theta}$ is a homeomorphism $(-\pi,\pi)\to S^1\backslash\{-1\}$ (as can be checked directly, where $S^1 = \{z : |z| = 1\}$), and $z\mapsto z/|z|$ is continuous on $\mathbb{C}\backslash\{0\}$ So $z\mapsto\mathrm{Log}(z)$ is continuous on U_1. We also have

$$e^{\mathrm{Log}(z)} = e^{\ln|z|+i\arg(z)} = e^{\ln|z|}\cdot e^{i\arg(z)} = |z|(\cos\arg(z) + i\sin\arg(z)) = z$$

So Log is a branch of logarithm in U_1

Note. Log does not have a continuous extension to $\mathbb{C}\backslash\{0\}$ (since $\arg(z)\to\pi$ as $z\to -1$ with $\mathrm{Im}(z) > 0$, and $\arg(z)\to-\pi$ as $z\to-1$ with $\mathrm{Im}(z) < 0$). We will later show there is no branch of log on $\mathbb{C}\backslash\{0\}$

Prop 1.7. (i) Log is holomorphic on U_q with $(\text{Log}(z))' = 1/z$

(ii) For $|z| < 1$

$$\text{Log}(1+z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} z^n}{n}$$

Proof. (i) Remark above

(ii) Note that r.o.c. of the series is 1, and $|z| < 1 \implies 1 + z \in U_1$, so both sides are defined on $|z| < 1$. Let $F(z) = \text{Log}(1+z) - \sum_{n=1}^{\infty} (-1)^{n+1} z^n / n, \ |z| < 1$. Then

$$F'(z) = \frac{1}{1+z} - \sum_{n=1}^{\infty} (-z)^{n-1} = 0$$

so

$$F(z) = \textit{constant} = F(0) = 0$$

Using exp and Log, we can define further useful functions

(i) For any constant $\alpha \in \mathbb{C}$, define

$$z^\alpha = e^{\alpha \text{Log}(z)}, \quad z \in U_1$$

This is the principal branch of z^α. We have that z^α is holomorphic on $U_!$ with $(z^\alpha)' = \alpha z^{\alpha-1}$

(ii) $\cos(z), \sin(z), \cosh(z), \sinh(z)$ can all be written in terms of exponentials. All entire derivatives fiven by the familiar expressions from the real variables

1.2 Conformality

Let $f : U \to \mathbb{C}$ be holomorphic (where $U \subset \mathbb{C}$ is open, as usual). Let $w \in U$ and suppose that $f'(w) \neq 0$. Take to C^1 curves $\gamma_i : [-1,1] \to U$ $(i = 1,2)$ such that $\gamma_i(0) = w$ and $\gamma_i \ \neq 0$ for $i = 1,2$. Then $f \circ \gamma_i$ are C^1 curves passing through $f(w)$. Moreover

$$(f \circ \gamma_i)'(0) = f'(w)\gamma_i'(0) \neq 0$$

So

$$\frac{(f \circ \gamma_1)'(0)}{(f \circ \gamma_2)'(0)} = \frac{\gamma_1'(0)}{\gamma_2'(0)}$$

and hence

$$\arg(f \circ \gamma_1)'(0) - \arg(f \circ \gamma_2)'(0) = \arg \gamma_1'(0) - \arg \gamma_2'(0)$$

This means that the angle that the curves γ_1, γ_2 make at w is the same as the angle their images $f \circ \gamma_1, f \circ \gamma_2$ make at $f(w)$, in size as well as in orientation, i.e. f is "angle-preserving at w" whenever $f'(w) \neq 0$ (In particular, if the curves γ_1, γ_2 are tangential at w, then the curves $f \circ \gamma_1, f \circ \gamma_2$ are tangential at $f(w)$)

Remark. If f is a C^1 map on U, the converse also holds, ie. if $w \in U, f$ has the property that $(f \circ \gamma)'(0) \neq 0$ for any C^1 curve γ with $\gamma(0) = w$ and $\gamma'(0) \neq 0$, and if f is angle preserving at w in the above sense then $f'(w)$ exists and $f'(w) \neq 0$. See example sheet 1

Definition. A holomorphic function $f : U \to \mathbb{C}$ on an open set U is said to be **conformal** at a point $w \in U$ if $f'(w) \neq 0$

Definition. Let $U, \tilde{U}$ be domains in $\mathbb{C}$. A map $f : U \to \tilde{U}$ is said to be a **conformal equivalence** between U and $\tilde{U}$ if f is a bijective holomorphic map with $f'(z) \neq 0$ for every $z \in U$

Remark. If f is holomorphic and injective, then $f'(z) \neq 0$ for each z. (We shall see this later in the course). So in the above definition, the requirement $f'(z) \neq 0$ is redundant

Remark. It is automatic that the inverse $f^{-1} : \tilde{U} \to U$ is holomorphic. This follows from the holomorphic inverse function theorem, which can be proved using the real inverse function theorem (see example sheet 1)

Examples. (i) Mobius maps

$$f(z) = \frac{az+b}{cz+d}, \quad a, b, c, d \in \mathbb{C}, \ ad - gb \neq 0$$

f is conformal on $\mathbb{C} \backslash \{-d/c\}$ if $c \neq 0$, and conformal on $\mathbb{C}$ if $c = 0$. Mobius maps sometimes seve as explicit conformal equivalences between subdomains of $\mathbb{C}$. e.g. Let $\mathbb{H}$ = open upper-half plane $= \{z : \text{Im}(z) > 0\} \cap \mathbb{C}$. Then

$$\begin{aligned} z \in \mathbb{H} &\iff z \text{ is closer to } i \text{ than to } -i \\ &\iff |z - i| < |z + i| \\ &\iff \left| \frac{z-i}{z+i} \right| < 1 \end{aligned}$$

Thus $g(z) = (z - i)/(z_i)$ maps $\mathbb{H}$ onto $D(0, 1)$, so $f : \mathbb{H} \to D(0, 1)$ is a conformal equivalence

(ii) $f : z \mapsto z^n, \ n \geq 1$ integer

$$f : \{z \in \mathbb{C} \backslash \{0\} : 0 < \arg(z) < \pi n\} \to \mathbb{H}$$

This is a conformal equivalence, with inverse $f^{-1}(z) = z^{1/n}$ (the principal branch of $z^{1/n}$)

$$\exp : \{z \in \mathbb{C} : -\pi < \text{Im}(z) < \pi\} \to \mathbb{C} \backslash \{x \in \mathbb{R} : x \leq 0\}$$

is a conformal equivalence, whose inverse is $\text{Log}(z)$

Aside:

Theorem 1.8. Any simply connected domain $U \subset \mathbb{C}$ with $U \neq \mathbb{C}$ is conformally equivalent to $D(0, 1)$

Proof. See e.g. Ahlfors, Complex Analysis, or Rudin, Real and Complex Analysis

Here $U \subset \mathbb{C}$ simply connected means that every continuous map $\gamma : S^1 = \partial D(0,1) \to U$ extends to a continuous map $\Gamma : \overline{D(0,1)} \to U$ with $\Gamma|_{\partial D(0,1)} = \gamma$. (We will discuss simply connected domains in detail later in the course; intuitively, simply connected domains are the ones "without holes")

Remark. The case $U = \mathbb{C}$ has to be excluded in the theorem in view of Liouville's theorem (which we will prove later)

2 Complex Integration

Moral. We aim to extend real (Riemann) integration to integration of complex functions $f : U \to \mathbb{C}, U \subset \mathbb{C}$, along curves in U.
First look at complex functions of a real variable

Definition. If $f : [a, b] \subset \mathbb{R} \to \mathbb{C}$ is a complex function, and if f is continuous (or more generally if f is Riemann integrable, i.e. $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$ are Riemann integrable), define

$$\int_a^b f(t)\,\mathrm{d}t = \int_a^b \mathrm{Re}(f(t))\,\mathrm{d}t + i\int_a^b \mathrm{Im}(f(t))\,\mathrm{d}t$$

In particular,

$$\int_a^b ig(t) = i\int_a^b g(t)\,\mathrm{d}t$$

for a real function $g : [a, b] \to \mathbb{R}$. Hence, by direct calculation

$$w\int_a^b f(t)\,\mathrm{d}t = \int_a^b wf(t)\,\mathrm{d}t$$

for any $w \in \mathbb{C}$

Prop 2.1 (Basic estimate)**.** If $f : [a, b] \to \mathbb{C}$ is continuous, then

$$\left| \int_a^b f(t)\,\mathrm{d}t \right| \leq \int_a^b |f(t)|\,\mathrm{d}t \leq (b-a) \sup_{t\in[a,b]} |f(t)|$$

with equality iff f is constant

Proof. If $\int_a^b f(t)\,\mathrm{d}t = 0$, then we are done. Else write $\int_a^b f(t)\,\mathrm{d}t = re^{i\theta}, \theta \in [0, 2\pi)$, and let $M = \sup_{t\in[a,b]} |f(t)|$. Then

$$\begin{aligned} \left| \int_a^b f(t)\,\mathrm{d}t \right| &= r = e^{-i\theta} \int_a^b f(t)\,\mathrm{d}t = \int_a^b e^{-i\theta} f(t)\,\mathrm{d}t \\ &= \int_a^b \mathrm{Re}(e^{-i\theta} f(t))\,\mathrm{d}t + i \int_a^b \mathrm{Im}(e^{-i\theta} f(t))\,\mathrm{d}t \end{aligned}$$

Since LHS is real

$$\begin{aligned} \left| \int_a^b f(t)\,\mathrm{d}t \right| &= \int_a^b \mathrm{Re}(e^{-i\theta} f(t))\,\mathrm{d}t \\ &\leq \int_a^b |e^{-i\theta} f(t)|\,\mathrm{d}t \\ &= \int_a^b |f(t)|\,\mathrm{d}t \\ &\leq (b-a)M \end{aligned}$$

Equality holds iff $|f(t)| = M$ and $\mathrm{Re}(e^{-i\theta} f(t)) = M$ for all $t \in [a, b]$, i.e. iff $|f(t)| = M$ and $\arg(f(t)) = \theta$ for all $t \in [a, b]$, i.e. iff $f = \mathit{const}$.

Definition. Let $U \subset \mathbb{C}$ be open and $U \to \mathbb{C}$ be continuous. Let $\gamma[a, b] \to U$ be a C^1 curve. Then the **integral of f along γ** is

$$\int_\gamma f(z)\,\mathrm{d}z = \int_a^b f(\gamma(t))\gamma'(t)\,\mathrm{d}t$$

Basic properties:

(i) Invariance under reparametrisation: let $\varphi : [a_1, b_1] \to [a, b]$ be a C^1 and injective with $\varphi(a_1) = a$ and $\varphi(b_1) = b$. Let $\delta = \gamma \circ \varphi : [a_1, b_1] \to U$. Then we have

$$\int_\delta f(z)\,\mathrm{d}z = \int_\gamma f(z)\,\mathrm{d}z$$

Proof.

$$\int_\delta f(z)\,\mathrm{d}z = \int_{a_1}^{b_1} f(\gamma\circ\varphi)\gamma'(\varphi(t))\varphi'(t)\,\mathrm{d}t = \int_a^b f(\gamma(s))\gamma'(s)\,\mathrm{d}s = \int_\gamma f(z)\,\mathrm{d}z$$

using $s = \varphi(t)$. The second equality follows from the definition of integral of a function of a real variable and the change of variables formula for integrals of real functions of a real variable

(ii) Linearity:

$$\int_\gamma (c_1 f_1(z) + c_2 f_2(z))\,\mathrm{d}z = c_1 \int_\gamma f_1\,\mathrm{d}z + c_2 \int_\gamma f_2(z)\,\mathrm{d}z$$

for constants $c_1, c_2 \in \mathbb{C}$

(iii) Additivity: If $\gamma : [a, b] \to U$ is a C^1 curve and $a < c < b$, then

$$\int_\gamma f(z)\,\mathrm{d}z = \int_{\gamma|_{[a,c]}} f(z)\,\mathrm{d}z + \int_{\gamma|_{[c,b]}} f(z)\,\mathrm{d}z$$

(iv) Inverse path: Define the inverse path $(-\gamma) : [-b, -a] \to U$ by $(-\gamma)(t) = \gamma(-t)$ for $-b \le t \le -a$. Then

$$\int_{(-\gamma)} f(z)\,\mathrm{d}z = -\int_\gamma f(z)\,\mathrm{d}z$$

(ii), (iii), (iv) are easy to check using the definitions

Definition. Let $\gamma : [a, b] \to \mathbb{C}$ be a C^1 curve. The **length of γ** is defined by

$$\text{length}(\gamma) = \int_a^b |\gamma'(t)|\,\mathrm{d}t$$

Definition. A **piecewise C^1 curve** is a continuous map $\gamma : [a, b] \to \mathbb{C}$ such that there exists a finite subdivision

$$a = a_0 < a_1 < a_2 < \cdots < a_{n-1} < a_n = b$$

with the property that $\gamma_j : \gamma|_{[a_{j-1}, a_j]} : [a_{j-1} \to a_j] \to \mathbb{C}$ is C^1 for $1 \le j \le n$.
If γ is a piecewise C^1 is piecewise C^1 curve as above, define:

$$\int_\gamma f(z)\,\mathrm{d}z = \sum_{j=1}^n \int_{\gamma_j} f(z)\,\mathrm{d}z$$

and

$$\text{length}(\gamma) = \sum_{j=1}^n \text{length}(\gamma_j) = \sum_{j=1}^n \int_{a_{j-1}}^{a_j} |\gamma'(t)|\,\mathrm{d}t$$

Note. Both definitions are independent of the subdivision, by the additivity property (iii) above. From now on, by a "curve", we shall mean a piecewise C^1 curve

Definition. If $\gamma_1 : [a,b] \to \mathbb{C}$ and $\gamma_2 : [c,d] \to \mathbb{C}$ are curves with $\gamma_1(b) = \gamma_2(c)$, we define the **sum of γ_1 and γ_2** to be the curve

$$(\gamma_1 + \gamma_2) : [a, b+d-c] \to \mathbb{C}$$

$$(\gamma_1 + \gamma_2)(t) = \begin{cases} \gamma_1(t) & a \leq t \leq b \\ \gamma_2(t-b+c) b \leq t \leq b+d-c \end{cases}$$

Prop 2.2. For any continuous function $f : U \to \mathbb{C}$ and any curve $\gamma : [a,b] \to \mathbb{C}$, we have that

$$\left| \int_\gamma f(z) \, \mathrm{d}z \right| \leq \operatorname{length}(\gamma) \sup_\gamma |f|$$

where $\sup_\gamma |f| = \sup_{t \in [a,b]} |f(\gamma(t))|$

Proof. If γ is C^1, then $|\int_\gamma f(z) \, \mathrm{d}z| = |int_a^b f(\gamma(t))\gamma'(t) \, \mathrm{d}t| \leq \int_a^b |f(\gamma(t))||\gamma'(t)| \, \mathrm{d}t \leq \sup_{t \in [a,b]} |f(\gamma(t))| \operatorname{length}(\gamma)|$. If γ is piecewise C^1 then the result follows from the definition $\int_\gamma f(z) \, \mathrm{d}z = \sum_{j=1}^n \int_{\gamma_j} f(z) \, \mathrm{d}z$ where γ_j is C^1, and the triangle inequality

We have the complex version of the fundamental theorem of calculus

Theorem 2.3. Suppose that $f : U \to \mathbb{C}$ is continuous, $U \subset \mathbb{C}$ open. If there is a function $F : U \to \mathbb{C}$ such that $F'(z) = f(z)$ for all $z \in U$, then for any curve $\gamma : [a,b] \to U$,

$$\int_\gamma f(z) \, \mathrm{d}z = F(\gamma(b)) - F(\gamma(a))$$

If additionally γ is a closed curve, i..e. $\gamma(b) = \gamma(a)$, then $\int_\gamma f(z) \, \mathrm{d}z = 0$

Proof.

$$\int_\gamma f(z) \, \mathrm{d}z = \int_a^b f(\gamma(t))\gamma'(t) \, \mathrm{d}t = \int_a^b \frac{\mathrm{d}}{\mathrm{d}t} F(\gamma(t)) \, \mathrm{d}t = F(\gamma(b)) = F(\gamma(a))$$

Definition. Such F as in Theorem 2.3 is called an anti-derivative of f

Note. We shall see late (by infinite differentiability of holomorphic functions) that if $F'(z) = f(z)$, then f is automatically continuous

Example. $\int_\gamma z^n \, \mathrm{d}z$ for n an integer, where $\gamma : [0,1] \to \mathbb{C}, \gamma(t) = Re^{2\pi i t}$ $t \in [0,1]$, for some $R > 0$. (The image of γ is the circle of radius R centered at the origin.)
For $n \neq -1 : z^{n+1}/(n+1)$ is an anti-derivative of z^n in $\mathbb{C}\backslash\{0\}$ (in $\mathbb{C}$ if $n \geq 0$), so by Theorem 2.3, $\int_\gamma z^n \, \mathrm{d}z = 0$ (since γ is a closed curve)/
For $n = -1$, use the definition of integral:

$$\int_\gamma \frac{1}{z} \, \mathrm{d}z = \int_0^1 \frac{1}{\gamma(t)} \gamma'(t) \, \mathrm{d}t = \int_0^1 \frac{1}{Re^{2\pi i t}} 2\pi i R e^{2\pi i t} \, \mathrm{d}t = 2\pi i$$

Since $\int_\gamma 1/z \, \mathrm{d}z \neq 0$, we can conclude that for any $R > 0$, $1/z$ has no anti-derivative in any open set containing the circle $\{|z| = R\}$.
In particular, since any branch $\lambda(z)$ of logarithm the derivative $\lambda'(z) = 1/z$, there is no branch of logarithm on $\mathbb{C}^* = \mathbb{C}\backslash\{0\}$

Theorem 2.4 (Converse to FTC)**.** Let $U \subset \mathbb{C}$ be a domain. If $f : U \to \mathbb{C}$ is continuous and if $\int_\gamma f(z)\,\mathrm{d}z = 0$ for every closed curve γ in U, then f has an anti-derivative. i.e. there is a holomorphic function $F : U \to \mathbb{C}$ such that $F'(z) = f(z)$ for each $z \in U$

Proof. Fix any $a_0 \in U$. For $w \in U$ define

$$F(w) = \int_{\gamma_w} f(z)\,\mathrm{d}z$$

where $\gamma_w : [0,1] \to \mathbb{C}$ is a curve with $\gamma_w(0) = a_0,\ \gamma_w(1) = w$. (Such a continuous curve exists since U is path-connected; starting with this we can find a piecewise C^1 curve, in fact a polygonal curve. Exercise.) The definition of F is independent of the choice of γ_w (connecting a_0 to w) since, by hypothesis, $\int_\gamma f(z)\,\mathrm{d}z = 0$ for all closed curves γ. So we have a well defined function $F : U \to \mathbb{C}$.
Fix $w \in U$. Since U is open, there is $r > 0$ such that $D(w,r) \subset U$. For $h \in \mathbb{C}$ with $0 < |h| < r$, let δ_h be the radial path $t \mapsto w + th,\ t \in [0,1]$. Let

$$\gamma = \gamma_w + \delta_h + (-\gamma_{w+h})$$

γ is a closed curve, so $\int_\gamma f(z)\,\mathrm{d}z = 0$, which implies that

$$\int_{\gamma_{w+h}} f(z)\,\mathrm{d}z = \int_{\gamma_w} f(z)\,\mathrm{d}z + \int_{\delta_h} f(z)\,\mathrm{d}z$$

In terms of F, this says that

$$F(w+h) = F(w) + \int_{\delta_h} f(z)f(z)\,\mathrm{d}z = F(w) + hf(w) + \int_{\delta_h} (f(z) - f(w))\,\mathrm{d}z$$

So

$$\begin{aligned}\left|\frac{F(w+h) - F(w)}{h} - f(w)\right| &= \frac{1}{|h|}\left|\int_{\delta_h} (f(z) - f(w))\,\mathrm{d}z\right| \\ &\leq \frac{1}{|h|} length(\delta_h) \sup_{z \in image(\delta_h)} |f(z) - f(w)| \\ &= \sup_{z \in image(\delta_h)} |f(z) - f(w)| \to 0\end{aligned}$$

as $h \to 0$, since f is continuous. Thus F is differentiable at w with $F'(w) = f(2)$

Definition. A domain U is **star-shaped** (or is a **star domain**) if there is $a_0 \in U$ such that for each $w \in U$, the straight-line segment $[a_0, w] \subset U$

Note. U is a disk $\implies$ U is convex $\implies$ U is star-shaped $\implies$ U is path-connected. None of the reverse implications hold

Definition. A **triangle** in $\mathbb{C}$ is the convex hull of three points in $\mathbb{C}$. The three points are the vertices of the triangle. Thus the triangle T whose vertices are $z_1, z_2, z_3 \in \mathbb{C}$ is the (closed) set

$$T = \{az_1 + bz_2 + cz_3 : 0 \leq a, b, c \leq 1,\ a + b + c = 1\}$$

Notation. For a triangle T, we write $\int_{\partial T} f(z)\,\mathrm{d}z$ to denote the integral of f along the piecewise affine closed curve $\gamma = \gamma_1 + \gamma_2 + \gamma_3$ where $\gamma_1, \gamma_2, \gamma_3$ are affine functions parametrising the three straight-line segments making up the boundary of T, with the parametrisations chosen so that T lies to the left of each directed segment

Corollary 2.5. If U is star-shaped, $f : U \to \mathbb{C}$ is continuous and $\int_{\partial T} f(z)\,\mathrm{d}z =$ for any triangle $T \subset U$, then f has an anti-derivative in U

Proof. Suppose U is star shaped with respect to a point $a_0 \in U$ and let $w \in U$ be an arbitrary point. Let γ_w be the affine function parametrising the (directed) line segment $[a_0, w]$, and let $F(w) = \int_{\gamma_w} f(z)\,\mathrm{d}z$. With h and δ_h as in the proof of Theorem 2.4, and $\gamma = \gamma_2 + \delta_h + (-\gamma_{w+h})$, we hthen have that $\int_\gamma f(z)\,\mathrm{d}z = \pm \int_{\partial T} f(z)\,\mathrm{d}z$ for a triangle $T \subset U$ (with the $-$ sign if T lies to the right of the directed boundary segments). Since $\int_{\partial T} f(z) = 0$ by hypothesis, we have that $\int_\gamma f(z)\,\mathrm{d}z = 0$. We can now proceed exactly as in the proof of Theorem 2.4

Recap:

(i) For a domain $U \subset \mathbb{C}$ and a continuous function $f : U \to \mathbb{C}$:
$\int_\gamma f(z)\,\mathrm{d}z = 0$ for any closed curve γ in $U \iff f$ has an anti-derivative in U ($\implies f$ is holomorphic in U; see later.)

(ii) The example $f(z) = 1/z$ on $U = \mathbb{C}\backslash\{0\}$ (or $U = D(0,R)\backslash\overline{D(0,r)}$ for $0 < r < R \leq \infty$) shows that f holomorphic on $U \not\Longrightarrow f$ has an anti-derivative in U

(iii) If U is a star domain, then for continuous $f : U \to \mathbb{C}$,
$\int_{\partial T} f(z)\,\mathrm{d}z = 0$ for ant triangle $T \subset U \iff f$ has an anti-derivative in U.

Moral. The validity of $\int_\gamma f(z)\,\mathrm{d}z = 0$ for any holomorphic f on U and any closed curve γ in U has important consequences, as we shall see.
A central aim in the rest of the course: answer the following question, and examine the consequences of the answer:
Whcihc domains $U \subset \mathbb{C}$ have the property that $\int_\gamma f(z)\,\mathrm{d}z = 0$ for any holomorphic $f : U \to \mathbb{C}$ and any closed curve $\gamma \in U$?
We will eventually give the complete short answer: simply connected domains. This is called Cauchy's theorem. First step: do the special case of star domains. This can be handled as an immediate corollory (Corollary 2.8 below) of the following

Theorem 2.6 (Cauchy's Theorem for triangles)**.** Let $U \subset \mathbb{C}$ be open and $f : U \to \mathbb{C}$ be holomorphic. Then $\int_{\partial T} f(z)\,\mathrm{d}z = 0$ for any triangle $T \subset U$

Proof. Let $\eta(T) = \int_{\partial T} f(z)\,\mathrm{d}z$. First key idea: subdivide T into four smaller triangles $T^{(1)}, T^{(2)}, T^{(3)}, T^{(4)}$ by joining the mid-points of the sides of T, and note that

$$\eta(T) = \int_{\partial T^{(1)}} f(z)\,\mathrm{d}z + \int_{\partial T^{(2)}} f(z)\,\mathrm{d}z + \int_{\partial T^{(3)}} f(z)\,\mathrm{d}z + \int_{\partial T^{(4)}} f(z)\,\mathrm{d}z$$

So by the triangle inequality

$$\left|\int_{\partial T^{(j)}} f(z)\,\mathrm{d}z\right| \geq \frac{|\eta(T)}{4}$$

for some $j \in \{1, 2, 3, 4\}$. Let $T_1 = T^{(j)}$ for this j, and write $T_0 = T$. So $|\eta(T_1)| \geq \frac{1}{4}|\eta(T_0)|$. Also $length(\partial T_1) = \frac{1}{2}length(\partial T_0)$.

Now repeate the process: subdivide T_1 and choose a new triangle $T_2 \subset T_1$ exactly the same way. Doing this indefinitely generates a sequence of triangles $T_0 \supset T_1 \supset T_2 \supset \dots$ satisfying for $n = 1, 2, 3, \dots$

$$\eta(T_n)| \geq \frac{1}{4}|\eta(T_{n-1}) \text{ and } length(\partial T_n) = \frac{1}{2}length(\partial T_{n-1})$$

Iterating these,

$$|\eta(T_n)| \geq \frac{1}{4^b}|\eta(T_0)|, \quad length(\partial T_n) = \frac{1}{2^n}length(\partial T_0) \text{ for } n = 1, 2, \dots$$

Since T_n are non-empty, nested closed subsets with $\operatorname{diam}(T_n) \to 0$, we have that $\bigcap_{n=1}^{\infty} T_n = \{z_0\}$ for some $z_0 \in \mathbb{C}$. (Exercise).

Now let $\varepsilon > 0$. Since f is differentiable at z_0, there is $\delta > 0$ such that

$$z \in U,\ |z - z_0| < \delta \implies |f(z) - f(z_0) - f'(z_0)(z - z_0)| \leq \varepsilon|z - z_0|$$

Second key idea: oberve that for any n,

$$\int_{\partial T_n} f(z)\,\mathrm{d}z = \int_{\partial T_n} (f(z) - f(z_0) - f'(z_0)(z - z_0))$$

Since $\int_{\partial T_n} 1\,\mathrm{d}z = \int_{\partial T_n} z\,\mathrm{d}z = 0$ by the FTC.

So choosing n with $T_n \subset D(z_0, \delta)$ (possible since $z_0 \in T_n$ for all n and $\operatorname{diam}(T_n) \to 0$)

$$\begin{aligned}
\frac{1}{4^n}|\eta(T_0) \leq |\eta(T_n)| &= \left|\int_{\partial T_n} f(z)\,\mathrm{d}z\right| \\
&= \left|\int_{\partial T_n} (f(z) - f(z_0) - f'(z_0)(z - z_0))\,\mathrm{d}z\right| \\
&\leq \left(\sup_{z \in \partial T_n} |f(z) - f(z_0 - f'(z_0)(z - z_0)|)\right) length(\partial T_n) \\
&\leq \varepsilon \left(\sup_{z \in \partial T_n} |z - z_0|\right) length(\partial T_n) \leq \varepsilon(length(\partial T_n))^2 = \frac{\varepsilon}{4^n}(length(\partial T_0))^2
\end{aligned}$$

Cancel $1/4^n$ on both sides and let $\varepsilon \to 0$. This gives $\eta(T_0) = 0$

For later applications, it is important to generalise theroem 2.6 to continuous f which are (a priori) holomorphic except at a finite number of points

Theorem 2.7. Let $U \subset \mathbb{C}$ be open and $f : U \to \mathbb{C}$ be continuous. Let $S \subset U$ be a finite set and suppose that f is holomorphic on $U \backslash S$. Then $\int_{partialT} f(z)\,\mathrm{d}s = 0$ for every triangle $T \subset U$

Proof. Subdivide T into a total of $N = 4^n$ smaller triangles by iterative procedure before, where at each step we join up the mid points of the sides of the triangles at the previous step. This time, we keep all the smaller triangles call them $T_1, T_2, \ldots, T_N$. (Note the notational difference from before.) Then, since the integrals along the sides (of the smaller triangles) that are interior to T cancel, we get that

$$\int_{\partial T} f(z)\,\mathrm{d}z = \sum_{j=1}^{N} \int_{\partial T_j} f(z)\,\mathrm{d}z$$

Now by the previous theorem, $\int_{\partial T_j} f(z)\,\mathrm{d}z = 0$ unless $T_j \cap S \neq \varnothing$. So letting $UI = \{j : T_j \cap S \neq \varnothing\}$, we have that $\int_{\partial T} f(z)\,\mathrm{d}z = \sum_{j\in I} \int_{\partial T_j} f(z)\,\mathrm{d}z$. Since any point can be in at most 6 smaller triangles and $\text{length}(\partial T_j) = \frac{1}{2^n}\,\text{length}(\partial T)$, we get that

$$\left| \int_{\partial T} \right| \leq 6|S| \left(\sup_{z\in T} |f(z)| \right) \frac{\text{length}(\partial T)}{2^n}$$

Let $n \to \infty$

Corollary 2.8. Let $U \subset \mathbb{C}$ be convex, or more generally, a star domain. Let $f : U \to \mathbb{C}$ be continuous and holomorphic in $U \backslash S$ where S is a finite set. Then $\int_\gamma f(z)\,\mathrm{d}z = 0$ for any closed curve γ in U

Proof. By Theorem 2.7, $\int_{\partial T} f(z)\,\mathrm{d}z = 0$ for any triangle $T \subset U$. Since U is a star domain and f is continuous, this means (by Corollary 2.5) that f has an anti-derivative in U. The result now follows from the FTC (Theorem 2.3)

Remark. We will see soon that if $f : U \to \mathbb{C}$ is continuous and holomorphic in $U \backslash S$ where S is finite, then f is holomorphic in U. Our proof of this fact will rely on Corollary 2.8
Now we are ready to draw a series of very nice corollaries of "convex Cauchy". The main corollary is a representation formula known as the Cauchy integral formula, from which the other results will follow.

Notation. For a disk $D(a, \rho)$, we will write $\int_{\partial D(a,\rho)} f(z)\,\mathrm{d}z$ to mean $\int_\gamma f(z)\,\mathrm{d}z$ where $\gamma : [0, 1] \to \mathbb{C}$ is the curve $\gamma(t) = a + \rho e^{2\pi i t}$ (which parametrises the boundary of the disk with positive orientation, i.e. so that the dsik lies to the left of the directed boundary circle.)

Theorem 2.9 (Cauchy Interal Formula (CIF) for a disk)**.** Let $D = D(a,r)$ and let $f : D \to \mathbb{C}$ be holomorphic. Then for any ρ with $0 < \rho < r$ and any $w \in D(a,\rho)$, we have that

$$f(w) = \frac{1}{2\pi i}\int_{\partial D(a,\rho)} \frac{f(z)}{z-w}\,\mathrm{d}z$$

In particular (taking $w = a$)

$$f(a) = \frac{1}{2\pi i}\int_{\partial D(a,\rho)} \frac{f(z)}{z-a} f(z)\,\mathrm{d}z = \int_0^1 f(a+\rho e^{2\pi i t})\,\mathrm{d}t$$

Proof. Fix $w \in D(a,\rho)$ and define $h : D \to \mathbb{C}$ by

$$h(z) = \begin{cases} \frac{f(z)-f(w)}{z-w} & z \neq w \\ f'(w) z = w \end{cases}$$

Then h is continuous on D and holomorphic in $D\backslash\{w\}$, so by "Convex Cauchy" (corollary 2.8)

$$\int_{\partial D(a,\rho)} h(z)\,\mathrm{d}z = 0$$

substituting for h, we get

$$f(w)\int_{\partial D(a,\rho)} \frac{\mathrm{d}z}{z-w} = \int_{\partial D(a,\rho)} \frac{f(z)\,\mathrm{d}z}{z-w}$$

Now we just have to show that $\int_{partial D(a,\rho)} \frac{1}{z-w}\,\mathrm{d}z = 2\pi i$. To do this, note that

$$\frac{1}{z-w} = \frac{1}{z-a+a-w} = \frac{1}{(z-a)(1-\frac{w-a}{z-a})} = \sum_{j=0}^{\infty} \frac{(w-a)^j}{(z-a)^{j+1}}$$

where the convergence is uniform for $z \in \partial D(a,\rho)$ by the Weierstrass M-test (since $|(\frac{w-a}{z-a})^j| = (\frac{|w-a|}{\rho})^j = M_j$ and $\sum M_j < \infty$). Therefore, by the above fact, we can interchange summation and integration to get

$$\int_{\partial D(a,\rho)} \frac{\mathrm{d}z}{z-w} = \sum_{j=0}^{\infty}(w-a)^j \int_{\partial D(a,\rho)} \frac{1}{(z-a)^{j+1}}\,\mathrm{d}z$$

Now for $j \geq 1$, the function $1/(z-a)^{j+1}$ has an anti-derivative $(= -\frac{1}{j(z-a)^j})$ in a neighbourhood of $\partial D(a,\rho)$, so by FTC, all integrals on the right for $j \geq 1$ are zero. For $j = 0$, by direct computation $\int_{\partial D(a,\rho)} \frac{1}{z-a}\,\mathrm{d}z = 2\pi i$. So $\int_{\partial D(a,\rho)} \frac{1}{z-w}\,\mathrm{d}z = 2\pi i$, and the proof of CIF is complete.
Taking $w = a$ in CIF, get $f(a) = \frac{1}{2\pi i}\int_{\partial D(a,\rho)} \frac{f(z)}{z-a}\,\mathrm{d}z/$ By direct computation using the parametrisation $t \mapsto a + \rho e^{2\pi i t},\ t \in [0,1]$, this gives $f(a) = \int_0^1 f(a+\rho e^{2\pi i t})\,\mathrm{d}t$

Remark. The result $f(a) = \int_0^1 f(a+\rho e^{2\pi i t})\,\mathrm{d}t$ is called the mean value property for holomorphic functions

For the proof above, we used the following fact:
if $\gamma : [a, b] \to \mathbb{C}$ is a curve and (f_n) is a sequence of continuous complex functions on image(γ) converging uniformly to a function f on image(γ) then $\int_\gamma f_n(z)\,\mathrm{d}z \to \int_\gamma f(z)\,\mathrm{d}z$. This is true because

$$\begin{aligned}\left|\int_\gamma f_n(z)\,\mathrm{d}z - \int_\gamma f(z)\,\mathrm{d}z\right| &= \left|\int_\gamma (f_n(z) - f(z))\,\mathrm{d}z\right| \\ &\leq \sup_{z\in\text{image}(\gamma)} |f_n(z) - f(z)|\text{length}(\gamma)\end{aligned}$$

Theorem 2.10 (Liouville's Theorem)**.** If $\mathbb{C} \to \mathbb{C}$ is entire (i.e. holomorphic everywhere on $\mathbb{C}$), and bounded (i.e. $|f(z)| \leq K$ for some fixed $K \geq 0$ and all $z \in \mathbb{C}$), then f is constant. More generally if f is entire with sub-linear growth (i.e. thre are constants $K \geq 0$ and $\alpha < 1$ such that $|f(z)| \leq K(1+|z|^\alpha)$ for all $z \in \mathbb{C}$), then f is constant

Proof. For any given $w \in \mathbb{C}$ and any $\rho > |w|$, we have by CIF that $f(w) = \frac{1}{2\pi i}\int_{\partial D(0,\rho)} \frac{f(z)}{z-w}\,\mathrm{d}z$ and

$$f(0) = \frac{1}{2\pi i}\int_{\partial D(0,\rho)} \frac{f(z)}{z}\,\mathrm{d}z$$

Thus

$$\begin{aligned}|f(w) - f(0)| &= \frac{1}{2\pi}\left|\int_{\partial D(0,\rho)} \frac{wf(z)}{z(z-w)}\,\mathrm{d}z\right| \\ &\leq \frac{|w|}{2\pi}\sup_{z\in\partial D(0,\rho)} \frac{|f(z)|}{|z||z|-|w||}\text{length}(\partial D(0,\rho)) \\ &\leq \frac{|w|K(1+\rho^\alpha)}{2\pi\rho(\rho-|w|)}2\pi\rho = \frac{|w|K(1+\rho^\alpha)}{\rho-|w|}\end{aligned}$$

Let $\rho \to \infty$ in this, keeping w fixed to conclude that $f(w) = f(0)$

Theorem 2.11 (Fundamental Theorem of Algebra)**.** Every non-constant polynomial with complex coefficients has a complex root

Proof. Let $p(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ be a complex polynomial of degree $n \geq 1$. Then $a_n \neq 0$, and for $z \neq 0$ we can write

$$p(z) = z^n\left(a_n + \frac{a_{n-1}}{z} + \cdots + \frac{a_0}{z^n}\right)$$

So by the triangle inequality

$$|p(z)| \geq |z|^n\left(|a_n| - \frac{|a_1|}{|z|} - \cdots - \frac{|a_0|}{|z|^n}\right)$$

This implies that we can find $R > 0$ such that $|p(z)| \geq 1$ for $|z| > R$ (in fact $|p(z)| \to \infty$ as $|z| \to \infty$).
Now if $p(z) \neq 0$ for all z, then $g(a) = 1/p(z)$ is entire. By the above, $|g(z)| \leq 1$ for $|z| > R$. By continuity of g, we also have that $|g(z)|$ bounded from above on the compact set $\{|z| \leq R\}$. thus g is a bounded entire function, so by Liouville's theorem g is constant. Since p is non-constant, this is impossible. Thus p must have a zero

Theorem 2.12 (Local maximimum modulus principle)**.** If $f : D(a, R) \to \mathbb{C}$ is holomorphic and if $|f(z)| \leq |f(a)|$ for all $z \in D(a, R)$, then f is constant

Proof. By the mean value property (Theorem 2.9), for any $\rho \in (0, R)$ we hae that

$$f(a) = \int_0^1 f(a + \rho e^{2\pi i t} \, \mathrm{d}t)$$

Therefore

$$|f(a)| = \left| \int_0^1 f(a + \rho e^{2\pi i t}) \, \mathrm{d}t \right| \leq \sup_{t \in [0,1]} |f(a + \rho e^{2\pi i t})| \leq |f(a)|$$

where the last inequality is by hypothesis. Thus both inequalities must be equality. Equality in the first inequality implies, by proposition 2.1 that $f(a + \rho e^{2\pi i t}) = c_p$ for some c_p and all $t \in [0, 1]$. But then by the first equality $|c_p| = |f(a)|$ for each $\rho \in (0, R)$. Thus $|f(a + \rho e^{2\pi i t})|$ is constant for all $\rho \in (0, R)$ and $t \in [0, 1]$, i.e. $|f(z)|$ is constant on $D(a, R)$. By the Cauchy-Riemann equations, it follows that f is constant.

We have seen that power series are a way to construct holomorphic functions on disks.
The next theorem says that in fact every holomorphic function on a disk arises this way

Theorem 2.13. Let $f : D(a,R) \to \mathbb{C}$ be holomorphic. Then f has a convergent series representation on $D(a,R)$. More precisely, there is a sequence of complex numbers $c_0, c_1, c_2 \ldots$ such that

$$f(w) = \sum_{n=0}^{\infty} c_n (w-a)^n$$

for all $w \in D(a,R)$. The coefficient c_n is given by

$$c_n = \frac{1}{2\pi i} \int_{\partial D(a,\rho)} \frac{f(z)}{(z-a)^{n+1}} \,\mathrm{d}z$$

for any $\rho \in (0,R)$

Proof. Let $0 < \rho < R$. Then for any $w \in D(a,\rho)$, we have that by CIF that

$$\begin{aligned} f(w) = \frac{1}{2\pi i} \int_{\partial D(a,\rho)} \frac{f(z)}{z-w} \,\mathrm{d}z \\ = \frac{1}{2\pi i} \int_{\partial D(a,\rho)} f(z) \sum_{n=0}^{\infty} \frac{w-a}{(z-a)^{n+1}} \,\mathrm{d}z \\ = \sum_{n=0}^{\infty} \left(\frac{1}{2\pi i} \int_{\partial D(a,\rho)} \frac{f(z)}{(z-a)^{n+1}} \right) \mathrm{d}z (w-a)^n \end{aligned}$$

The last equality is true by uniform convergence of the series $\sum_{n=0}^{\infty} \frac{w-a}{(z-a)^{n+1}}$ for $z \in \partial D(a,\rho)$. Write (temporarily) $c_n(\rho) = \frac{1}{2\pi i} \int_{\partial D(a,\rho)} \frac{f(z)}{(z-a)^{n+1}} \,\mathrm{d}z$. Then we have shown that $f(w) = \sum_{n=0}^{\infty} c_n(\rho)(w-a)^{n+1}$ for $w \in D(a\rho)$.
So by Theorem 1.5, the function f (being given by a power series in $D(a,\rho)$) has derivatives of all orders in $D(a,\rho)$, and the coefficient $c_n(\rho) = \frac{f^{(n)(a)}}{n!}$. In particular, $c_n(\rho)$ is independent of ρ, so call it c_n. Since $\rho \in (0,R)$ is arbitrary, we then have that

$$f(w) = \sum_{n=0}^{\infty} c_n (w-a)^n$$

for all $w \in D(amR)$, where

$$c_n = \frac{f^{(n)}(a)}{n!} = \frac{1}{2\pi i} \int_{\partial D(a,\rho)} f(z)/(z-a)^{n+1} \,\mathrm{d}z$$

for any $\rho \in (0,R)$

Corollary 2.14 (Higher order differentiability)**.** If f is holomorphic on an open set $U \subset \mathbb{C}$, then f has derivatives of all orders in U which are themselves holomorphic on U

Proof. f has a power series representation near every point, so its derivatives of all orders exist everywhere This also of course means that the derivatives of all orders are holomorphic

Remarks.

(i) Suppose $D(a, R) \subset U$. In the proof of Theorem 2.13, we have established the formula valid for any $\rho \in (0, R)$. This is a special case of a Cauchy integral formula for derivatives (Theorem 2.16 below)

(ii) Taking $n = 1$ in the above formula and estimating, we get

$$|f'(a)| \leq \frac{1}{\rho}\left(\sup_{z\in\partial D(a,\rho)} |f(z)|\right)$$

This estimate can be thought of as a "localisation of Liouville's Theorem," and it directly implies Liouville's theorem: if f is entire and bounded, we can choose any $a \in \mathbb{C}$, apply the estimate and let $\rho \to \infty$ to conclude that $f' = 0$ on $\mathbb{C}$ and hence f is constant

(iii) A funtion f (real or complex) is said to be analytic at a point a if, in a neighbourhood of a, f is given by a convergent power series about a. We know that f analytic at $a \implies f$ has derivatives of all orders near a.
Corollary 2.14 says that if f is a complex function, then: f is analytic at $a \iff f$ has complex derivatives of all orders in a neighbourhood of $a \iff f$ is complex differentiable once in a neighbourhood of a (i.e. f is holomorphic at a).
For real functions, existence of derivatives of all orders $\not\Longrightarrow$ analyticity. (e.g. $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = e^{-1/x^2}$ for $x \neq 0$ and $f(0) = 0$; this function is differentiable to any order and has $f^{(n)} = 0$ for all n, so f is not given by a conergent power series near 0.) From now on we use "analytic" and "holomorphic" interchangably.

(iv) Let $U \subset \mathbb{C}$ be open. We now have (from Corollary 2.14) that $f = u + iv$ is holomorphic in $U \iff u, v$ have continuous (first) partial derivatives in U (i.e. u, v are C^1 in U) and u, v satisfy the Cauchy-Riemann equations

(v) Corollary 2.14 provides C^2 regularity of u and v if $f = u + iv$ is holomorphic in U. So we now have fully justified that real and imaginary parts of a holomorphic function are harmonic functions

Corollary 2.14 (higher order differentiability) leads to the following very useful integral criteroion for holomorphicity of a continuous function

Theorem 2.15 (Morera's Theorem)**.** Let $U \subset \mathbb{C}$ be open. If $f : U \to \mathbb{C}$ is continuous and $\int_\gamma f(z) = 0$ for every clused curve γ in U, then f is holomorphic in U

Proof. By Theorem 2.5, f has an antiderivative F on U. Such F is of course holomorphic. By corollar 3.14 then F is twice differentiatble in U. Since $F' = f$, this means that f is holomorphic

Corollary 2.16. Let $U \subset \mathbb{C}$ be open. If $U \to \mathbb{C}$ is continuous and holomrophic in $U \backslash S$ where S is a finite set, then f is holomorphic in U

Proof. For each $a \in U$, there is $r > 0$ such that $D = D(a, r) \subset U$. Since D is convex, we can apply Corollary 2.8 (convex Cauchy) to see that $\int_\gamma f(z)\,\mathrm{d}z = 0$ for any closed curve in D. By Morera's theorem f is holomorphic in D.

As a further application of Taylor series, we next show that zeros of a non-zero holomorphic function are isolated points.
Suppose f is a non-zero holomorphic function on a disk $D = D(a, R)$. Then by the Taylor series theorem, there are constants c_n such that

$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n \text{ for all } z \in D$$

Since $f \not\equiv 0$ in D, there is n such that $c_n \neq 0$. Let $m = \min\{n : c_n \neq 0\}$ Then

$$f(z) = (z-a)^m g(z)$$

where $g(z) = \sum_{n=m}^{\infty} c_n (z-a)^{n-m}$. The function g is holomorphic on D since it is given by a power series, and $g(a) = c_m \neq 0$.

Notation. If $m \geq 1$, we say that f has a zero of order m at $z = a$. From the formula $c_n = f^{(n)}(a)/n!$, it is clear that m is the smallest of the positive integers n such that $f^{(n)}(a) \neq 0$

Note. Recall that if $S \subset \mathbb{C}$, then a point $w \in S$ is an isolated point of S if there is $r > 0$ such that $S \cap D(w, r) = \{w\}$

Theorem 2.17 (Principle of isolated zeros)**.** Let $f : D(a, R) \to \mathbb{C}$ be holomrophic and not identically zero. Then there is r with $0 < r \leq R$ such that $f(z) \neq 0$ whenever $0 < |z-a| < r$

Proof. If $f(a) \neq 0$, then by continuity of f we can fund $r > 0$ such that $f(z) \neq 0$ for $z \in D(a, r)$ and we are done.
If $f(a) = 0$, then by preceding discussion, there is an integer $m \geq 1$ such that $f(z) = (z-a)^m g(z)$ for $z \in D(a, R)$, where g is holomorphic with $g(a) \neq 0$.
So again we find $r > 0$ such that $g(z) \neq 0$ for $z \in D(a, r)$, and hence $f(z) \neq 0$ for $z \in D(a, r) \backslash \{a\}$

Remarks.

(i) If $f(a) = 0$, the theorem says that $\{z : f(z) = 0\} \cap D(a, r) = \{a\}$, i.e. that a is an isolated point of the zero set (unless $f \equiv 0$). So e.g. there is no non-zero holomorphic function vanishing on a line segment or a half-disk

(ii) This theorem allows us to csee that certain familar identities from real analysis hold for complex functions: e.g. $\sin^2 z + \cos^2 z = 1$. $g(z) = \sin^2 + \cos^2 z - 1$ is holomorphic and vanishes on the real line. So $g \equiv 0$ by the principle of isolated zeros

(iii) It is possible that the zero set may have an accumulation point on the boundary of the domain of f. Consider for example $f(z) = \sin 1/z,\ z \in D(1, 1)$. If $a_n = \frac{1}{2n\pi},\ n = 1, 2, 3, \ldots,$ then $a_n = \frac{1}{2n\pi} \in D(1, 1),\ f(a_n) = 0$ and $a_n \to 0 \in \partial D(1, 1)$

2.1 Unique Continuation for analytic functions

Consider a holomrphic function $f : D(a,r) \to \mathbb{C}$ defined on a disk. We know by Taylor series theorem, f is uniquely determined by its values in any arbitrary small disk $D(a,\rho) \subset D(a,r)$ (because the Taylor series coefficient $c_n = \frac{f^{(n)}(a)}{n!}$ for $n = 0,1,2,\dots$)
This can be generalised to arbitrary domains

Theorem 2.18 (Unique Continuation for analytic functions)**.** Let U, V be domains such that $U \subset V$. If $g_1, g_2 : V \to \mathbb{C}$ are analytic and $g_1 = g_2$ on U, then $g_1 = g_2$ on V.
Equivalently, if $f : U \to \mathbb{C}$ is analytic, then there is at most one analytic function $g : V \to \mathbb{C}$ such that $g = f$ on U

Proof. Let $g_1, g_2 : V \to \mathbb{C}$ be analytic with $g_1|_U = g_2|_U$. Then $h = g_1 - g_2 : V \to \mathbb{C}$ is analytic and $h(z) = 0$ on U. We want to show that $h \equiv 0$ on V.
Let
$$V_0 = \{z \in V : h \text{ is identically zero in some open disk } D(z,r),\ r > 0\}$$
and
$$V_1 = \{z \in V : h^{(n)(z) \neq 0 \text{ for some } n \geq 0}\}$$
Let $z \in V$ and suppose $z \notin V_0$. Then for any disk $D = D(z,r) \subset V$, we have that $h \not\equiv 0$ in D. Hence by Taylor series, $h^{(n)(z)} \neq 0$ for some n, so $z \in V_1$. Thus $V = V_0 \cup V_1$.
We also have that $V_0 \cap V_1 = \varnothing$. Moreover, V_0 is open by definition, and V_1 is open by continuity of the derivatives $h^{(n)}$. Hence by connectedness of V, one of V_0 or V_1 must be empty. But $U \subset V_0$, so $V_1 = \varnothing$. Hence $V = V_0$. Therefore $h = 0$ on V

Notation. Given analytic $f : U \to \mathbb{C}$, f a function g as inthe theorem exists, it is called the analytic continuation of f to V

Remark. The above proof relies on analyticity of h, i.e. the property of having a convergent Taylor series about every point. So the theorem holds for real analytic functions aswell. E.g. for harmonic functions (which are C^2 to begin with, but are automatically real analytic by verture of "elliptic regularity;" see Part II, Analysis of Functions)

Given a holomorphic function f on a disk, we can ask for the largest domain containing the disk to which there is an analytic continuation of f.
In general this is a difficult quesiton to answer, but we can illustrate it with some not so difficult examples

Example. $f(z) = \sum_{n=0}^{\infty} z^n$. This sereis has r.o.c. $= 1$, os f is analytic in $D(0,1)$, and there is no larger disk $D(0,r) \supset D(0,1), r > 1$, such that g has ana alytic continuation to $D(0,r)$. (If there is then r.o.c. > 1.) However, since $f(z) = 1/(1-z)$ for $z \in D(0,1)$, and the function $1/(1-z)$ is analytic in $\mathbb{C}\backslash\{1\}$, f does have analytic continuation to a domain containing $D(0,1)$, namely to $\mathbb{C}\backslash\{1\}$.

Example. $f(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} z^n}{n}$. Again r.o.c. $= 1$, so f is analytic on $D(0,1)$. It has the analytic continuation $\text{Log}(1+z)$ to the domain $\mathbb{C} \backslash \{x \in \mathbb{R} : x \leq -1$ containing $D(0,1)$.
Analytic continuation to a larger domain need not always exist

Example. $f(z) = \sum_{n=0}^{\infty} z^{n!}$. The power series has r.o.c $= 1$. So f is analytic in $D(0,1)$. But f has no analytic continuation to any larger domain containing $D(0,1)$. See example sheet 2. ($\partial D(0,1)$ is called the "natural boundary" of f)

Note. The non-extendability pheneomenon illustrated by this example is not special to functions defined by power series on disks; it is in fact unavoidable in the following sense: for any given domain $U \subset \mathbb{C}$, there is a holomorphic funtion $f : U \to \mathbb{C}$ which has no analytic continuation to a domain properly containing U. (See ES2)

Remark. The failure of analytic continuation in some cases can be explained as the result of loss of "regularity" (i.e. boundedness, continuity, differentiability etc) on approach to the boundary of the domain (as in third example above); however, this is not alyways the reason, and analytic continuation may still fail even when the function is well behaved up to the boundary in terms of regularity. This is quite remarkable at a first glace since holomrophicity after all is a regularity requirement. However, it is not so surprising form a PDE theoretic point of view, i.e. if one looks at analytic continuation as extendability of real functions as solutions ot a set of PDEs

Example. $f(z) = \sum_{n=0}^{\infty} e^{2^{n/2}} z^{2^n}$. This has r.o.c. $= 1$. The function and its derivatives of any order are uniformly continuous on the closed disk $\overline{D(0,1)}$ (as can be seen by checking uniform convergence of the corresponding series on the closed disk). However, by the following theroem (which we don't prove), this has natural boundary $\partial D(0,1)$: Ostrowski-Hadamard gap theorem: Let (p_n) be a sequence of positive integers suhc that $p_{n+1} > (1+\delta)p_n$ for all n and some fixed $\delta > 0$. If (c_n) is a sequence of complex numbers such that the power series $f(z) = \sum_{n=0}^{\infty} c_n z^{p_n}$ has r.o.c. $= 1$, then $\partial D(0,1)$ is the natural boundary of f.

Recall: $w \in S$ is a non-isolated point of S if for every $\varepsilon > 0$, $S \cap (D(w,\varepsilon) \backslash \{w\}) \neq \varnothing$

Corollary 2.19. Let $f, g : U \to \mathbb{C}$ be holomorphic in a domain U. If the set $S = \{z \in U : f(z) = g(z)\}$ contains a non-isolated point, then $f = g$ in U

Proof. Let $h = f - g$ os that $S = \{z \in U : h(z) = 0\}$. Suppose S has a non-isolated point w. If for some $r > 0$ the function h is not identically zero in $D(w,r)$, then by the principle of isolated zeros (Theorem 2.17) we can find $\varepsilon > 0$ s.t. $f(z) \neq 0$ whenever $0 < |z - w| < \varepsilon$, i.e. $S \cap D(w,\varepsilon) \backslash \{w\} = \varnothing$. This directly contradicts the assumption that w is a non-isolated point of S. Therefore $h \equiv 0$ on $D(w,r)$ for every $D(w,r) \subset U$. Theorem 2.18 then says that $h \equiv 0$ on U, i.e. $f = g$ on U

Corollary 2.20 (Global maximum principle)**.** Let U be a bounded open set. Let $\overline{U}$ be the closure of U, and suppsoe that $f : \overline{U} \to \mathbb{C}$ is a continuous function such that f is holomorphic in U. Then $|f|$ attains its maximum on $\partial U = \overline{U} \backslash U$

Proof. $\overline{U}$ is a closed, bounded subset of $\mathbb{R}^2$, and $|f|$ is continuous on $\overline{U}$. so there is a point $w \in \overline{U}$ such that $|f(w)| = \max_{z \in \overline{U}} |f(z)|$. If $w \notin U$, then $w \in \partial U$ and we are done. So suppose that $w \in U$, and choose a disk $D = D(w, r) \subset U$. Since $|f(z)| \leq |f(w)|$ for all $z \in D$, it follows from the local maximum principle (Theorem 2.12) that $f = c$ on D for some constant c. Hence by the identity principle (applied with $g = c$), $f = c$ on the connected component of U containing D. If this component is U', then by continuity we have that $f = c$ on $\overline{U'}$. In particular $|f(z)| = |c| = |f(w)|$ for any $z \in \partial U' \subset \partial U$, so the conlusion holds again

Next we turn to the question of limits of sequences of holomorphic functions. We can consider limits in various topologgies, but uniform limits are an important and natural first question to study. We'd like to understand differentiablity of the limits, so we start by deriving a respresentation formula for the derivatives

Theorem 2.21 (CIF for derivatives and Cauchy estimates)**.** (i) Let $f : D(a,R) \to \mathbb{C}$ be holomorphic. If $f^{(n)}$ denotes the n-th derivative, then for any $\rho \in (0,R)$ and any $w \in D(a,\rho)$, we have that

$$f^{(n)}(w) = \frac{n!}{2\pi i}\int_{\partial D(a,\rho)} \frac{f(z)}{(z-w)^{n+1}}\,\mathrm{d}z$$

(ii)

$$\sup_{z\in D(a,R/2)} |f^{(k)}(z)| \leq \frac{C}{R^k}\sup_{z\in D(a,R)} |f(z)|$$

where $C = C(k)$ is a constant depending only on k; in fact we can take $C = k!2^{k+1}$

Proof. (i) Case $k = 0$ is the usual CIF. For the case $k = 1$, consider $g(z) = f(z)/(z-w)$. This is holomorphic in $D(a,R)\backslash\{w\}$, with derivative

$$g'(z) = \frac{f'(z)}{z-w} - \frac{f(z)}{(z-w)^2}$$

Since $\partial D(a,\rho) \subset D(a,R)\backslash\{w\}$, $\int_{\partial D(a,\rho)} g'(z)\,\mathrm{d}z = 0$ by FTC. So

$$\int_{\partial D(a,\rho)} \frac{f'(z)}{z-w}\,\mathrm{d}z = \int_{\partial D(a,\rho)} \frac{f(z)}{(z-w)^2}\,\mathrm{d}z$$

By the usual CIF formula applied to f'

$$f'(w) = \frac{1}{2\pi i}\int_{\partial D(a,\rho)} \frac{f'(z)}{z-w}\,\mathrm{d}z$$

Combining these gives the result for $k = 1$.
For general $k \geq 2$, use this idea plus induction on k. So fix $k \geq 2$, and suppose the formula is valid (i.e. $f^{(k)}(w) = \frac{k!}{2\pi i}\int_{\partial D(a,\rho)} \frac{f(z)}{(z-w)^{k+1}}\,\mathrm{d}z$) for that k and all holomorphic $f : D(a,R) \to \mathbb{C}$ (induction hypothesis)
Given any holomorphic $f : D(a,R) \to \mathbb{C}$, consider $g(z) = \frac{f(z)}{(z-w)^{k+1}}$ which has derivative

$$g'(z) = \frac{f'(z)}{(z-w)^{k+1}} - \frac{(k+1)f(z)}{(z-w)^{k+2}}$$

in $D(a,R)\backslash\{w\}$. Since

$$\int_{\partial D(a,\rho)} g'(z)\,\mathrm{d}z = 0$$

we get

$$\int_{\partial D(a,\rho)} \frac{f'(z)}{(z-w)^{k+1}}\,\mathrm{d}z = (k+1)\int_{\partial D(a,\rho)} \frac{f(z)}{(z-w)^{k+2}}$$

But by the induction hypothesis with f' in place of f

$$f^{(k+1)}(w) = \frac{k!}{2\pi i}\int_{\partial D(a,\rho)} \frac{f'(z)}{(z-w)^{k+1}}\,\mathrm{d}z$$

Combining the preceding two expressions,

$$f^{(k+1)}(w) = \frac{(k+1)!}{2\pi i}\int_{\partial D(a,\rho)} \frac{f(z)}{(z-w)^{k+2}}\,\mathrm{d}z$$

This completes the induction step and the proof of part (i)

(ii) We may assume that $\sup_{z\in D(a,R)} |f(z)| < \infty$ (else there is nothing to prove). Pick any $\rho \in (R/2, R)$. Then by part (i), for any $w \in D(,R/2)$, we have that

$$|f^{(k)}(w)| \leq \frac{k!}{2\pi}\left(\sup_{z\in\partial D(a,\rho)} \frac{|f(z)|}{|z-w|^{k+1}}\right)\text{length}(\partial D(a,\rho))$$

Since $|z-w| \geq \rho - R/2$ for any $z \in \partial D(a,\rho)$ and any $w \in D(a,R/2)$, this implies

Remarks.

(i) If we directly apply CIF to $f^{(n)}$ (can do this as $f^{(n)}$ is holomorphic), we get a formula for $f^{(n)}(w)$ in terms of an integral involving $f^{(n)}$. The significance of Theorem 2.21 is that right hand side involves only f, and not any of its derivatives

(ii) We have already seen the special case $w = a$ of this formula in the proof of Taylor series (see Remark 1 after Corollary 2.14)

2.2 Uniform Limits of Holomorphic Functions

Definition. Let $U \subset \mathbb{C}$ be open, and let $f_n : U \to \mathbb{C}$ be a sequence of functions. We say that (f_n) converges **locally uniformly on U** if for each $a \in U$, there is $r > 0$ such that (f_n) converges uniformly on $D(a, r)$.

Example. $f_n(z) = z^n$. Then $f_n \to 0$ locally uniformly on $D(0, 1)$, but not uniformly on $D(0, 1)$. In fact $f_n \to 0$ uniformly on any disk $D(0, r)$ if $r < 1$. But $\sup_{z \in D(0,1)} |f_n(z)| = 1$ for each n

Prop 2.22. (f_n) converges locally uniformly on $U \iff (f_n)$ converges uniformly on each compact subset $K \subset U$

Proof. "$\implies$": A straightforward exercise using the definition of compactness (every open cover has a finite subcover)
"$\impliedby$": Clear since $\forall a \in U$, $\exists$ compact disk $\overline{D(a, r)} \subset U$

Theorem 2.23 (Uniform limit of holomorphic functions)**.** Let $U \subset \mathbb{C}$ be open, and $f_n : U \to \mathbb{C}$ be holomorphic for each $1, 2, 3, \dots$. If (f_n) converges locally uniformly on U ofr some function $f : U \to \mathbb{C}$, then f is holomorphic. Moreover, $f_n' \to f'$ locally uniformly on U. (By applying this iteratively, we get that for each k, $f_n^{(k)} \to f^{(k)}$ locally uniformly on U as $n \to \infty$)

Proof. The first part is an immediate consequence of convex Cauchy and Morera theorems. Indeed, let $a \in U$, and choose $r > 0$ such that $\overline{D(a,r)} \subset U$ and $f_n \to f$ uniformly on $D(a,r)$. Since f_n are continuous, by a result from Analysis & Topology, we have that f is continuous in $\overline{D(a,r)}$.
Let γ be a closed curve in $D(a,r)$.
Since $D(a,r)$ is convex, we have by "convex Cauchy" $\int_\gamma f_n(z)\,\mathrm{d}z = 0$.
Since $f_n \to f$ uniformly on $D(a,r)$, it follows that

$$\int_\gamma f(z)\,\mathrm{d}z - \lim_{n\to\infty} \int_\gamma d_n(z)\,\mathrm{d}z = 0$$

Since f is continuous and γ is an arbitrary closed curve, by Morera's theorem (Theorem 2.15) f is holomorphic in $D(a,r)$. Since a is arbitrary f is holomrophic on U.
To see that $f_n' \to f'$ locally uniformly on U, let $a \in U$ be arbitrary and let $D(a,r)$ be as above (so $f_n \to f$ uniformly on $\overline{D(a,r)}$). Apply the Cauchy estimate (Theorem 2.21 (ii)) with $k = 1$ (the first derivative), $R = r$ and with $f_n - f$ in place of f. This gives

$$\sup_{z\in D(a,r/2)} |f_n'(z) - f'(z)| \leq \frac{4}{r} \sup_{z\in D(a,r)} |f_n(z) - f(z)|$$

Since the $RHS \to 0$ as $n \to \infty$, the claim follows.

Remark. This result spectacularly fails for real functions, as seen by the following theorem:
Weierstrass approximation theorem: Let $f : [a,b] \to \mathbb{R}$ be a continuous function on a compact interval $[a,b] \subset \mathbb{R}$. Then there is a sequence of polynomials (p_n) converging uniformly to f on $[a,b]$ (see Part II, Linear Analysis).
There exist continuous nowhere differentiable functions $f : [a,b] \to \mathbb{R}$. (IB Analysis & Topology). Applying the Weierstrass approximation theorem to such f shows that the uniform limit of real analytic functions need not have a single point of differentiability

Aside: links with the theory of harmonic functions:
Some of the key results we have proves for holomorphic functions have direct analogues for (real) harmonic function son domains not just in $\mathbb{R}^2$ but in $\mathbb{R}^n$ for any n. For example:

- if $u : \mathbb{R}^n \to \mathbb{R}$ is a bounded harmonic function, then u is constant (Liouville's theoerm)
- if $u : D = D(a, r) \to \mathbb{R}$ is a C^2 harmonic function on an open ball $D(a, r) = \{x \in \mathbb{R}^n : \|x - a\| < r\}$ and if $u(x) \le u(a)$ (or $u(x) \ge u(a)$) for all $x \in D$, then u is constant (local maximum principle, also known as the strong maximum principle)
- global maximum principle also holds (also known as the weak maximum principle): a harmonic function on a bounded open set U that is conitnuous on $\overline{U}$ attains its maximum (and its minimum) on the boundary $\partial U = \overline{U} \setminus U$
- harmonic functions are real analytic
- unique continuation principle holds (follows from analyticity exactly as in the case of holomorphic functions)
- uniform limits of harmonic functions are harmonic
- derivative estimates hold: if $u : D(a, R) \subset \mathbb{R}^n \to \mathbb{R}$ is harmonic, then $\sup_{D(a,R/2)} |D^k u| \le CR^{-k} \sup_{D(a,R)} |u|,\ C = C(n, k)$

When $n = 2$, results for harmonic functions often can be deduced from the corresponding results for holomorphic funtions.
For instance, for the Liouville theorem, given u a harmonic function on $\mathbb{R}^2$, find a function $f : \mathbb{R}^2 \to \mathbb{R}$ (the harmonic conjugate of u) such that $f = u + iv$ is holomorphic on $\mathbb{C}$ [Warning: a harmonic conjugate need not always exist; it does exist if the domain is simply connected (ex. sheet 3).] Then $g = e^f$ is holomorphic with $|g| = e^u$; so if u is bounded then g is bounded, so by Liouville for holomorphic functions, g and hence f is constant. The proof of Liouville's theorem for harmonic functions in higher dimensions will have to be different.
Exercise (ex. sheet 3): give a Complex Analysis proof of the derivative estimate for harmonic $u : D(a, R) \subset \mathbb{R}^2 \to \mathbb{R}$

3 Complex Integration: Part II

Recall the version of Cauchy's theorem we have proved: if U is a star-shaped domain, then $\int_\gamma f(z)\,\mathrm{d}z = 0$ for any holomorphic $f : U \to \mathbb{C}$ and any closed curve in U ("convex Cauchy").
There are domains for which the conclusion of Cauchy's theorem fails for some holomorphic functions and some closed curves (e.g. $U = \mathbb{C} \setminus \{0\}$ then $\int_{\partial D(0,1)} \frac{\mathrm{d}z}{z} = 2\pi i$)
Next goal:

(i) For a given domain, characterise the closed curves in it for which Cauchy's theorem holds for all holomorphic functions
(ii) Use this characterisation to enlarge the class for which Cauchy's theorem holds (for all holomorphic functions and all closed curves).

To do this, we will utilise a notion called the winding number of a closed curve about a point not in its image. Informally, this is the number of times the curve γ "winds around" the point.
Let $\gamma : [a, b] \to \mathbb{C}$ be a closed (piecewise C^1) curve, and let $w \notin \text{Image}(\gamma)$.
For each t, there is $r(t) > 0$ and $\theta(t) \in \mathbb{R}$ such that $\gamma(t) = w + r(t)e^{i\theta t}$. This is true because $\gamma(t) - w \ne 0$ for all t.
Then the function $r : [a, b] \to \mathbb{R}$ is given by $r(t) = |\gamma(t) - w|$, so it is uniquely determined (by γ and w), and is piecewise C^1

Definition. If we have a continuous choice of $\theta : [a, b] \to \mathbb{R}$ such that $\gamma(t) + r(t)e^{i\theta(t)}$, then define the **winding number** or the **index** of γ about w as

$$l(\gamma; w) = \frac{\theta(b) - \theta(a)}{2\pi}$$

Remark. $l(\gamma; w)$ is an integer $((\gamma(a) = \gamma(b))$, so $r(a) = r(b)$ and $e^{i(\theta(b)-\theta(a))} = 1$; hence $\theta(b) - \theta(a) = 2\pi n$ for some $n \in \mathbb{Z}$.)
If $\theta_1 : [a, b] \to \mathbb{C}$ is also a continuous function such that $\gamma(t) = w + r(t)e^{i\theta_1(t)}$ then, $e^{i(\theta(t)-\theta_1(t))} = 1$ so $\frac{\theta_1(t)-\theta(t)}{2\pi} \in \mathbb{Z}$. Since $\theta_1 - \theta$ is continuous, it must be constant. Hence $l(\gamma; w)$ is well defined, independent of the (continuous) choice of θ

Lemma 3.1. If $w \in \mathbb{C}, \gamma : [a,b] \to \mathbb{C}\backslash\{w\}$ is a piecewise C^1 curve, then there exists a piecewise C^1 function $\theta : [a,b] \to \mathbb{R}$ such that $\gamma(t) = w + r(t)e^{i\theta(t)}$ where $r(t) = |\gamma(t) - w|$. Moreover, if γ is closed, then

$$l(\gamma; w) = \frac{1}{2\pi i}\int_\gamma \frac{\mathrm{d}z}{z - w}$$

Motivation: if γ is C^1 and there is a C^1 function θ such that $\gamma(t) = w + r(t)e^{i\theta(t)}$, then $\gamma'(t) = (r'(t) + ir(t)\theta'(t))e^{i\theta(t)} = \left(\frac{r'(t)}{r(t)} + i\theta'(t)\right) r(t)e^{i\theta(t)} = \left(\frac{r'(t)}{r(t)} + i\theta'(t)\right)(\gamma(t) - w)$. Hence

$$\theta'(t) = \mathrm{Im}\frac{\gamma'(t)}{\gamma(t) - w}$$

and so

$$\theta(t) = \theta(a) + \mathrm{Im}\int_a^t \frac{\gamma'(s)}{\gamma(s) - w}\,\mathrm{d}s$$

Proof. Let $h(t) = \int_a^t \frac{\gamma'(s)}{\gamma(s)-w}\,\mathrm{d}s$. The integrand is bounded on $[a,b]$, and is continuous except at the finite number of points where γ' may be discontinuous. So $h : [a,b] \to \mathbb{C}$ is continuous. Moreover, h is differentiable with $h'(t) = \frac{\gamma'(t)}{\gamma(t)-w}$ at each t where γ' is continuous (so for all $t \in [a,b]$ except possibly for a finite set). This also shows that h is piecewise C^1. Thus we have an ode for $(\gamma(t) - w)$ in the form

$$(\gamma(t) - w)' - (\gamma(t) - w)h'(t) = 0$$

valid for $t \in [a,b]$ except possibly for a finite set. This says that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left((\gamma(t) - w)e^{-h(t)}\right) = \gamma'(t)e^{-h(t)} - (\gamma(t) - w)e^{-h(t)}h'(t) = 0$$

except possibly for finitely many t's.
Hecne $(\gamma(t) - w)e^{-h(t)}$ is continuous, it must be constant, and equal to its value at $t = a$. So $\gamma(t) - w = (\gamma(a) - w)e^{h(t)} = (\gamma(a) - w)e^{\mathrm{Re}h(t)} \cdot e^{i\mathrm{Im}h(t)} = |\gamma(a) - w|e^{\mathrm{Re}h(t)}e^{i(\alpha + \mathrm{Im}h(t))}$ for a choice of α such that

$$e^{i\alpha} = \frac{\gamma(a) - w}{|\gamma(a) - w|}$$

So just set $\theta(t) = \alpha + \mathrm{Im}h(t)$.
To see the second part, not that

$$l(\gamma; w) = \frac{\theta(b) - \theta(a)}{2\pi} = \frac{\mathrm{Im}(h(b) - h(a))}{2\pi} = \frac{\mathrm{Im}h(b)}{2\pi}$$

Since $\gamma(t) - w = (\gamma(a) - w)e^{h(t)}$ and $\gamma(b) = \gamma(a)$, we have $e^{h(b)} = 1$ so $\mathrm{Re}h(b) = 0$ and $\mathrm{Im}h(b) = h(b)/i$. Hence

$$l(\gamma; w) = \frac{1}{2\pi i}h(b) = \frac{1}{2\pi i}\int_a^b \frac{\gamma'(s)}{\gamma'(s) - w}\,\mathrm{d}s = \int_\gamma \frac{\mathrm{d}z}{z - w}$$

Remark. The 'continuous version' of the first part of the lemma is also true. That is if γ is merely continuous, then it is true that there is a continuous θ such that $\gamma(t) = w + r(t)e^{i\theta(t)}$. We do not need this, so will omit its proof. (The formula for $l(\gamma; w)$ is not meaningful for continuous γ)

Prop 3.2. If $\gamma : [a,b] \to \mathbb{C}$ is a closed curve then the function $w \mapsto l(\gamma; w)$ is continuous on $\mathbb{C}\backslash\text{Image}(\gamma)$. Hence (since $l(\gamma;w)$ is integer-valued), $l(\gamma;w)$ is locally constant, or equivalently, constant on each connected component of (the open set) $\mathbb{C}\backslash\text{Image}(\gamma)$

Proof. Exercise. Use the formula for $l(\gamma;w)$. For a different proof, see ES2, Q11

Prop 3.3. (i) If $\gamma : [a,b] \to D(z_0,R)$ is a closed curve, then $l(\gamma;w) = 0$ for any $w \in \mathbb{C}\backslash D(z_0,R)$

(ii) If $\gamma : [a,b] \to \mathbb{C}$ is a closed curve, then there is a unique unbounded connected component Ω of $\mathbb{C}\backslash\gamma([a,b])$, and $l(\gamma;w) = 0$ for all $w \in \Omega$

Proof. (i) If $w \in \mathbb{C}\backslash D(z_0,R)$, then the function $z \mapsto \frac{1}{z-w}$ is holomorphic in $D(z_0,R)$. So $l(\gamma;w) = \frac{1}{2\pi i}\int_\gamma \frac{\mathrm{d}z}{z-w} = 0$ by convex Cauchy.

(ii) Since $\gamma([a,b]) \subset \mathbb{C}$ is compact (by continuity of γ), there is a $R > 0$ such that $\gamma([a,b]) \subset D(0,R)$. Since $\mathbb{C}\backslash D(0,R)$ is a connected subset of $\mathbb{C}\backslash\gamma([a,b])$, there is a component Ω of $\mathbb{C}\backslash\gamma([a,b])$ such that $\mathbb{C}\backslash D(0,R) \subset \Omega$, so is contained in $D(0,R)$ and hence is bounded. So the unbounded component is unique. Since $l(\gamma;w)$ is locally constant and zero on $\mathbb{C}\backslash D(0,R)$ (by (i)), it is zero on Ω.

We will soon need the following lemma for the proof of the general Cauchy theorem

Lemma 3.4. Let $f : U \to \mathbb{C}$ be holomorphic, and define $g : U \times U \to \mathbb{C}$ by

$$g(z,w) = \begin{cases} \frac{f(z)-f(w)}{z-w} & z \neq w \\ f'(w) & z = w \end{cases}$$

Then g is continuous. Moreover, if γ is a closed curve in U, then the function $h(w) = \int_\gamma g(z,w)\,\mathrm{d}z$ is holomorphic on U

Proof. Continuity of g at (z,w) is clear if $z \neq w$. To check continuity at $(a,a) \in U \times U$, pick any $\varepsilon > 0$ and choose $\delta > 0$ such that $D(a,\delta) \subset U$ and $|f'(z) - f'(a)| < \varepsilon$ for all $z \in D(a,\delta)$ (possible by continuity of f').
Let $z,w \in D(a,\delta)$. If $z = w$, then

$$|g(z,w) - g(a,a)| = |f'(z) - f'(a)| < \varepsilon$$

If $z \neq w$, we have $tz + (1-t)w \in D(a,\delta)$ for $t \in [0,1]$ (by convexity of $D(a,\delta)$). So

$$\begin{aligned} f(z) - f(w) &= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t} f(tz + (1-t)w)\,\mathrm{d}t \\ &= \int_0^1 f'(tz + (1-z)w)(z-w)\,\mathrm{d}t \\ &= (z-w)\int_0^1 f'(tz + (1-t)w)\,\mathrm{d}t \end{aligned}$$

Proof (continued)**.** Thus

$$\begin{aligned} |g(z,w)-g(a,a)| &= \left| \frac{f(z)-f(w)}{z-w} - f'(a) \right| \\ &= |\int_0^1 (f'(tz+(1-t)w) - f'(a))\,\mathrm{d}t \\ &\le \sup_{t\in[0,1]} |f'(tz+(1-t)w)0f'(a)| < \varepsilon \end{aligned}$$

So we've shown that $|(z,w)-(a,a) < \delta \implies |g(z,w)-g(a,a)| < \varepsilon$, i.e. g is continuous at (a,a).
To show h is holomorphic, first check h is continuous. So fix $w_0 \in U$ and suppose that $w_n \to w_0$. Choose $\delta > 0$ such that $D(w_0,\delta) \subset U$. The function g is continuous on $U \times U$, so it is uniformly continuous on the compact subset $\text{Image}(\gamma) \times D(w_0,\delta) \subset U \times U$. This means that if we let $g_n(z) = g(z,w_n)$ and $g_0(z) = g(z,w_0)$ for $z \in \text{Image}(\gamma)$, then $g_n \to g_0$ uniformly on $\text{Image}(\gamma)$. So $\int_\gamma g_n(z)\,\mathrm{d}z \to \int_\gamma g_0(z)\,\mathrm{d}z$, i.e. $h(w_n) \to h(w_0)$. Thus h is continuous.
Now use convex Cauchy and Morera theorems to check h is holomorphic on U. Specifically, given $w_0 \in U$ choose disk $D(w_0,\delta) \subset U$. Suppose that γ is parametrized over $[a,b]$ and let $\beta : [c,d] \to D(w,\delta)$ be any closed curve. Then $h(w) = \int_\gamma g(z,w)\,\mathrm{d}z = \int_a^b g(\gamma(t),w)\gamma'(t)\,\mathrm{d}t$, so,

$$\begin{aligned} \int_\beta h(w)\,\mathrm{d}w &= \int_c^d \left(\int_a^b g(\gamma(t),\beta(s))\gamma'(t)\beta'(s)\,\mathrm{d}t \right) \mathrm{d}s \\ &= \int_a^b \left(\int_c^d g(\gamma(t),\beta(s))\gamma'(t)\beta'(s)\,\mathrm{d}s \right) \mathrm{d}t \\ &= \int_\gamma \left(\int_\beta g(z,w)\,\mathrm{d}w \right) \mathrm{d}z \end{aligned}$$

by Fubini's theorem - Lemma 3.5 below - (applied on each C^1 piece of the curves).
But by Theorem 2.16, for each fixed $z \in U$, the function $w \mapsto g(z,w)$ is holomorphic in $D(w_0,\delta)$ (in fact in U), because it is continuous in U and holomorphic except at one point (namely z). Hence by convex Cauchy, $\int_\beta g(z,w)\,\mathrm{d}w = 0$. So $\int_\beta h(w)\,\mathrm{d}w = 0$ and hence by Morera's theorem h is holomorphic in $D(w_0,\delta)$, and hence on U.

Lemma 3.5 (Fubini's theorem, special case)**.** If $\varphi : [a,b] \times [c,d] \to \mathbb{R}$ is continuous, then the function $f_1 : s \mapsto \int_c^d \varphi(s,t)\,\mathrm{d}t$ is continuous on $[a,b]$, the function $f_2 : t \mapsto \int_a^b \varphi(s,t)\,\mathrm{d}s$ is continuous on $[c,d]$, and

$$\int_a^b \left(\int_c^d \varphi(s,t)\,\mathrm{d}t \right) \mathrm{d}s = \int_c^d \left(\int_a^b \varphi(s,t)\,\mathrm{d}s \right) \mathrm{d}t$$

Proof. Since φ is continuous on the compact set $[a,b] \times [c,d]$, it is uniformly continuous. So given $\varepsilon > 0$, there is $\delta > 0$ such that $|s_1 - s_2| < \delta \implies |\varphi(s_1,t) - \varphi(s_2,t)| < \varepsilon \ \forall t \in [c,d] \implies |f_1(s_1) - f_1(s_2)| < (d-c)\varepsilon$. i.e. f_1 is continuous. Similarly, f_2 is continuous. To see the equality of the iterated integrals, note that since φ is uniformly continuous, it is the uniform limit of a sequence of step functions, i.e. functions of the form $g(x,y) = \sum_{j=1}^N \alpha_j \chi_{R_j}(x,y)$ where α_j are constants; R_j, $j = 1, \dots N$ are sub-rectangles of the form $R_j = [a_j,b_j) \times [c,d)$, and χ_{R_j} such that $\bigcup R_j$ is a (finite) partition of $[a,b) \times [c,d)$, and χ_{R_j} is the characteristic function of R_j (so $\chi_{R_j}(x,y) = 1$ if $(x,y) \in R_j$ and $\chi_{R_j}(x,y) = 0$ if $(x,y) \notin R_j$). But for step functions it is trivial to check the validity of the iterated integral.

Definition. Let $U \subset \mathbb{C}$ be open. A closed curve $\gamma : [a, b] \to U$ is said to be **homologous to zero** in U if $I(\gamma; w) = 0$ for every $w \in \mathbb{C}\backslash U$.

Moral. The concept of a closed curve being "homologous to zero" is precisely what is needed for the definitive versions of Cauchy's theorem and the CIF

Theorem 3.6 (General Cauchy theorem and the Cauchy Integral Formula)**.** Let U be an open subset of $\mathbb{C}$ and $f : U \to \mathbb{C}$ a holomorphic function. If γ is a closed curve in U homologous to zero in U then

(i)

$$I(\gamma; w)f(w) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-w}\,\mathrm{d}z \text{ for every } w \in U\setminus \text{Image}(\gamma)$$

(ii)

$$\int_\gamma f(z)\,\mathrm{d}z = 0$$

Proof. Part (ii) follows from part (i) (remark (ii) below) so we only need to prove part (i).
Part (i) $\iff \int_\gamma \frac{f(z)-f(w)}{z-w}\,\mathrm{d}z = 0\ \forall w \in U\setminus \text{Image}(\gamma)$

$$\iff \int_\gamma g(z,w)\,\mathrm{d}z = 0 \quad \forall w \in U\setminus \text{Image}(\gamma)$$

where $g(z,w) = \frac{f(z)-f(w)}{z-w}$ for $z \neq w$. Extend g to $U \times U$ by setting $g(w,w) = f'(w)$, so

$$g(z,w) = \begin{cases} \frac{f(z)-f(w)}{z-w} & z \neq w \\ f'(w) & z = w \end{cases}$$

Define $h : U \to \mathbb{C},\ h(w) = \int_\gamma g(z,w)\,\mathrm{d}z$. By Lemma 3.4, h is holomorphic on U. Our goal is to show $h = 0$.
Key idea: show that

- h extends to all of $\mathbb{C}$ as a holomorphic function H and
- $H(w) \to 0$ as $w \to \infty$

Then by Liouville's theorem $H = 0$.
To extend h to $\mathbb{C}$, note that by the definition of γ being homologous to zero in U, we have $\mathbb{C}\setminus U \subset V = \{w \in \mathbb{C}\setminus \text{Image}(\gamma) : I(\gamma; w) = 0\}$. So $\mathbb{C} = U \cup V$, and also V is open (since $I(\gamma; \cdot)$ is locally constant).
For $w \in U \cap V,\ h = \int_\gamma \frac{f(z)-f(w)}{z-w}\,\mathrm{d}z = \int_\gamma \frac{f(z)}{z-w}\,\mathrm{d}z$ (since $\int_\gamma \frac{\mathrm{d}z}{z-w} = 2\pi i I(\gamma; w) = 0$ if $w \in V$). This says that on $U \cap V$, the function h agrees with

$$h_1 : V \to \mathbb{C}, \quad h_1(w) = \int_\gamma \frac{f(z)}{z-w}\,\mathrm{d}z$$

h_1 is holomorphic on V.[In fact on $\mathbb{C}\setminus$ Image(γ); either use argument of Lemma 3.4 (working in a disk about any point $w_0 \in \mathbb{C}\setminus$ Image(γ) using Convex Cauchy + Morera), or use more elementary reasoning that h_1 has a power series expansion about every point $w_0 \in \mathbb{C}\setminus$ Image(γ), by expanding $\frac{1}{z-w}$ appropriately. See also ES2 Q8]. Hence the function

$$H : \mathbb{C} \to \mathbb{C}, \quad H(w) = \begin{cases} h(w) & w \in U \\ h_1(w) & w \in V \end{cases}$$

is well defined and holomorphic.
Claim: $H(w) \to 0$ as $|w| \to \infty$.
To see this, fix $R > 0$ such that Image$(\gamma) \subset D(0,R)$ (possible since Image(γ) is compact). By Prop 3.3, $\mathbb{C}\setminus D(0,R) \subset V$. If $|w| > R$,

$$|H(w)| = |h_1(w)| = \left|\int_\gamma \frac{f(z)}{z-w}\,\mathrm{d}z\right| \leq \frac{1}{|w|-R}\left(\sup_{z\in\text{Image}(\gamma)} |f(z)|\right)\text{length}(\gamma)$$

which shows that $H(w) \to 0$ as $|w| \to \infty$, as claimed.
So H is bounded (Since H is continuous, and by the claim $|H(w)| \leq 1$ outside some closed disk $\overline{D(0,R_1)}$). By Liouville's theorem, H is constant, and by the claim $H = 0$. In particular $h = 0$.

Remarks.

(i) Cauchy's theorem says that if $\int_\gamma f(z)\,\mathrm{d}z = 0$ for a special family of holomorphic functions on U, namely for $f(z) = \frac{1}{z-w}$, $w \in \mathbb{C}\backslash U$, then $\int_\gamma f(z)\,\mathrm{d}z = 0$ for any holomorphic $f : U \to \mathbb{C}$

(ii) Parts (i) and (ii) of the theorem are equivalent statements.
"(i) $\implies$ (ii)": given any holomorphic $f : U \to \mathbb{C}$, pick any $w \in U\backslash$ Image(γ) and apply (i) with $F(z) = (z-w)f(z)$ in place of f. Since $F(w) = 0$, it follows that $\int_\gamma f(z)\,\mathrm{d}z = 0$
"(ii) $\implies$ (i)": if $f : U \to \mathbb{C}$ is holomorphic, then for any $w \in U$, the function $g(z) = \frac{f(z)-f(w)}{z-w}$, $z \neq w$, $g(w) = f'(w)$ is holomorphic in U. (Corollary 2.16; application of Convex Cauchy + Morera). So (ii) $\implies \int_\gamma g(z)\,\mathrm{d}z = 0$, which says

$$\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-w}\,\mathrm{d}z = I(\gamma; w)f(w)$$

whenever $w \notin$ Image(γ)

(iii) Let γ be a closed curve in U. We have: γ being hologous in U is equivalent to Cauchy's theorem being valid (or to CIF being valid by remark (ii)) with respect to γ for all holomorphic $f : U \to \mathbb{C}$. The non-trivial direction of this is Theorem 3.7 (γ homologous to zero $\implies \int_\gamma f(z)\,\mathrm{d}z = 0$); the other direction is obvious: given any $w \in \mathbb{C}\backslash U$, we can just apply the Cauchy theorem to the function $f(z) = \frac{1}{z-w}$, which is holomorphic on U, to get $I(\gamma; w) = 0$.
So in fact have: γ homologous to zero in $U \iff$ (i) $\iff$ (ii)

Note. This proof actually gives a more general theorem involving several curves:

Corollary 3.7. Let $U \subset \mathbb{C}$ be open and $\gamma_1, \gamma_2, \dots, \gamma_n$ be closed curves in U such that $\sum +j = 1^n I(\gamma_j; w) = 0$ for all $w \in \mathbb{C}\backslash U$. Then for any holomorphic $f : U \to \mathbb{C}$, we have

(i)
$$f(w)\sum_{j=1}^n I(\gamma_j; w) = \sum_{j=1}^n \frac{1}{2\pi i}\int_{\gamma_j} \frac{f(z)}{z-w}\,\mathrm{d}z$$

for every $w \in U\backslash \bigcup_{j=1}^n$ Image(γ_j) and

(ii) $\sum_{j=1}^n \int_{\gamma_j} f(z)\,\mathrm{d}z = 0$

Proof. For part (i), define $g(z, w)$ as before, but take

$$V = \{w \in \mathbb{C}\backslash \bigcup_{j=1}^n \text{Image}(\gamma_j) : \sum_{j=1}^n I(\gamma_j; w) = 0\}$$

In the definitions of h and h_1, use the sum of the integrals over γ_j. Then process as above. Part (ii) follows from part (i) as before.

Corollary 3.8. Let $U \subset \mathbb{C}$ be open and let β_1, β_2 be two closed curves in U such that $I(\beta_1, w) = I(\beta_2, w)$ for all $w \in \mathbb{C} \backslash U$. Then

$$\int_{\beta_1} f(z)\,\mathrm{d}z = \int_{\beta_2} f(z)\,\mathrm{d}z$$

for any holomorphic function $f : U \to \mathbb{C}$

Proof. Apply Corollary 3.7(ii) with $n = 2$, $\gamma_1 = \beta_1$ and $\gamma_2 = (-\beta_2)$ (the inverse path to β_2), noting that $I((-\beta_2); w) = -I(\beta_2, w)$ for any $w \notin \text{Image}(\beta_2) = \text{Image}(-\beta_2)$

Concerning the question "for which closed curves in a given domain U is the Cauchy theorem valid", we have the definitive answer: curves that are homologous to zero in U. This condition may be difficult to check.
There is a more restrictive but more geometric and easier-to-visualise condition, called being null-homotopic, that implies being homologous to zero. We want to explore this next.

Definition. Let $U \subset \mathbb{C}$ be a domain, and let $\gamma_0, \gamma_1 : [a, b] \to U$ be closed curves. We say that γ_0 is homotopic to γ_1 in U if there is a continuous map $H : [0, 1] \times [a, b] \to U$ suhc that

$$\begin{aligned} H(0, t) &= \gamma_0(t) \quad \forall t \in [a, b] \\ H(1, t) &= \gamma_1(t) \quad \forall t \in [a, b] \text{ and} \\ H(s, a) &= H(s, b) \forall s \in [0, 1] \end{aligned}$$

Such a map H is called a **homotopy between γ_0 and γ_1**.
For $0 \le s \le 1$, if we let $\gamma_s : [a, b] \to U$ be defined by $\gamma_s(t) = H(s, t)$ for $t \in [a, b]$, then the above conditions imply that $\{\gamma_s : s \in [0, 1]\}$ is a family of continuous closed curves in U which "deforms γ_0 to γ_1 continuously without ever leaving U"

Definition. A closed curve $\gamma : [a, b] \to U$ is said to be **null-homotopic in U** if it is homotopic to a constant curve in U, i.e. homotopic to a curve with image equal to one point in U

Theorem 3.9. If $\gamma_0,\ \gamma_1 : [a,b] \to U$ are homotopic closed curves in U, then $I(\gamma_0; w) = I(\gamma_1; w)$ for every $w \in \mathbb{C}\backslash U$. In particular, if a closed curve γ in U is null-homotopic in U, then it is homologous to zero in U

Proof. Let $H : [0,1] \times [a,b] \to U$ be a homoropy between γ_0 and γ_1. Since H is continuous and $[0,1] \times [a,b]$ is compact, $K = H([0,1] \times [a,b])$ is a compact subset of the open set U. Therefore, there exists $\varepsilon > 0$ such that

$$|w - H(s,t)| > 2\varepsilon \text{ for each } (s,t) \in [0,1] \times [a,b] \tag{1}$$

Also H is uniformly continuous on $[0,1] \times [a,b]$, so we can choose a positive integer n such that

$$(s,t),(s',t') \in [0,1] \times [a,b],\ |s-s'| + |t-t'| \leq \frac{1}{n} \implies |H(s,t) - H(s',t')| < \varepsilon \tag{2}$$

For $k = 0,1,2,\dots,n$, set $\Gamma_k(t) = H(\frac{k}{n}, t), a \leq t \leq b$. Then $\Gamma_k : [a,b] \to U$ are closed continuous curves, with $\Gamma_0 = \gamma_0$ and $\Gamma_n = \gamma_1$.
By (1) and (2), for each $k = 1,2,\dots,n$

$$|\Gamma_{k-1}(t) - \Gamma_k(t)| < |w - \Gamma_{k-1}(t)| \text{ for all } t \in [a,b] \tag{3}$$

If Γ_k are piecewise C^1, then this implies $I(\Gamma_{k-1}; w) = I(\Gamma_k; w)$ for $k = 1,2,\dots,n$, and hence $I(\gamma_0; w) = I(\gamma_1; w)$ (by ES2 Q11, which says $|\gamma(t) - \tilde{\gamma}(t)| < |w - \gamma(t)| \forall t \implies I(\gamma; w) = I(\tilde{\gamma}; w)$).
Since H is only assumed to be continuous, Γ_k need not be piecewise C^1. But this is easily handled as we can approximate Γ_k by a polygonal closed curve. Specifically, take in place of Γ_k the curve $\tilde{\Gamma}_k : [a,b] \to U$ defined by

$$\tilde{\Gamma}_k(t) = \left(1 - \frac{n(t - a_{j-1})}{b-a}\right) H(\frac{k}{n}, a_{j-1}) + \frac{n(t-a_{j-1})}{b-a} H(\frac{k}{n}, a_j)$$

for $a_{j-1} \leq t \leq a_j$ where $a_j = a + \frac{(b-a)j}{n}$ for $j = 0,1,\dots,n$. These still satisfy (3).

Remark. For both being null-homotopic and being homologous to zero, the domain matters. E.g. the circle $\gamma(t) = e^{2\pi i t},\ t \in [0,1]$. This is null-homotopic (and homologous to zero) in $\mathbb{C}$, but not homologous to zero in $U = \mathbb{C}\backslash\{0\}$ (and hence not null-homotopic in U by the theorem)

Remark. As mentioned, theorem 3.9 gives that γ null-homotopic in $U \implies \gamma$ is homologous to zero in U. The converse of this is false, i.e. for a given closed curve γ in U,

$$\gamma \text{ is homologous in } U \not\Longrightarrow \gamma \text{ is null-homotopic in } U$$

For instance, take $U = \mathbb{C}\backslash\{w_1, w_2\}$ for distinct points w_1, w_2, and let $U_1 = U \cup \{w_1\} = \mathbb{C}\backslash\{w_2\}$ and $U_2 = U \cup \{w_2\} = \mathbb{C}\backslash\{w_1\}$. Consider a curve γ not null-homotopic in U, but null-homotpic in each of the larger domains U_1 and U_2. (ES3: draw a picture of suhc a curve). Then γ is homologous to zero in $U_j,\ j = 1,2$ (Theorem 3.9). This means that $I(\gamma; w_j) = 0$ for $j = 1,2$, so γ is homologous to zero in U

Corollary 3.10. If $\gamma_0, \gamma_1 : [a,b] \to U$ are homotopic closed curves in U, then $\int_{\gamma_0} f(z)\,\mathrm{d}z = \int_{\gamma_1} f(z)\,\mathrm{d}z$ for any holomorphic function $f : U \to \mathbb{C}$

Proof. Immediate from Theorem 3.9 (γ_0, γ_1 homotopic $\implies I(\gamma_0; w) = I(\gamma_1; w)\ \forall w \in \mathbb{C}\backslash U$) and Corollary 3.8 ($I(\gamma_0; w) = I(\gamma_1; w)\ \forall w \in \mathbb{C}\backslash U \implies \int_{\gamma_0} f(z)\,\mathrm{d}z = \int_{\gamma_1} f(z)\,\mathrm{d}z$)

Remark. Using Corollary 3.8 to prove Corollary 3.10 is actually overkill!
Direct proof of Corollary 3.10: With $\tilde{\Gamma}_k$ as above, the closed curve made up of $\tilde{\Gamma}_k\big|_{[a_{j-1}, a_j]}$, the line segment $[\tilde{\Gamma}_{k-1}(a_j), \tilde{\Gamma}_k(a_j)]$, the curve $(-\tilde{\Gamma}_k\big|_{[a_{j-1}, a_j]})$ and the line segment $[\tilde{\Gamma}_k(a_{j-1}), \tilde{\Gamma}_{k-1}(a_{j-1})]$ is contained in the disk $D(\tilde{\Gamma}_{k-1}(a_{j-1}), \varepsilon) \subset U$; apply convex Cauchy to this curve and sum over j to get $\int_{\tilde{\Gamma}_{k-1}} f(z)\,\mathrm{d}z = \int_{\tilde{\Gamma}_k} f(z)\,\mathrm{d}z$. Similar reasoning also gives

$$\int_{\tilde{\Gamma}_0} f(z)\,\mathrm{d}z = \int_{\gamma_0} f(z)\,\mathrm{d}z$$

and

$$\int_{\tilde{\Gamma}_n} f(z)\,\mathrm{d}z = \int_{\gamma_1} f(z)\,\mathrm{d}z$$

Definition. A domain U is said to be **simply connected** if every clused curve in U is null-homotopic in U

Example. A star domain (in particular a convex domain) Ω is simply connected. (proof: there is $a \in \Omega$ such that the line segment $[a, z] \subset \Omega$ for each $z \in \Omega$. If $\gamma : [a,b] \to \Omega$ is a closed curve, set $H(s,t) = (1-s)a + s\gamma(T) \in \Omega \in [0,1] \times [a,b]$. Then $H(s,t) \in U$, and H is a homotopy between γ and the constant curve $\gamma_0(t) = a$)

Theorem 3.11 (Cauchy's theorem for simply connected domains)**.** If U is simply connected, then $\int_\gamma f(z)\,\mathrm{d}z = 0$ for every holomorphic function $f : U \to \mathbb{C}$ and every closed curve γ in U

Proof. Immediate from Corollary 3.10 (γ_0, γ_1 homotopic $\implies \int_{\gamma_0} f(z)\,\mathrm{d}z = \int_{\gamma_1} f(z)\,\mathrm{d}z$) and the fact that $\int_\gamma f(z)\,\mathrm{d}z = 0$ if γ is a constant curve

Remark. The converse of Theorem 3.11 is also true (but is harder to prove). Thus:
U is simply connected $\iff$ $\int_\gamma f(z)\,\mathrm{d}z = 0$ for every closed curve in U and every holomorphic function function f on U.
This should spark some surprise: one condition (simply connected) is topological, based only on continuous deformation of curves in the domain. The other (validity of Cauchy thm) is analytic, involving the behaviour of differential functions on the domain; these are equivalent!
The "$\impliedby$" direction of the above together with Theorem 3.6 (Cauchy for curves homologous to zero in U) says that U is simply connected iff every closed curve in U is homologous to zero in U.

Contras tthis to the Remark preceding Corollary 3.10: for a given closed curve γ in U,
γ is homologours to zero in U $\not\Longrightarrow$ γ is null-homotopic in U.

4 Isolated Singularities, Laurent Series and the Residue Theorem

Moral. Let U be open, and γ be a closed curve in U homologous to zero in U. If $f : U \to \mathbb{C}$ is holomorphic then we have CIF:

$$\int_\gamma \frac{f(z)}{z-a}\,\mathrm{d}z = 2\pi i I(\gamma;a)f(A) \text{ for } a \in U\backslash \text{ Image}(\gamma)$$

We can look at this formula in the following way. It lets us compute $\int_\gamma g(z)\,\mathrm{d}z$ for a holomorphic function $g : U\backslash\{a\} \to \mathbb{C}$ when γ does not pass through the "bad point a" (the singular point of g) provided g satisfies a condition: $(z-a)g(z)$ is the restriction to $U\backslash\{a\}$ of a holomorphic function $f : U \to \mathbb{C}$. What if we drop this condition? i.e. how can we compute $\int_\gamma g(z)\,\mathrm{d}z$ for arbitrary holomorphic $g : U\backslash\{a\} \to \mathbb{C}$ where $a \in U$ and γ misses a? E.g. $g(z) = e^{1/z}$, $U = \mathbb{C}$, $a = 0, \gamma = \partial D(0,1)$? (Note: $zg(z) = ze^{1/z}$ is not holomorphic. not even continous at $z = 0$.) More generally, what about the case of several bad points, i.e. $\int_\gamma g(z)\,\mathrm{d}z$ for holomorphic $g : U\backslash\{a_1,\dots,a_k\} \to \mathbb{C}$ where $a_1,\dots,a_k \in U$, with γ missing all a_j? The answer is an important theorem (the Residue Theorem), which we will prove. We first discuss types of behaviour of a holomorphic function g on $U\backslash\{a\}$ on approach to a

4.1 Isolated Singularities

Definition. Let $U \subset \mathbb{C}$ be open. If $a \in U$ and $f : U\backslash\{a\} \to \mathbb{C}$ is holomorphic, then we say f has an **isolated singularity** at a (or a is an isolated singularity of f)

Definition. An isolated singularity a of f is a **removable singularity** of f if f can be defined at a so that the extended function is holomorphic on U

Prop 4.1 (characterising removable singularities)**.** Suppose U is open, $a \in U$ and $f : U\backslash\{a\} \to \mathbb{C}$ is holomorphic. Then the following are equivalent:

(i) f has a removable singularity at a

(ii) $\lim_{z\to a} f(z)$ exists in $\mathbb{C}$

(iii) There is a disk $D(a,\varepsilon) \subset U$ such that $|f(z)|$ is bounded in $D(a,\varepsilon)\backslash\{a\}$

(iv) $\lim_{z\to a}(z-a)f(z) = 0$

Proof. The implication (i) $\implies$ (ii) is clear: if a is a reovable singularity of f, then by definition there is holomorphic $g : U \to \mathbb{C}$ such that $f(z) = g(z)\ \forall z \in U\backslash\{a\}$. Then $\lim_{z\to a} f(z) = \lim_{z\to a} g(z) = g(a) \in \mathbb{C}$. Also the implications (ii) $\implies$ (iii) and (iii) $\implies$ (iv) are clear.

To check (iv) $\implies$ (i), consider $h : U \to \mathbb{C}$ defined by

$$h(z) = \begin{cases} (z-a)^2 f(z) & z \neq a \\ 0 & z = a \end{cases}$$

We have $\lim_{z\to a} \frac{h(z)-h(a)}{z-a} = \lim_{z\to a}(z-a)f(z) = 0$ where the second equality holds by assumption. So h is differentiable at a with $h'(a) = 0$. Since h is clearly differentiable in $U\backslash\{a\}$, it follows that h is holomorphic in U. Since $h(a) = h'(a) = 0$, Taylor series theorem gives $r > 0$ and holomorphic $g : D(a,r) \to \mathbb{C}$ such that $h(z) = (z-a)^2 g(z)$ for $z \in D(a,r)$. Comparing this to the definition of h, we have that $f(z) = g(z)$ for $z \in D(a,r)\backslash\{a\}$. Define $f(a) = g(a)$. then f is differentiable at a (with $f'(a) = g'(a)$). So a is a removable singularity of f

Example. $f(z) = \frac{e^z-1}{z}$. Then f is holomorphic in $\mathbb{C}\backslash\{0\}$, and $\lim_{z\to 0} zf(z) = 0$. So $z = 0$ is a removable singularity by the proposition. We also see directly, by the Taylow series of e^z at $z = 0$, that $f(z) = \sum_{k=1}^{\infty} \frac{z^{k-1}}{k!}$ for $z \neq 0$; the series on the right defines a holomorphic function on all of $\mathbb{C}$

Remark (Removable singularities of harmonic functions)**.** If $u : D(0,1)\backslash\{0\} \to \mathbb{R}$ is a C^2 harmonic function, when can we say that $z = 0$ is a removable singularity, i.e. that u extends to $z = 0$ as a harmonic function (equivalently, u has an extension of class C^2)?

One way to answer this is to relate it to holomorphic functions. However, unlike with some of the parallels we've already seen between harmonic and holomorphic functions (e.g. Liouville's theorem), here one needs to proceed with care. The naive idea of finding a harmonic conjugate (i.e. harmonic v on $U = D(0,1)\backslash\{0\}$ such that $f(z) = u(z) + iv(z)$ is holomorphic in U) does not work; the problem is that U is not simply connected, so a conjugate function need not exist a priori.

Still, the answer has a close parallel: if $\lim_{z\to 0} u(z)$ exists (i.e. if u extends continuously to $z = 0$), then the extended function is C^2 and harmonic. More generally, if u is bounded near $z = 0$, then there is a harmonic extension. We can also ask, in parallel with the holomorphic case, what if $\lim_{z\to 0} |z||u(z)| = 0$? (Exercise: give complex analysis proofs; see ex. sheet 3.)

By Prop 4.1, if f has a non-removable singularity at a, then f is not bounded in $D(a,r)\backslash\{a\}$ for any $r > 0$. We analyse this case next.

Definition. If $a \in U$ is an isolated singularity of f, then a is a **pole** of f if $\lim_{z\to a} |f(z)| = \infty$

Example. $f(z) = (z-a)^{-k}$ for a constant integer $k \geq 1$. This has a pole at a

Definition. If $a \in U$ is an isolated singularity of f, then a is an **essential singularity** of f if a is neither a removable singularity nor a pole

Remark. a is an essential singularity $\iff \lim_{z \to a} |f(z)|$ does not exist in $[0, \infty]$. This follows from Proposition 3.1 and the definition of pole.

Example. $f(z) = e^{1/z}$. We have $|f(iy)| = 1$ for all $y \in \mathbb{R} \backslash \{0\}$, while $\lim_{x \to 0^+} f(x) = \infty$. So $z = 0$ is an essential singularity of f

Prop 4.2 (characterising poles)**.** Let $f : U \backslash \{a\} \to \mathbb{C}$ be holomorphic. The following are equivalent:

(i) f has a pole at a

(ii) There is $\varepsilon > 0$ and holomorphic $h : D(a, \varepsilon) \to \mathbb{C}$ with $h(a) = 0$ and $h(z) \neq 0$ for $z \neq a$ such that $f(z) = \frac{1}{h(z)}$ for $z \in D(a, \varepsilon) \backslash \{a\}$

(iii) $\exists$ a unique integer $k \geq 1$ and a unique holomorphic $g : U \to \mathbb{C}$ with $g(a) \neq 9$ such that $f(z) = (z-a)^{-k} g(z)$ for $z \in U \backslash \{a\}$

Proof. (i) $\implies$ (ii): Since $\lim_{z \to a} |f(z)| = \infty$, there is $\varepsilon > 0$ such that $0 < |z - a| < \varepsilon \implies |f(z)| \geq 1$. Hence $1/f(z)$ is holomorphic and bounded in $D(a, \varepsilon) \backslash \{a\}$. By Prop 4.1, $1/f$ has a removable singularity at a, i.e. there is holomorphic $h : D(a, \varepsilon) \to \mathbb{C}$ such that $1/f(z) = h(z)$, or equivalently $f(z) = 1/h(z)$ for $z \in D(a, \varepsilon) \backslash \{a\}$. Since $|f(z)| \to \infty$ as $z \to a$, we also have $h(a) = 0$.

(ii) $\implies$ (iii): Let ε and h be as in (ii). By Taylor series, there is an integer $k \geq 1$ and a holomorphic $h_1 : D(a, \varepsilon) \to \mathbb{C}$ with $h_1(z) \neq 0 \; \forall z \in D(a, \varepsilon)$ such that $h(z) = (z-a)^k h_1(z)$. If we let $g_1 = 1/h_1$, then g_1 is holomorphic in $D(a, \varepsilon)$, $g_1 \neq 0$ in $D(a, \varepsilon)$ and

$$f(z) = (z-a)^{-k} g_1(z) \text{ in } D(a, \varepsilon) \backslash \{a\} \tag{*}$$

Define $g : U \to \mathbb{C}$ by $h_1(z)$ for $z \in D(a, \varepsilon)$ and $g(z) = (z-a)^k f(z)$ for $z \in U \backslash \{a\}$.
By (*), the definitions agree on $D(a, \varepsilon) \backslash \{a\}$, so g is well-defined and holomorphic in U, and $g(a) = g_1(a) \neq 0$. This proves the existence of an integer $k \geq 1$ and a holomorphic $g : U \to \mathbb{C}$ with $g(a) \neq 0$ such that

$$f(z) = (z-a)^{-k} g(z) \text{ for all } z \in U \backslash \{a\}$$

To prove uniqueness of k and g, suppose there is an integer $\tilde{k} \geq 1$ and a holomorphic function $\tilde{g} : U \to \mathbb{C}$ with $\tilde{g}(a) \neq 0$ such that $f(z) = (z-a)^{-\tilde{k}} \tilde{g}(z)$ for all $z \in U \backslash \{a\}$. Then we must have $g(z) = (z-a)^{\tilde{k}-k} \tilde{g}(z)$ for $z \in U\{a\}$. Since $g, \tilde{g}$ are holomorphic with $g(a) \neq 0$ and $\tilde{g}(a) \neq 0$, this can only be true if $\tilde{k} = k$, in which case we also have $\tilde{g} = g$ (first on $U \backslash \{a\}$, and hence also at $z = a$ by continuity).

(iii) $\implies$ (i): this is clear

Remark. The implication (i) $\implies$ (iii) says, remarkably, the following: there is no holomorphic function on a punctured disk
$f : D(a, R)\backslash\{a\} \to \mathbb{C}$ such that $|f(z)| \to \infty$ as $z \to a$ "at the rate of a negative, non-integer power of $|z-a|$," i.e. with $c|z-a|^{-s} \leq |f(z)| \leq C|z-a|^{-s}$ for some constants $s \in (0,\infty)\backslash\mathbb{N}$, $c > 0$, $C > 0$ and all $z \in D(a, R)\backslash\{a\}$)

Notation. (i) If f has a pole at $z = a$, then the unique positive integer k given by Proposition 4.3 is the **order of the pole** at a. If $k = 1$, then f has a **simple pole** at a.

(ii) Let U be open and $S \subset U$ be a discrete subset of U (which means that all points of S are isolated points). If $f : U\backslash S \to \mathbb{C}$ is holomorphic and each $a \in S$ is either a removable singularity or a pole of f, then f is said to be a meromorphic function on U. (This includes the possibility $S = \varnothing$, in which case f is holomorphic.)

Note. If $f : U\backslash\{a\} \to \mathbb{C}$ is holomorphic and the singularity $z = a$ is a pole of f, we can regard f as a continuous mapping $f : U \to \hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, where $\hat{\mathbb{C}}$ is the Riemann sphere, by setting $f(a) = \infty$. As such f is in fact holomorphic on U. [Holomorphicity of the extended map near the pole a follows from (i) in a a punctured disk about a, $1/f$ has the form $(z-a)^k/g(z)$ for some holomorphic g with $g(z) \neq 0$ near a; and (ii) any function h defined in a neighbourhood of $\infty \in \hat{\mathbb{C}}$ is holomorphic, by definition, if the function $\tilde{h}(z) = h(1/z)$ if $z \neq 0$, $\tilde{h}(0) = h(\infty)$ is holomorphic near 0. These two facts make $h \circ f = \tilde{h} \circ (1/f)$ holomorphic near a whenever h is a holomorphic function in a neighbourhood of ∞ in $\hat{\mathbb{C}}$]
This way, any meromorphic function $f : U\backslash S \to \mathbb{C}$ becomes a holomorphic function $f : U \to \hat{\mathbb{C}}$.
Thus, from this geometric point of view, poles are not singularityies at all, and the only genuine isolated singularityies are the essential singularities. (Note that the above reasoning cannot be carried out if the singularity is essential). See Part II, Riemann Surfaces for more on this.

Behaviour near an essential singularity: Suppose $z = a$ is an essential singularity of holomorphic $f : U\backslash\{a\} \to \mathbb{C}$. Then there is a sequence of points $a_n \in U\backslash\{a\}$, $a_n \to a$ such that $f(a_n) \to \infty$ (else $z = a$ would be a removable singularity by Proposition 4.1), and there is another sequence of points $b_n \in U\backslash\{a\}$, $b_n \to a$ such that $(f(b_n))$ is bounded (else $z = a$ would be a pole). In fact much more can be said:

Theorem 4.3 (Casorati–Weierstrass theorem)**.** If $f : U\backslash\{a\} \to \mathbb{C}$ is holomorphic and $a \in U$ is an essential singularity of f, then for any $\varepsilon > 0$, the set $f(D(a, \varepsilon)\backslash\{a\})$ is dense in $\mathbb{C}$

Proof. ES2

Even more remarkably, we have the following (more difficult) result:

Theorem 4.4 (Picard's Theorem)**.** If $f : U\backslash\{a\} \to \mathbb{C}$ is holomorphic and $a \in U$ is an essetnial singularity of f, then there is $w \in \mathbb{C}$ such that for any $\varepsilon > 0$, $\mathbb{C}\backslash\{w\} \subset f(D(a\varepsilon)\backslash\{a\})$. i.e. in any neighbourhood $D(a, \varepsilon)\backslash\{a\}$, f attains all possible complex numbers except possibly one

Proof. Omitted

Note. Picard's theorem is optimal: the function $f(z) = e^{1/z}$ does not attain $w = 0$

4.2 Laurent Series

Moral. If $z = a$ is a removable singularity of f, then for some $R > 0$, f is given by a power series $\sum_{n=0}^{\infty} c_n(z-a)^n$ (the Taylor series of the holomorphic extension of f to $D(a, R)$) for all $z \in D(a, R)\backslash\{a\}$. If a is a pole of some order $k \geq 1$, then for some $R > 0$, $f(z) = (z-a)^{-k}g(z)$ for some holomorphic $g : D(a, R) \to \mathbb{C}$ and all $z \in D(a, R)\backslash\{a\}$, so using the Taylor series of g, we get a series of the form $f(z) = \sum_{n=-k}^{\infty} c_n(z-a)^n$, $z \in D(a, R)\backslash\{a\}$. When a is an essential singularity, we still have a series expansion, now of the form $f(z) = \sum_{n=-\infty}^{\infty} c_n(z-a)^n$. In fact we have more generally the following

Theorem 4.5 (Laurent expansion)**.** Let f be holomorphic on an annulus $A = \{z \in \mathbb{C} : r < |z-a| < R\}$ where $0 \le r < R \le \infty$. Then:

(i) f has a unique convergent series expansion

$$f(z) = \sum_{z=-\infty}^{\infty} c_n(z-a)^n (\equiv \sum_{n=1}^{\infty} c_{-n}(z-a)^{-n} + \sum_{n=0}^{\infty} c_n(z-a)^n)$$

where c_n are constants

(ii) For any $\rho \in (r, R)$, the coefficient c_n is given by

$$c_n = \frac{1}{2\pi i} \int_{\partial D(a,\rho)} \frac{f(z)}{(z-a)^{n+1}} \, \mathrm{d}z$$

(iii) If $r < \rho' \le \rho < R$, then the series in (i) converges uniformly (i.e. the two series separate;y converge uniformly) on the set $\{z \in \mathbb{C} : \rho' \le |z-a| \le \rho\}$

Proof. Fix $w \in A$ and consider the function

$$g(z) = \begin{cases} \frac{f(z)-f(w)}{z-w} & z \neq w \\ f'(w) & z = w \end{cases}$$

Then g is continuous in A and clearly holomorphic in $A \backslash \{w\}$ and hence (by Theorem 2.16) holomorphic in A. Choose ρ_1, ρ_2 such that $r < \rho_1 < |w-a| < \rho_2 < R$. The two positively oriented curves $\partial D(a, \rho_1)$ and $\partial D(a, \rho_2)$ (i.e. the curves $\gamma_1, \gamma_2 : [0,1] \to \mathbb{C}$ given by $\gamma_j = a + \rho_j e^{2\pi i t},\ j = 1, 2$) are homotopic in A. So by Corollary 3.10

$$\int_{\partial D(a,\rho_1)} g(z) \, \mathrm{d}z = \int_{\partial D(a,\rho_2)} g(z) \, \mathrm{d}z$$

Substituting for g, this gives

$$\int_{\partial D(a,\rho_1)} \frac{f(z)}{z-w} \, \mathrm{d}z - 2\pi i I(\partial D(a,\rho_1); w) f(w) = \int_{\partial D(a,\rho_2)} \frac{f(z)}{z-w} \, \mathrm{d}z - 2\pi i I(\partial D(a,\rho_2); w) f(w)$$

Since $I(\partial D(a,\rho_1); w) = 0$ (Prop. 3.3) and $I(\partial D(a,\rho_2); w) = I(\partial D(a,\rho_2); a) = 1$ (since a, w are in the same component of $\mathbb{C} \backslash \partial D(a, \rho_2)$), this says

$$f(w) = \frac{1}{2\pi i} \int_{\partial D(a,\rho_2)} \frac{f(z)}{z-w} \, \mathrm{d}z - \frac{1}{2\pi i} \int_{\partial D(a,\rho_1)} \frac{f(z)}{z-w} \, \mathrm{d}z$$

For the first integral, use the expansion

$$\frac{1}{z-w} = \frac{1}{z-a-(w-a)} = \sum_{n=0}^{\infty} \frac{(w-a)^n}{(z-a)^{n+1}}$$

where the series converges uniformly over $z \in \partial D(a, \rho_2)$.
For the second integral, use

$$\frac{1}{z-w} = \frac{1}{z-a-(w-a)} = -\frac{1}{(w-a)(1-\frac{z-a}{w-a})} = -\sum_{n=0}^{\infty} \frac{(z-a)^n}{(w-a)^{n+1}}$$

where the series converges uniformly over $z \in \partial D(a, \rho_1)$.
Substituting these and switching integration and summation (OK by uniform convergence), get

$$f(w) = \sum_{n=0}^{\infty} c_n (w-a)^n - \sum_{m=1}^{\infty} d_m (w-a)^{-m}$$

Proof (continued)**.** where

$$c_n = \frac{1}{2\pi i}\int_{\partial D(a,\rho_2)} \frac{f(z)}{(z-a)^{n+1}} \text{ for } n \geq 0 \text{ and}$$
$$d_m = \frac{1}{2\pi i}\int_{\partial D(a,\rho_1)} f(z)(z-a)^{m-1}\,\mathrm{d}z \text{ for } m \geq 1$$

Writing $d_n = c_{-n}$ for $n \geq 1$, we have (i) (existence).
For (ii) and (iii), suppose there are constants c_n such that

$$f(z) = \sum_{-\infty}^{\infty} c_n(z-a)^n \tag{*}$$

for $z \in A$, and lt $r < \rho' \leq \rho < R$. Then the power series $\sum_{n=0}^{\infty} c_n(z-a)^n$ converges for $z \in A$, so it must have r.o.c. $\geq R$ and converge uniformly for $|z-a| \leq \rho$. also, the series $\sum_{n=1}^{\infty} c_{-n}(z-a)^{-n}$ converges on A. Putting $\zeta = (z-a)^{-1}$, this means that the power series $\sum_{n=1}^{\infty} c_{-n}\zeta^n$ converges for $1/R < |\zeta| < 1/r$, so it must have r.o.c. $\geq 1/r$ and converge unfirmly for $|\zeta| \leq 1/\rho'$. thus hte series $\sum_{n=1}^{\infty} c_{-n}(z-a)^{-n}$ converges uniformly for $|z-a| \geq \rho'$. So (*) converges uniformly in the common region $\rho' \leq |z-a| \leq \rho$. Hence

$$\int_{\partial D(a,\rho)} \frac{f(z)}{(z-a)^{m+1}}\,\mathrm{d}z = \sum_{n=-\infty}^{\infty} c_n \int_{\partial D(a,\rho)} (z-a)^{n-m-1}\,\mathrm{d}z$$

By the FTC, the only non-zero integral on the right occurs when $n-m-1 = -1$, i.e. when $n = m$. Computing this integral gives

$$c_m = \frac{1}{2\pi i}\int_{\partial D(a,\rho)} \frac{f(z)}{(z-a)^{m+1}}\,\mathrm{d}z$$

for any $\rho \in (r, R)$. This formula also implies the uniqueness of the coefficients c_n so that the series expansion is valid.

Remark. If f is the restriction to A of a holomorphic function g on the full disk $D(a,R)$, then by the formula in part (ii), we have for any negative integer $n = -m,\ m \geq 1$, the coefficient

$$c_{0m} = \int_{\partial D(a,\rho)} g(z)(z-a)^{m-1}\,\mathrm{d}z$$

is zero by Cauchy's theorem.
So in this case, Laurent series of f is the Taylor series of g restricted to A. The new content of the theorem is when f has no holomorphic extension to $D(a,r)$

Remark. The proof of the theorem shows that if $f : A = D(a,R)\backslash\overline{D(a,r)} \to \mathbb{C}$ is holomorphic, then there is a holomorphic function $f_1 : D(a,R) \to \mathbb{C}$ and a holomorphic function $f_2 : \mathbb{C} \to \overline{D(a,r)} \to \mathbb{C}$ such that

$$f = f_1 + f_2 \text{ on } A$$

Indeed, with c_n as in the theorem, we can take $f_1(z) = \sum_{n=0}^{\infty} c_n(z-a)^n$ and $f_2(z) = \sum_{n=1}^{\infty} c_{-n}(z-a)^{-n}$. This decomposition is not unique (since we can also take $f_! + g$ in place of f_1 and $f_2 - g$ in place of f_2 for any entire function g). If we also require $f_2(z) \to 0$ as $z \to \infty$ then the decomposition is unique (ES3). (For the above choice of f_2, we do have $f_2(z) \to 0$ as $z \to \infty$ by uniform convergence of $\sum_{n=1}^{\infty} c_{-n}(z-a)^{-n}$ in $\{|z-a| \geq \rho'\}$ for any $\rho' > r$).
Question: in the above proof, what if A is a domain (bounded or unbounded) whose boundary is two disjoint non-concentric circles? (ES3)

4.2.1 Isolated Singularities and Laurent Coefficients

Suppose $f : D(a,R)\backslash\{a\} \to \mathbb{C}$ is holomorphic (so $z = a$ is an isolated singularity of f). Then by the Laurent series (taken with $r = 0$), we have a unique set of complex numbers c_n so that

$$f(z) = \sum_{n=-\infty}^{\infty} c_n(z-a)^n \text{ for } z \in D(a,R)\backslash\{a\}$$

Classification of the singularity $z = a$ (as removable, pole or essential) has evidently the following formulation in terms of the coefficients c_n:

(i) $c_n = 0\ \forall n < 0 \iff f(z) = \sum_{n=0}^{\infty} c_n(z-a)^n \equiv g(z)\ \forall z \in D(a,R)\backslash\{a\}$, and g is holomorphic on $D(a,R) \iff z = a$ is a removable singularity. (This uses uniqueness of Laurent series, Taylor series of g and the definition of removable singularity.)

(ii) $c_{-k} \neq 0$ for some $k \geq 1$, and $c_{-n} = 0$ for all $n \geq k+1 \iff$

$$f(z) = \frac{c_{-k}}{(z-a)^k} + \frac{c_{-k+1}}{(z-a)^{k-1}} + \cdots + \frac{c_{-1}}{(z-a)} + \sum_{n=0}^{\infty} c_n(z-a)^n\ \forall z \in D(a,R)\backslash\{a\} \text{ and } c_{-k} \neq 0$$

$$\iff f(z) = (z-a)^{-k} g(z)\ \forall z \in D(a,R)\backslash\{a\}$$

where g is holomorphic on $D(a,R)$ with $g(a) = c_{-k} \neq 0$ (This uses Taylor series of g and uniqueness of Laurent series.)

$$\iff z = a \text{ is a pole of order } k \text{ (by Proposition 4.2)}$$

(iii) $c_n \neq 0$ for infinitely many $n < 0 \iff z = a$ is an essential singularity. (This follows from (i) and (ii))

Notation. Let $f : D(a,R)\backslash\{a\} \to \mathbb{C}$ be holomorphic. The coefficient c_{-1} of the Laurent series of f in $D(a,R)\backslash\{a\}$ is called the **residue of f at a** denoted $\mathrm{Res}_f(a)$.
$f_P \equiv \sum_{n=1}^{\infty} c_{-n}(z-a)^{-n}$ is called the **principal part of f at a**

Recall (see remarks following the proof of the Laurent series theorem) that the principal part f_P is holomorphic on $\mathbb{C}\backslash\{a\}$, with the series defining f_P converging uniformly on compact subsets of $\mathbb{C}\backslash\{a\}$. By the Laurent series, $f = f_P + h$ on $D(a,R)\backslash\{a\}$, with h holomorphic on $D(a,R)$. Let γ be a closed curve in $D(a,R)$, with $a \notin \text{Image}(\gamma)$. Then $\int_\gamma h(z)\,\mathrm{d}z = 0$ by Cauchy's theorem, and hence $\int_\gamma f(z)\,\mathrm{d}z = \int_\gamma f_P(z)\,\mathrm{d}z = 2\pi i I(\gamma;a)\text{Res}(a)$, where the last equality is by uniform convergence of the series for f_P and the FTC.
This reasoning can easily be extended to the case of more than one isolated singularity, and leads to the following imporant result:

4.3 Residue Theorem

Theorem 4.6 (Residue Theorem)**.** Let U be an open set $\{a_1, \alpha_2, \dots, a_k\} \subset U$ a finite set, and $f : U\backslash\{a_1, a_2, \dots, a_k\} \to \mathbb{C}$ be holomorphic. If γ is any closed curve in U homologous to zer oin U, and if $a_j \notin \text{Image}(\gamma)$ for each j, then

$$\frac{1}{2\pi i}\int_\gamma f(z)\,\mathrm{d}z = \sum_{j=1}^{k} I(\gamma;a_j)\text{Res}_f(a_j)$$

Proof. Let $f_P^{(j)} = \sum_{n=1}^{\infty} c_{-n}^{(j)}(z-a_j)^{-n}$ be the principal part of f at a_j. Then $f_P^{(j)}$ is holomorphic in $\mathbb{C}\backslash\{a_j\}$, and hence in $\mathbb{C}\backslash\{a_1, a_2, \dots, a_k\}$. So $h \equiv f - (f_P^{(1)} + f_P^{(2)} + \dots + f_P^{(k)})$ is holomorphic in $U\backslash\{a_1, a_2, \dots, a_k\}$. Fix a j. The function $f - f_P^{(j)}$ has a removable singularity at $z = a_j$. For each $l \neq j$, $f_p^{(l)}$ is holomorphic at a_j. Hence h has a removable singularity at a_j. This is true for every j, so h extends to all of U as a holomorphic function and hence by Cauchy's theorem, $\int_\gamma h(z)\,\mathrm{d}z = 0$. So

$$\frac{1}{2\pi i}\int_\gamma f(z)\,\mathrm{d}z = \sum_{j=1}^{k}\int_\gamma f_P^{(j)}\,\mathrm{d}z$$

But

$$\frac{1}{2\pi i}\int_\gamma f_P^{(j)}(z)\,\mathrm{d}z = I(\gamma;a_j)\text{Res}(a_j)$$

(by term-wise integration of the series for $f_P^{(j)}$, which converges uniformly on compact subsets of $\mathbb{C}\backslash\{a_j\}$)

Notes. Useful facts for residue calculation:

(i) If f has a simple pole at $z = a$, then $\mathrm{Res}_f(a) = \lim_{z\to a}(z-a)f(z)$. (Near a, we have $f(z) = (z-a)^{-1}g(z)$ with g holomorphic and $g(a) \neq 0$, so by Taylor series of g, $\mathrm{Res}_f(a) = g(a)$)

(ii) If f has a pole of order k at a, then near a we have $f(z) = (z-a)^{-k}g(z)$ with g holomorphic and $g(a) \neq 0$. In this case

$$\mathrm{Res}_f(a) = \text{coefficient of } (z-a)^{k-1} \text{ of the Taylor series of } g \text{ at } a = \frac{g^{(k-1)}(a)}{(k-1)!}$$

(iii) If $f = g/h$ with g, h holomorphic at $z = a$, $g(a) \neq 0$, and h has a simple zero (i.e. a zero of order 1) at $z = a$, then

$$\mathrm{Res}_f(a) = \frac{g(a)}{h'(a)}$$

This follows from (i):

$$\mathrm{Res}_f(a) = \lim_{z\to a}\frac{(z-a)g(z)}{h(z)} = \lim_{z\to a}\frac{g(z)}{\frac{h(z)-h(a)}{z-a}} = \frac{g(a)}{h'(a)}$$

Remark. Note that this generalises the CIF (Theorem 3.6)

Two useful lemmas for computing $\int_\gamma f(z)\,\mathrm{d}z$:

- Integrals on large semi-circles
- Integrals on small circular arcs

Lemma 4.7 (Jordan's lemma)**.** Let f be a continuous complex-values function on the semi-circle $C_R^+ = \text{Image}\gamma_R^+$ in the upper-half plae, where $R > 0$, and $\gamma_R^+(t) = Re^{it},\ 0 \le t \le \pi$. Then for $\alpha > 0$

$$\left| \int_{\gamma_R^+} f(z) e^{i\alpha z} \, \mathrm{d}z \right| \le \frac{\pi}{\alpha} \sup_{z \in C+R^+} |f(z)|$$

In particular, if f is continuous in $H^+ \backslash D(0, R_0)$ for some $R_0 > 0$ where $H^+ = \{z : \text{Im}(z) \ge 0\}$, and if $\sup_{z \in C_R^+} |f(z)| \to 0$ as $R \to \infty$, then for each $\alpha > 0$,

$$\int_{\gamma_R^+} f(z) e^{i\alpha z} \, \mathrm{d}z \to 0 \text{ as } R \to \infty$$

Proof. Letting $M_R = \sup_{z \in C_R^+} |f(z)|$, we have

$$\begin{aligned}
\left| \int_{\gamma_R^+} f(z) e^{i\alpha z} \, \mathrm{d}z \right| &= \left| \int_0^\pi f(Re^{it}) e^{-\alpha R \sin t + i\alpha R \cos t} iRe^{it} \, \mathrm{d}t \right| \\
&\le RM_R \int_0^\pi e^{-\alpha R \sin t} \, \mathrm{d}t \\
&= RM_R \left(\int_0^{\pi/2} e^{-\alpha R \sin t} \, \mathrm{d}t + \int_{\pi/2}^\pi e^{-\alpha R \sin t} \, \mathrm{d}t \right) \\
&= 2RM_R \int_0^{\pi/2} e^{-\alpha R \sin t} \, \mathrm{d}t \\
&\le 2RM_R \int_0^{\pi/2} e^{-2\alpha R t/\pi} \, \mathrm{d}t \\
&= \frac{\pi M_R}{\alpha} (1 - e^{-2\alpha R}) \\
&\le \frac{\pi M_R}{\alpha}
\end{aligned}$$

where we have used the fact that for $t \in (0, \pi/2],\ \varphi(t) \equiv \frac{\sin t}{t} \ge \frac{2}{\pi}$ (since $\varphi(\pi/2) = 2/\pi$ and $\varphi'(t)) \le 0$ on $[0, \pi/2]$

Remark. A similar statement holds for $\alpha < 0$ and for the semi-circle C_R^- =Image(γ_R^-) in the lower half-plane, where $\gamma_R^-(t) = -Re^{it}$ for $R > 0$ and $0 \le t \le \pi$

Lemma 4.8. Let f be holomorphic in $D(a, R)\backslash\{a\}$ with a simple pole at $z = a$. If $\gamma_\varepsilon : [\alpha, \beta] \to \mathbb{C}$ is the circular arc $\gamma_\varepsilon(t) = a + \varepsilon e^{it}$, then

$$\lim_{\varepsilon\to 0^+} \int_{\gamma_e} f(z)\,\mathrm{d}z = (\beta - \alpha)i\mathrm{Res}_f(a)$$

Proof. Write $f(z) = \frac{c}{z-a} + g(z)$ with g holomorphic in $D(a, R)$ and $c = \mathrm{Res}_f(a)$. Then

$$\begin{aligned}\left|\int_{\gamma_\varepsilon} g(z)\,\mathrm{d}z\right| &= \left|\int_\alpha^\beta g(a + \varepsilon e^{it})\varepsilon i e^{it}\,\mathrm{d}t\right| \\ &\leq \varepsilon(\beta - \alpha) \sup_{t\in[\alpha,\beta]} |g(a + \varepsilon e^{it})| \to 0\end{aligned}$$

as $\varepsilon \to 0^+$. and

$$\int_{\gamma_\varepsilon} \frac{c}{z-a}\,\mathrm{d}z = (\beta - \alpha)ic$$

by direct calculation. The claim follows.

Examples. (i) $\int_0^\infty \frac{\sin x}{x}\,\mathrm{d}x$. Let $f(z) = e^{iz}/z$. Consider the integral $\int_\gamma f(z)\,\mathrm{d}z$ over the curve $\gamma = \gamma_R + \gamma_1 + \gamma_\varepsilon + \gamma_2$, where $\gamma_R(t) = Re^{it}$ for $0 \le t \le \pi$, $\gamma_1(t) = t$ for $-R \le t \le -\varepsilon$, $\gamma_\varepsilon(t) = \varepsilon e^{-it}$ for $-\pi \le t \le 0$ and $\gamma_2(t) = t$ for $\varepsilon \le t \le R$ (Draw a picture!)

By Jordan's lemma, $\int_{\gamma_R} f(z)\,\mathrm{d}z \to 0$ as $R \to \infty$.

f has a simple pole at $z = 0$ with $\mathrm{Res}_f(0) = \lim_{z\to 0} zf(z) = 1$. So by Lemma 4.7, $\int_{-\gamma_\varepsilon} f(z)\,\mathrm{d}z \to \pi i$ as $\varepsilon \to 0^+$.

Now f is holomorphic in $U = \mathbb{C}\backslash\{0\}$ and the curve γ is homologous to zer oin U (either convince yourself that γ is null-homotopic in U or use the fact that $\mathbb{C}\backslash U = \{0\}$, and $I(\gamma; 0) = 0$ since 0 is in the unbounded component of $\mathbb{C}\backslash$ Image(γ)). Hence by Cauchy's theorem $\int_\gamma f(z)\,\mathrm{d}z = 0$. So $\int_{\gamma_R} f(z)\,\mathrm{d}z + \int_{-R}^{\varepsilon} \frac{e^{it}}{t}\,\mathrm{d}t + \int_{\gamma_\varepsilon} f(z)\,\mathrm{d}z + \int_e^R \frac{e^{it}}{t}\,\mathrm{d}t = 0$. Combining the two integrals on the real acis after a simple change of variables, this gives

$$\int_\varepsilon^R \frac{e^{it} - e^{-it}}{t}\,\mathrm{d}t + \int_{\gamma_R} f(z)\,\mathrm{d}z + \int_{\gamma_\varepsilon} f(z)\,\mathrm{d}z = 0$$

Letting $R \to \infty,\ \varepsilon \to 0^+$, this gives

$$\int_0^\infty \frac{\sin t}{t}\,\mathrm{d}t = \frac{\pi}{2}$$

(ii) A proof of $\sum_{n\ge 1} \frac{1}{n^2} = \frac{\pi^2}{6}$.
For this, consider the function

$$f(z) = \frac{\pi\cot(\pi z)}{z^2} = \frac{\pi\cos(\pi z)}{z^2\sin(\pi z)}$$

which is holomorphic in $\mathbb{C}$ except for simple poles at each point in $\mathbb{Z}\backslash\{0\}$, and an order 3 pole at 0. Near $n \in \mathbb{Z}\backslash\{0\}$, we have $f(z) = g(z)/h(z)$, where $g(n) \ne 0$ and h has a simple zero at n and so

$$\mathrm{Res}_f(n) = \frac{g(n)}{h'(n)} = \frac{1}{n^2}$$

To compute $\mathrm{Res}_f(0)$, use:

$$\begin{aligned}\cot(z) = \cos(z)\cdot[\sin(z)]^{-1} &= \left(1 - \frac{z^2}{2} + O(z^4)\right)\cdot\left(z - \frac{z^3}{6} + O(z^5)\right) \\ &= \frac{1}{z} - \frac{z}{3} + O(z^2)\end{aligned}$$

which gives

$$\frac{\pi\cot(\pi z)}{z^2} = \frac{1}{z^3} - \frac{\pi^2}{3z} + \dots$$

and so $\mathrm{Res}_f(0) = 0\pi^2/3$. Now for $N \in \mathbb{N}$, take γ_N to be the positively oriented boundary of the square defined by the lines $x = \pm(N + 1/2)$ and $y = \pm i(N + 1/2)$. Then by the residue theorem

$$\int_{\gamma_N} f(z)\,\mathrm{d}z = 2\pi i\left[2\left(\sum_{n=1}^N \frac{1}{n^2}\right) - \frac{\pi^2}{3}\right] \qquad (*)$$

Examples (continued)**.** (ii) But since $\text{length}(\gamma_N) = 4(2N+1)$, we have

$$\left|\int_{\gamma_N} f(z)\,\mathrm{d}z\right| \le \sup_{\gamma_N}\left|\frac{\pi\cot(\pi z)}{z^2}\right| \cdot 4(2N+1)$$
$$\le \sup_{\gamma_N}|\cot(\pi z)| \cdot \frac{4(2N+1)\pi}{(N+1/2)^2}$$
$$= \frac{16\pi}{(2N+1)} \cdot \sup_{\gamma_N}|\cot(\pi z)|$$

On γ_N, $\cot(\pi z)$ is bounded independently on N (exercise), and hence $\int_{\gamma_N} f(z)\,\mathrm{d}z \to 0$ as $N \to \infty$. Letting $N \to \infty$ in (*), we get

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

5 The Argument Principle, Local Degree and Rouché's Theorem

Prop 5.1. If f has a zero (pole) of order $k \ge 1$ at $z = a$, then f'/f has a simple pole at $z = a$ with residue $\text{Res}_{f'/f}(a) = k$ ($-k$ resp.)

Proof. If $z = a$ is a zero of order k, then there si a disk $D(a, r)$ such that $f(z) = (z-a)^k g(z)$ for $z \in D(a, r)$, where $g : D(a, r) \to \mathbb{C}$ is holomorphic with $g(z) \ne 0\ \forall z \in D(a, r)$. So

$$f'(z) = k(z-a)^{k-1}g(z) + (z-a)^k g'(z)$$

and

$$\frac{f'(z)}{f(z)} = \frac{k}{z-a} + \frac{g'(z)}{g(z)} \quad \forall z \in D(a, r)\backslash\{a\}$$

Since g'/g is holomorphic in $D(a, r)$, the claim follows from this.
In the case of a pole of order k, use the same argument and the fact that $f(z) = (z-a)^{-k}g(z)$ in $D(a, r)\backslash\{a\}$.

Notation. Denote by $\text{ord}_f(a)$, the order of the zero or pole of f at $z = a$

Theorem 5.2 (Argument Principle)**.** Let f be a meromorphic function on a domain U with finitely many zeros $a_1, a_2, \ldots . a_k$ and finitely many poles $b_1, b_2, \ldots, b_l$. If γ is a closed curve in U homologous to zero in U, and if $a_i, b_j \notin \text{Image}(\gamma)$ for all i, j then

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} = \sum_{i=1}^{k} I(\gamma; a_i)\text{ord}_f(a_i) - \sum_{j=1}^{l} I(\gamma; b_j)\text{ord}_f(b_j)$$

Proof. Apply the residue theorem to $g = f'/f$. IF $z_0 \in U$ is not a pole of f, then f, and also f', is holomorphic near z_0. If additionally z_0 not a zero of f, this makes g holomorphic near z_0. So the set of singularities of g is precisely $\{a_1, \ldots, a_k\} \cup \{b_1, \ldots, b_l\}$. By Prop 5.1, $\text{Res}_g(a_i) = \text{ord}_f(a_i)$ for each i and $\text{Res}_g(b_j) = -\text{ord}_f(b_j)$ for each j.

Remark. Let f, γ be as in the theorem, and let $\Gamma(t) = f(\gamma(t))$. Then Γ is a closed curve with $\text{Image}(\Gamma) \subset \mathbb{C}\backslash\{0\}$ (since no zeros or poles of f on $\text{Image}(\gamma)$ so $f(\gamma(t)) \neq 0, \infty$); moreover,

$$I(\Gamma; 0) = \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} \, \mathrm{d}z$$

(Proof: If $[a, b]$ is the domain of γ. $I(\Gamma; 0) = \frac{1}{2\pi i} \int_\Gamma \frac{\mathrm{d}z}{z} = \frac{1}{2\pi i} \int_a^b \frac{\Gamma'(t)}{\Gamma(t)} = \frac{1}{2\pi i} \int_a^b \frac{f'(\gamma(t))\gamma'(t)}{f(\gamma(t))} \, \mathrm{d}t = \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} \, \mathrm{d}z$.) Thus $\frac{1}{2\pi i} \frac{f'(z)}{f(z)} \, \mathrm{d}z$ is the number of times the image curve $f \circ \gamma$ winds around 0 as we move along γ.

Definition. Let Ω be a domain and let γ be a closed curve in $\mathbb{C}$. We say that γ **bounds** Ω if $I(\gamma; w) = 1 \; \forall w \in \Omega$ and $I(\gamma; w) = 0 \; \forall w \in \mathbb{C}\backslash(\Omega \cup \text{Image}(\gamma))$

Example. $\partial D(0, 1)$ bounds $D(0, 1)$ but it does not bound $D(0, 1)\backslash\{0\}$

Remark. If γ bounds a domain Ω, then

- Ω is bounded [proof: choose a disk $D(a, R)$ such that $\text{Image}(\gamma) \subset D(a, R)$. Then $I(\gamma; w) = 0$ for $w \in \mathbb{C}\backslash D(a, R)$. Since $I(\gamma; w) = 1$ for each $w \in \Omega$, we must have $\Omega \subset D(a, R)$]
- The topological boundary $\partial\Omega \subset$Image(γ). (Exercise to check); it need not be true that $\partial\Omega = \text{Image}(\gamma)$.

There is a large class of closed curves that bound domains, namely, simple closed curved, i.e. curves $[a,b] \to \mathbb{C}$ with $\gamma(a) = \gamma(b)$ such that $\gamma(t_1) = \gamma(t_2) \implies t_1 = t_2$ or $\{t_1, t_2\} = \{a, b\}$ (so no self-intersections)..
That a simple closed curve bounds a domain is a highly non-trivial fact guaranteed by the Jordan curve theorem: if γ is a simple closed curve, then $\mathbb{C}\backslash\text{Image}(\gamma)$ consists precisely of two connected components, one unbounded and the other bounded, and moreover, γ (or $-\gamma$) bounds the bounded component (in the sense defined above), and Image(γ) is the boundary of each of these two componenets. [Thus if Ω_1 is the bounded component (the "inside" of γ) and Ω_2 is the unbounded component (the "outside"), then, after possibly changing the orientation of γ, we have $I(\gamma; w) = 1$ for $w \in \Omega_1$ and $I(\gamma; w) = 0$ for $w \in \Omega_2$; this last assertion is easy: for any disk $D(a, R) \supset \text{Image}(\gamma)$, $I(\gamma; w) = 0 \ \forall w \in \mathbb{C}\backslash D(a, R)$.]
For a domain bounded by a closed curve, the argument principle says tha following

Corollary 5.3 (argument principle for domains bounded by closed curves)**.** Let γ be a closed curve bounding a domain Ω, and let f be meromorphic in an open set U with $\Omega\cup$ Image$(\gamma) \subset U$. Suppose that f has no zeros or poles on Image(γ), and precisely N zeros and P poles in Ω, both counted with multiplicity. Then N and P are finite, and

$$N - P = \frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,\mathrm{d}z = I(\Gamma; 0)$$

where $\Gamma = f \circ \gamma$ is the image of γ under f

Proof. Since f is meromorphic in U, its singularities form a discrete set $S \subset U$ consisting of poles or removable singularities. Since γ bounds Ω, we have that Ω is bounded and hence $\overline{\Omega}$ is compact; also, $\overline{\Omega} \subset \Omega \cup \text{Image}(\gamma) \subset U$. If $\overline{\Omega} \cap S$ is infinite, then (by compactness of $\overline{\Omega}$) there is a point $w \in \overline{\Omega}$ and distinct points $w_j \in \overline{\Omega} \cap S$ such that $w_j \to w$. If $w \notin S$, then f is defined and holomorphic near w which is impossible since $w_j \in S$ and $w_j \to w$. So $w \in S$, but this is impossible since S is a discrete set. So $\overline{\Omega} \cap S$ is finite. In particular P is finite.
If f has infinitely many zeros in Ω, then (again by compactness of $\overline{\Omega}$) there is $z \in \overline{\Omega} \subset U$ and distinct zeros $z_j \in \Omega$ such that $z_j \to z$. Then either $z \in U\backslash S$ or (if $z \in S$), z must be a removable singularity (since otherwise z would be a pole and hence $|f(\zeta)| \to \infty$ as $\zeta \to z$ which is impossible since $z_j \to z$ and $f(z_j) = 0$.) In either case, by the principle of isolated zeros, f must be identically zero in $D(z, \rho)\backslash\{z\}$ for some $\rho > 0$. Since f is holomorphic in $\Omega\backslash S$ and $\Omega\backslash S$ is connected (since $\Omega \cap S$ is finite and Ω is connected), it follows from the unique continuation principle that $f \equiv 0$ in Ω. But this is not possible since f has no zeros in Image(γ). Hence N must be finite.
By the defininition of "γ bounding Ω", we have that $I(\gamma; w) = 1$ for every $w \in \Omega$ and $I(\gamma; w) = 0$ for every $w \in \mathbb{C}\backslash(\Omega \cup$ Image$(\gamma))$. In particular, this makes γ homologous to zer oin U. The final conclusions now follows from Theorem 5.2 and the remark after its proof (which says that Γ is a closed curve in $\mathbb{C}\backslash\{0\}$ and $I(\Gamma; 0) = \frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,\mathrm{d}z$)

Note. "zeros counted with multiplicity" means each zero is counted as many times as its order; same for poles

An important consequence of this is the local degree theorem below:

Definition. Let f be a holomorphic function on a disk $D(a,R)$ and assume that f is non-constant. The **local degree** of f at a, denoted $\deg_f(a)$, is the order of the zero of $f(z)-f(a)$ at $z=a$. This is a (finite) positive integer.
E.g. $f(z)=(z-1)^4+1$ has $\deg_f(1)=4$

Theorem 5.4 (Local degree theorem)**.** Le $f: D(a,R)\to\mathbb{C}$ be holomorphic and non-constant, with $\deg_f(a)=d$. There exists $r_0>0$ with the following property: for any $r\in(0,r_0]$, there is $\varepsilon>0$ such that for every w with $0<|f(a)-w|<\varepsilon$, the equation $f(z)=w$ has precisely d distinct roots in $D(a,r)\backslash\{a\}$

Proof. Let $g(z)=f(z)-f(a)$. Since g is non-constant $g'\not\equiv 0$ in $D(a,R)$. Apply the principle of isolated zeros (to g' and g to get $r_0\in(0,R)$ such that $g(z)\neq 0$ and $g'(z)\neq 0$ for $z\in\overline{D(a,r_0)}\backslash\{a\}$.
Claim: the conclusion holds with this r_0.
To see this, let $r\in(0,r_0]$, and for $r\in[0,1]$, let $\gamma(t)=a+re^{2\pi it}$ and $\Gamma(t)=g(\gamma(t))$. Note that Image(Γ) is compact and hence cosed in $\mathbb{C}$, and $0\notin$ Image(Γ) (since $g\neq 0$ on $\partial D(a,r)$). So there is $\varepsilon>0$ such that $D(0,\varepsilon)\subset\mathbb{C}\backslash$ Image(Γ). This is our choice of ε (corresponding to r).
To check that it works, fix any w with $0<|w-f(a)|<\varepsilon$. Then $w-f(a)\in D(0,\varepsilon)\subset\mathbb{C}\backslash$ Image(Γ). Since $z\mapsto I(\Gamma;z)$ is locally constant, it is constant on $D(0,\varepsilon)$m so we have $I(\Gamma;w-f(a))=I(\Gamma;0)$.
By direct calculation

$$I(\Gamma;w-f(a))=\frac{1}{2\pi i}\frac{g'(\gamma(t))\gamma'(t)}{g(\gamma(t))-(w-f(a))}\,\mathrm{d}t=\frac{1}{2\pi i}\int_{\partial D(a,r)}\frac{f'(z)}{f(z)-w}\,\mathrm{d}z$$

By the argument principle (Cor. 5.3), $I(\Gamma;0)=d$ ($I(\Gamma;0)$ is the number of zeros of g in $D(a,r)$ counted with multiplicity; the zero of g at $z=a$ has order d, and it is the only zero in $D(a,r)$). So we have

$$\frac{1}{2\pi i}\int_{\partial D(a,r)}\frac{f'(z)}{f(z)-w}\,\mathrm{d}z=d$$

By Cor 5.3 again, this implies that the number of zeros of $f(z)-w$ in $D(a,r)$ is d. None of these zeros are equal to a since $w\neq f(a)$. Since $f'(z)=g'(z)\neq 0$ in $D(a,r)\backslash\{a\}$, it follows (from the Taylor series) that these zeros are simple (i.e. have order 1). Thus $f(z)-w$ has d distinct zeros in $D(a,r)\backslash\{a\}$

Corollary 5.5 (Open mapping theorem)**.** A non-constant holomorphic function maps open sets to opens

Proof. Let $f: U\to\mathbb{C}$ be holomorphic and non-constant, and let $V\subset U$ be open. Let $b\in f(V)$. Then $b=f(a)$ for some $a\in V$. Since V is open, there is $r>0$ such that $D(a,r)\subset V$. By Theorem 5.4, if r is sufficiently small, there is $\varepsilon>0$ such that $w\in D(f(a),\varepsilon)\backslash\{f(a)\}\implies w=f(z)$ for some $z\in D(a,r)\backslash\{a\}$, i.e. $D(f(a),\varepsilon)\backslash\{f(a)\}\subset f(D(a,r)\backslash\{a\})$. This implies that

$$D(b,\varepsilon)=D(f(a),\varepsilon)\subset f(D(a,r))\subset f(V)$$

Thus for every $b\in f(V)$ there is a disk $D(b,\varepsilon)\subset f(V)$, so $f(V)$ is open

Theorem 5.6 (Rouché's theorem)**.** Let γ be a closed curve bounding a domain Ω, and let f, g be holomorphic functions on an open set U containing $\Omega\cup$ Image(γ). If $|f(z) - g(z)| < |g(z)|$ for every $z \in$ Image(γ), then f and g have the same number of zeros in Ω, counted with multiplicity

Proof. The strict inequality $|f - g| < |g|$ on Image(γ) implies that f and g are never zero on Image(γ), and hence never zero in some open set V containing Image(γ). So $h = f/g$ is holomorphic and never zero in V.
In particular, g is not identically zero in Ω, and hence the zeros of g in $\Omega \cup V$ are isolated. So h is heromorphic in $\Omega \cup V$, and h has no zeros or poles on Image(γ). Also f and g have finitely many zeros in Ω.
Now $|h(z)-1| < 1$ for $z \in$ Image(γ). This means that the curve $\Gamma = h\circ\gamma$ has image contained in $D(1,1)$. since 0 is outside this disk, $I(\Gamma; 0) = 0$, and so by Corollary 5.3,

$$\sum_{w\in\mathcal{P}} \mathrm{ord}_h(w) = \sum_{w\in\mathcal{Z}} \mathrm{ord}_h(w)$$

where $\mathcal{P}$ and $\mathcal{Z}$ denote the sets of distinct poles and zeros of h respectively, and the sums are finite. Now $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$ where

$$P_1 = \{w \in \Omega : g(w) = 0, f(w) \neq 0\}$$
$$P_2 = \{w \in \Omega : g(w) = f(w) = 0, \mathrm{ord}_g(w) > \mathrm{ord}_f(w)\}$$

and $\mathcal{Z} = \mathcal{Z} \cup \mathcal{Z}_2$m where

$$Z_1 = \{w \in \Omega : f(w) = 0, g(w) \neq 0\}$$
$$Z_2 = \{w \in \Omega : f(w) = g(w) = 0, \mathrm{ord}_f(w) > \mathrm{ord}_g(w)\}$$

So by the above

$$\sum_{w\in\mathcal{P}_1} \mathrm{ord}_g(w) + \sum_{w\in\mathcal{P}_2} (\mathrm{ord}_g(w) - \mathrm{ord}_f(w)) = \sum_{w\in\mathcal{Z}_1} \mathrm{ord}_f(w) + \sum_{w\in\mathcal{Z}_2} (\mathrm{ord}_f(w) - \mathrm{ord}_g(w))$$

or equivalently

$$\sum_{w\in\mathcal{P}_1} \mathrm{ord}_g(w) + \sum_{w\in\mathcal{P}_2} \mathrm{ord}_g(w) + \sum_{w\in\mathcal{Z}_2} \mathrm{ord}_g(w) = \sum_{w\in\mathcal{Z}_1} \mathrm{ord}_f(w) + \sum_{w\in\mathcal{Z}_2} \mathrm{ord}_f(w) + \sum_{w\in\mathcal{P}_2} \mathrm{ord}_f(w)$$

adding $\sum_{w\in\mathcal{R}} \mathrm{ord}_g(w)$ to the left-hand side of this and the equal number $\sum_{w\in\mathcal{R}} \mathrm{ord}_f(w)$ to the right hand side, where $\mathcal{R} = \{w \in \Omega : f(w) = g(w) = 0,\ \mathrm{ord}_f(w) = \mathrm{ord}_g(w)\}$, we conclude

$$\sum_{\{w\in\Omega: g(w)=0} \mathrm{ord}_g(w) = \sum_{\{w\in\Omega: f(w)=0} \mathrm{ord}_f(w)$$

Example. $z^4 + 6z + 3$ has 3 roots (counted with multiplicity) in $\{1 < |z| < 2\}$

Proof. Let $f(z) = z^4 + 6z + 3$. On $|z| = 2$, we have $|z|^4 = 16$ and $|6z + 3| \leq 6|z| + 3 = 15$ so $|z|^4 > |6z + 3|$. So by Rouché's theorem, f has the same number of roots inside $\{|z| < 2\}$ as z^4 (counting with multiplicity). Thus all roots of $f(z) = z^4 + 6z + 3$ lie inside $\{|z| < 2\}$ (we know that this is all the roots as f is polynomial). On $|z| = 1$, we have $|6z| = 6$ and $|z^4 + 3| \leq |z|^4 + 3 = 4$, so $|6z| > |z^4 + 3|$, and again by Rouché's theorem, we see that f has jsut one root inside $\{|z| < 1\}$ as $6z$ has one roots there, from our strinct inequalities, no roots lie on $|z| = 2$ or $|z| = 1$. So 3 roots of f must lie in $\{z \in \mathbb{C} : 1 < |z| < 2\}$

Example. For $0 < \alpha < 1$, sho that $\int_0^\infty \frac{x^{-\alpha}}{1+x}\,\mathrm{d}x = \frac{\pi}{\sin \pi\alpha}$.
Let $g(z) = z^{-\alpha}$ be the brainch of $z^{-\alpha}$ defined by $g(z) = e^{-\alpha l(z)}$ where $l(z)$ is the (holomorphic) branch of logarithm on $U = \mathbb{C}\backslash\{x \in \mathbb{R} : x \geq 0\}$ given by $l(z) = \log|z| + i\arg(z)$, with $\arg(z)$ the argument of z in $(0, 2\pi)$. Let $f(z) = g(z)/(1+z)$. Then

$$f(z) = \frac{|z|^{-\alpha}e^{-i\alpha\arg(z)}}{1+z}$$

and f is holomorphic in $U\backslash\{-1\}$ iwth $z = -1$ a simple pole with $\mathrm{Res}_f(-1) = \lim_{z\to-1}(z+1)f(z) = e^{-i\pi\alpha}$.
Choose ε, R such that $0 < \varepsilon < 1 < R$ and $\theta > 0$ small. Let γ be positively oriented "key-hole contour" determined by the two circular arcs $\gamma_R, \gamma_\varepsilon : [\theta, 2\pi - \theta] \to U,\ \gamma_R(t) = Re^{it},\ \gamma_\varepsilon(t) = \varepsilon e^{i(2\pi - t)}$, and the two line segments $\gamma_1, \gamma_2 : [\varepsilon, R] \to U,\ \gamma_1(t) = te^{i\theta},\ \gamma_2(t) = te^{i(2\pi-\theta)}$. The domain U is simply connected (in fact star-shaped) and hence γ is homologous to zero in U. And $I(\gamma; -1) = 1$ (directly from the definitions of γ and winding number). By the residue theorem,

$$\int_\gamma f(z)\,\mathrm{d}z = 2\pi i e6{-}i\pi\alpha$$

Now

$$\int_{\gamma_1} f(z)\,\mathrm{d}z = \int_\varepsilon^R f(te^{i\theta})e^{i\theta}\,\mathrm{d}t - \int_\varepsilon^R \frac{t^{-\alpha}e^{i(1-\alpha)\theta}}{1+te^{i\theta}}\,\mathrm{d}t$$

and

$$\int_{\gamma_2} f(z)\,\mathrm{d}z = \int_\varepsilon^R f(te^{i(2\pi-\theta)})e^{i(2\pi-\theta)}\,\mathrm{d}t = \int_\varepsilon^R \frac{t^{-\alpha}e^{i(1-\alpha)(2\pi-\theta)}}{1+te^{i(2\pi-\theta)}}\,\mathrm{d}t$$

As $\theta \to 0^+$, the integrands (on the right) converge uniformly on $[\varepsilon, R]$ to $\frac{t^{-\alpha}}{1+t}$ and $\frac{e^{-2i\pi\alpha}t^{-\alpha}}{1+t}$ respectively (exercise to check). So

$$\lim_{\theta\to0^+}\left(\int_{\gamma_1} f(z)\,\mathrm{d}z + \int_{(-\gamma_2)} f(z)\,\mathrm{d}z\right) = (1 - 2^{-i2\pi\alpha})\int_\varepsilon^R \frac{t^{-\alpha}}{1+t}\,\mathrm{d}t$$

$|f(z)| \leq \frac{R^{-\alpha}}{R-1}\ \forall z \in \mathrm{Image}(\gamma_R)$, and $|f(z)| \leq \frac{\varepsilon^{-\alpha}}{1-\varepsilon}\ \forall z \in \mathrm{Image}(\gamma_\varepsilon)$. So

$$\left|\int_{\gamma_R} f(z)\,\mathrm{d}z + \int_{\gamma_\varepsilon} f(z)\,\mathrm{d}z\right| \leq \frac{2\pi R^{1-\alpha}}{R-1} + \frac{2\pi\varepsilon^{1-\alpha}}{1-\varepsilon}$$

(Note: RHS is independent of θ, even though $\gamma_R, \gamma_\varepsilon$ depend on θ). Since

$$\int_\gamma f(z)\,\mathrm{d}z - (\int_{\gamma_1} f(z)\,\mathrm{d}z + \int_{(-\gamma_2)} f(z)\,\mathrm{d}z) = \int_{\gamma_R} f(z)\,\mathrm{d}z + \int_{\gamma_\varepsilon} f(z)\,\mathrm{d}z$$

we then have

$$\left|2\pi i e^{-i\pi\alpha} - (\int_{\gamma_1} f(z)\,\mathrm{d}z + \int_{(-\gamma_2)} f(z)\,\mathrm{d}z)\right| \leq \frac{2\pi R^{1-\alpha}}{R-1} + \frac{2\pi\varepsilon^{1-\alpha}}{1-\varepsilon}$$

First letting $\theta \to 0^+$ in this, and then letting $\varepsilon \to 0^+$ and $R \to \infty$, we conclude

$$(1 - e^{-2\pi i\alpha}\int_0^\infty \frac{t^{-\alpha}}{1+t}\,\mathrm{d}t = 2\pi i e^{-i\pi\alpha})$$

so

$$\int_0^\infty \frac{t^{-\alpha}}{1+t} = \frac{\pi}{\sin \pi\alpha}$$