Differential Equations

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Contents

1 Differentiation

1.1 L'Hopital's Rule

Theorem (L'Hopital)**.** If:

$$\lim_{x \to x_0} f(x) = f(x_0) = 0$$

$$\lim_{x \to x_0} g(x) = g(x_0) = 0$$

Then:

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}$$

(if $g'(x) \neq 0$)

Proof. Consider $f(x)$ expanded to $f'(x_0)$ in Taylor series.

1.2 Partial Differentiation

Notes.
Shorthand Notation:

- $\frac{\partial f}{\partial x} = f_x$
- $\frac{\partial^2 f}{\partial x \partial y} = f_{xy}$

1.2.1 Chain rule

Equation (Chain rule)**.** Given $f(x, y)$

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t}$$

if x and y vary with t

Proof. Consider definition of derivative/ partial derivative

Equation. if $f(x, y(x))$,

$$\begin{aligned}\frac{\mathrm{d}f}{\mathrm{d}x} &= \frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}x} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}x} \\ &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}x}\end{aligned}$$

Proof. Consider $t = x$ in previous eqn

Equation (Infinitesimal form)**.**

$$\mathrm{d}f = \frac{\partial f}{\partial x}\mathrm{d}x + \frac{\partial f}{\partial y}\mathrm{d}y$$

Equation. Integrating along path:

$$\int \mathrm{d}f = \int \frac{\partial f}{\partial x}\,\mathrm{d}x + \int \frac{\partial f}{\partial y}\,\mathrm{d}y$$

Note. Visualising path going along x to x_2 then up y to y_2

$$f(x_2, y_2) - f(x_1, y_1) = \int_{x_1}^{x_2} \frac{\partial f}{\partial x}\,\mathrm{d}x + \int_{y_1}^{y_2} \frac{\partial f}{\partial y}\,\mathrm{d}y$$

1.3 Polar Co-ordinates Transform

Equation. $x = r\cos\theta$, $y = r\sin\theta$

$$\left.\frac{\partial f}{\partial r}\right|_\theta = \left.\frac{\partial f}{\partial x}\right|_y \left.\frac{\partial x}{\partial r}\right|_\theta + \left.\frac{\partial f}{\partial y}\right|_x \left.\frac{\partial y}{\partial r}\right|_\theta$$

$$\implies \left.\frac{\partial f}{\partial r}\right|_\theta = \left.\frac{\partial f}{\partial x}\right|_y \cos\theta + \left.\frac{\partial f}{\partial y}\right|_x \sin\theta$$

Note. $f_r = f_x\cos\theta + f_y\sin\theta$
Similarly, $f_\theta = r(f_y\cos\theta - f_x\sin\theta)$

1.4 Surfaces

Equation. if $f(x, y, z(x, y)) = c$:
We have:

$$\mathrm{d}f = \left.\frac{\partial f}{\partial x}\right|_{y,z} \mathrm{d}x + \left.\frac{\partial f}{\partial y}\right|_{z,x} \mathrm{d}y + \left.\frac{\partial f}{\partial z}\right|_{x,y} \mathrm{d}z$$

$$\implies \left.\frac{\partial z}{\partial x}\right|_y = -\frac{\left.\frac{\partial f}{\partial x}\right|_{y,z}}{\left.\frac{\partial f}{\partial z}\right|_{x,y}}$$

By taking partial wrt x holding y on both sides and rearranging.

Note. LHS at top becomes zero as the function on x, y is constant. The function on x, y, z independent takes value at any point in 3-D space. So it can be a bit bruh at first sight lol.

1.5 Reciprocal Rule

Equation. Reciprocal rule holds as long as same variables held fixed:

$$\left.\frac{\partial r}{\partial x}\right|_y = \frac{1}{\left.\frac{\partial x}{\partial r}\right|_y}$$

1.6 Differentiating Integrals

Equation.

$$I(\alpha) = \int\limits_{a(\alpha)}^{b(\alpha)} f(x;\alpha)\,\mathrm{d}x$$

$$\implies \frac{\mathrm{d}I}{\mathrm{d}\alpha} = \int_{a(\alpha)}^{b(\alpha)} \frac{\partial f}{\partial \alpha}\,\mathrm{d}x + f(b;\alpha)\frac{\mathrm{d}b}{\mathrm{d}\alpha} - f(a;\alpha)\frac{\mathrm{d}a}{\mathrm{d}\alpha}$$

Proof. Consider definition of derivative applied to α.

Note. If $b(\alpha)$ or $a(\alpha)$ constant then $\frac{\mathrm{d}b}{\mathrm{d}\alpha}$ or $\frac{\mathrm{d}a}{\mathrm{d}\alpha}$ respectively is 0 so can remove term.

2 First order DEs

2.1 Definitions

Definition. **ODE**: DE involving function of one variable

Definition. **PDE**: DE involving functions of more than one variable (and partial derivative)

Definition. **Linear**: dependent variable appears linearly e.g. $x^2\frac{\mathrm{d}y}{\mathrm{d}x} + \sin(x)y = e^x$

2.2 Linear

Note. Linear case trivial □

2.3 Non-linear

Note. General form:

$$Q(x,y)\frac{\mathrm{d}y}{\mathrm{d}x} + P(x,y) = 0$$

Definition. Equation **separable** if it can be written in the form:

$$q(y)\mathrm{d}y = p(x)\mathrm{d}x$$

Then can solve for $y(x)$ by integrating both sides.

Definition. Equation **exact** iff $Q(x,y)\mathrm{d}y + P(x,y)\mathrm{d}x$ is an exact differential of a function $f(x,y)$ i.e.

$$\mathrm{d}f = Q(x,y)\mathrm{d}y + P(x,y)\mathrm{d}x$$

Can easily solve if:

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

Method. Solving such DEs:
Have: $\frac{\partial f}{\partial x} = P(x,y)$ and $\frac{\partial f}{\partial y} = Q(x,y)$ for some f (chain rule).

(i) Integrate P w.r.t. x giving constant $h(y)$
(ii) Substitute f into equation for Q to find $h(y)$

Definition. **Isocline**: curve along which $f = \dot{y} =$ constant

Note. When drawing isoclines, have arrows pointing in same direction along line.

2.4 Perturbation analysis

Method. To determine stability of fixed point, let $y = a + \varepsilon(t)$. If $\frac{\mathrm{d}y}{\mathrm{d}t} = f(y,t)$, Taylor Series approx gives:

$$\frac{\mathrm{d}\varepsilon}{\mathrm{d}t} \simeq \varepsilon \frac{\partial f}{\partial y}$$

If $\frac{\partial f}{\partial y} > 0$ unstable.
If $\frac{\partial f}{\partial y} < 0$ stable.
If $\frac{\partial f}{\partial y} = 0$ need higher order terms.

Method. Plotting 2D phase portrait:
$\frac{\mathrm{d}y}{\mathrm{d}t}$ on vertical axis, y on horizontal axis i.e. how $\frac{\mathrm{d}y}{\mathrm{d}t}$ varies with y

Method. Plotting 1D phase portrait:
y on the horizontal axis, arrows to show sign of $\frac{\mathrm{d}y}{\mathrm{d}t}$, solid circle shows stable fixed point, hollow circle shows unstable fixed point.

2.4.1 Discrete fixed points

Method. To find stability of fixed point in discrete equation:

$$x_{n+1} = f(x_n)$$

Expand $f(x)$ in Taylor Series to see:
x_f is stable if $\left|\frac{\mathrm{d}f}{\mathrm{d}x}\right| < 1$ at x_f (goes closer to x_f)
x_f is unstable if $\left|\frac{\mathrm{d}f}{\mathrm{d}x}\right| > 1$ at x_f (goes further from x_f)
Need higher order terms if $\left|\frac{\mathrm{d}f}{\mathrm{d}x}\right| = 1$ at x_f

3 Higher order DEs

3.1 Detuning

Method. Finding second solution when repeated roots:
(i) Consider slightly modified equation e.g. $y'' - 4y' + (4 - \varepsilon^2)y = 0$
(ii) Solve this equation
(iii) Expand Taylor Series to $O(\varepsilon)$
(iv) Substitute using boundary conditions

3.2 Reduction of order

Method. Given y_1 a solution to a DE, let $y_2 = vy_1$ to reduce the order. Trivial algebra leads to:

$$v''y_1 + (2y_1 + py_1)v' = 0$$

Which we can solve as 1st order in v' (as we are given y_1 solution.)

3.3 Wronskian

Definition.

$$W(x) = \begin{vmatrix} \uparrow & \uparrow & & \uparrow \\ \mathbf{Y}_1 & \mathbf{Y}_2 & \dots & \mathbf{Y}_n \\ \downarrow & \downarrow & & \downarrow \end{vmatrix}$$

Note. Usually, $\mathbf{Y}_i$ are solutions to DE.

Warning. $W(x) \neq 0$ sufficient for independent solutions but NOT necessary.

Method. Can find $W(x)$ without solving DE:

$$W(x) = W(x_0)e^{-\int_{x_0}^{x} p(u)\,\mathrm{d}u}$$

Method. Finding $W(x)$ if $\mathbf{Y}' + A\mathbf{Y} = \mathbf{0}$:
Then $W(x) = W(x_0)e^{-\int_{x_0}^{x} \mathrm{Tr}(A)\,\mathrm{d}u}$

3.4 Equidimensional equation

Definition. Form of equidimensional DE:

$$x^2\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + Ax\frac{\mathrm{d}y}{\mathrm{d}x} + By = 0$$

Method. To solve equidimensional DE, try solutions of form x^k

3.5 Forced equations

Method. Determining y_p:
(i) Guess P.I. form and check
(ii)

$$y_p = y_2\int^x \frac{y_1(t)f(t)}{W(t)}\,\mathrm{d}t - y_1\int^x \frac{y_2(t)f(t)}{W(t)}\,\mathrm{d}t$$

Note. Can derive equation by supposing $\mathbf{Y}_p = u(x)\mathbf{Y}_1 + v(x)\mathbf{Y}_2$ then subbing y_p into DE and solving for u, v.

3.6 Forced oscillating systems

3.6.1 Damping

Method. To analyse DE of form $\ddot{y} + \frac{L}{M}\dot{y} + \frac{k}{M}y = \frac{F(t)}{M}$:

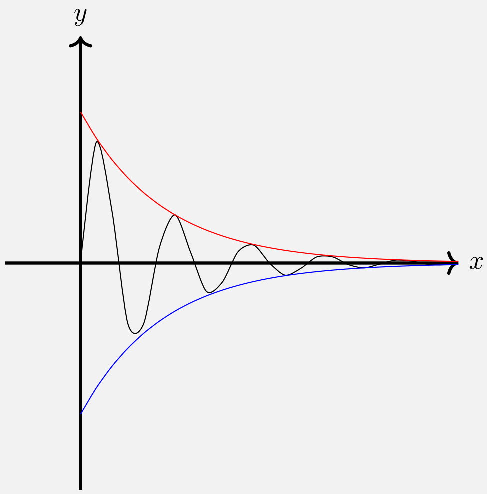
(i) Let $\tau \equiv \sqrt{\frac{k}{M}}t$ to transform to $y'' + 2Ky' + y = f(\tau)$
Where $y' \equiv \frac{\mathrm{d}y}{\mathrm{d}\tau}$, $K \equiv \frac{L}{2\sqrt{kM}}$, $f \equiv \frac{F}{k}$
Have: $\lambda = -K \pm \sqrt{K^2 - 1}$
(ii) Consider value of K to determine response:

Name	Value of K	Roots of char. eq.	Equation
Underdamped	$K < 1$	λ_1, λ_2 complex	$y = e^{-K\tau}[A\sin(\omega\tau) + B\cos(\omega\tau)]$
Critically Damped	$K = 1$	$\lambda_1 = \lambda_2 = -K$ degenerate	$y = (A + B\tau)e^{-k\tau}$
Overdamped	$K > 1$	$\lambda_1, \lambda_2 < 0$, real	$y = Ae^{\lambda_1\tau} + Be^{\lambda_2\tau}$

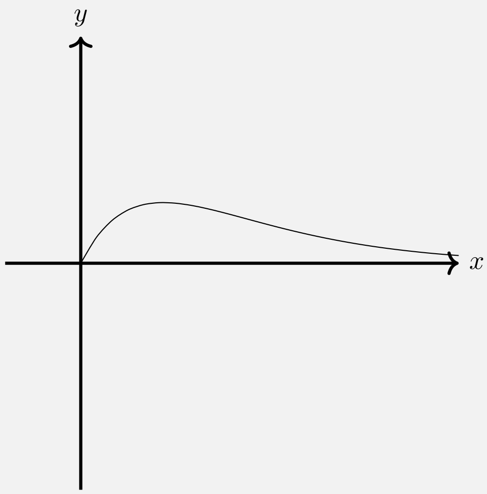
Where $\omega = \sqrt{1 - K^2}$

Note. Damped oscillator has period $\frac{2\pi}{\sqrt{1-K^2}}$

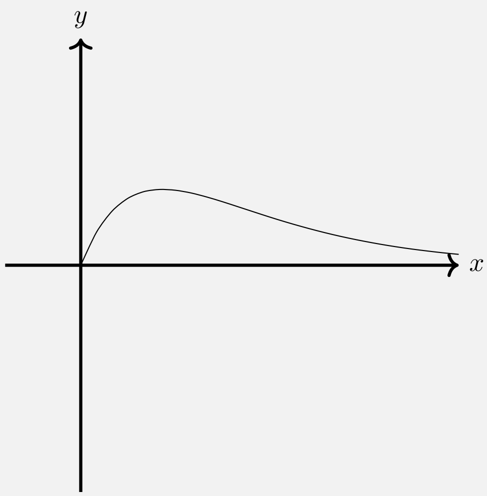
Underdamped



Critically damped



Overdamped



3.6.2 Transients

Definition. Dirac δ function properties:

(i) $\delta(x) = 0\,\forall x \neq 0$

(ii) $\int_{-\infty}^{\infty} \delta(x)\,\mathrm{d}x = 1$

(iii) Sampling property:

$$\int_{-\infty}^{\infty} g(x)\delta(x)\,\mathrm{d}x = g(0)\int_{-\infty}^{\infty} \delta(x)\,\mathrm{d}x = g(0)$$

$$\int_{a}^{b} g(x)\delta(x-x_0)\,\mathrm{d}x = \begin{cases} g(x_0) & a \leq x_0 < b \\ 0 & x_0 < a \text{ or } x_0 > b \end{cases}$$

Definition.

$$H(x) = \int_{-\infty}^{\infty} \delta(t)\,\mathrm{d}t$$

$$\frac{\mathrm{d}H}{\mathrm{d}x} = \delta(x) \text{ from F.T.C}$$

Properties:

(i) $H(x) = 0$ for $x < 0$

(ii) $H(x) = 1$ for $x > 0$

(iii) $H(0)$ undefined

Definition.

$$r(x) = \int_{-\infty}^{\infty} H(t)\,\mathrm{d}t$$

Note. Functions get smoother as we integrate

Method. Solving δ function forcing:

(i) Solve for $x < x_0$ and $x > x_0$, 2 cases, giving 4 unknown constants

(ii) Use 2 jump conditions and 2 ICs/ BCs to solve

Note.

$$\lim_{\varepsilon \to 0} [y]_{\frac{\pi}{2}-\varepsilon}^{\frac{\pi}{2}+\varepsilon} = [y]_{\frac{\pi}{2}^-}^{\frac{\pi}{2}^+}$$

Shorthand.

Method. Solving Heaviside step function forcing:

$$y'' + py' + q = 0 \text{ for } x < x_0$$

$$y'' + py' + q = 1 \text{ for } x > x_0$$

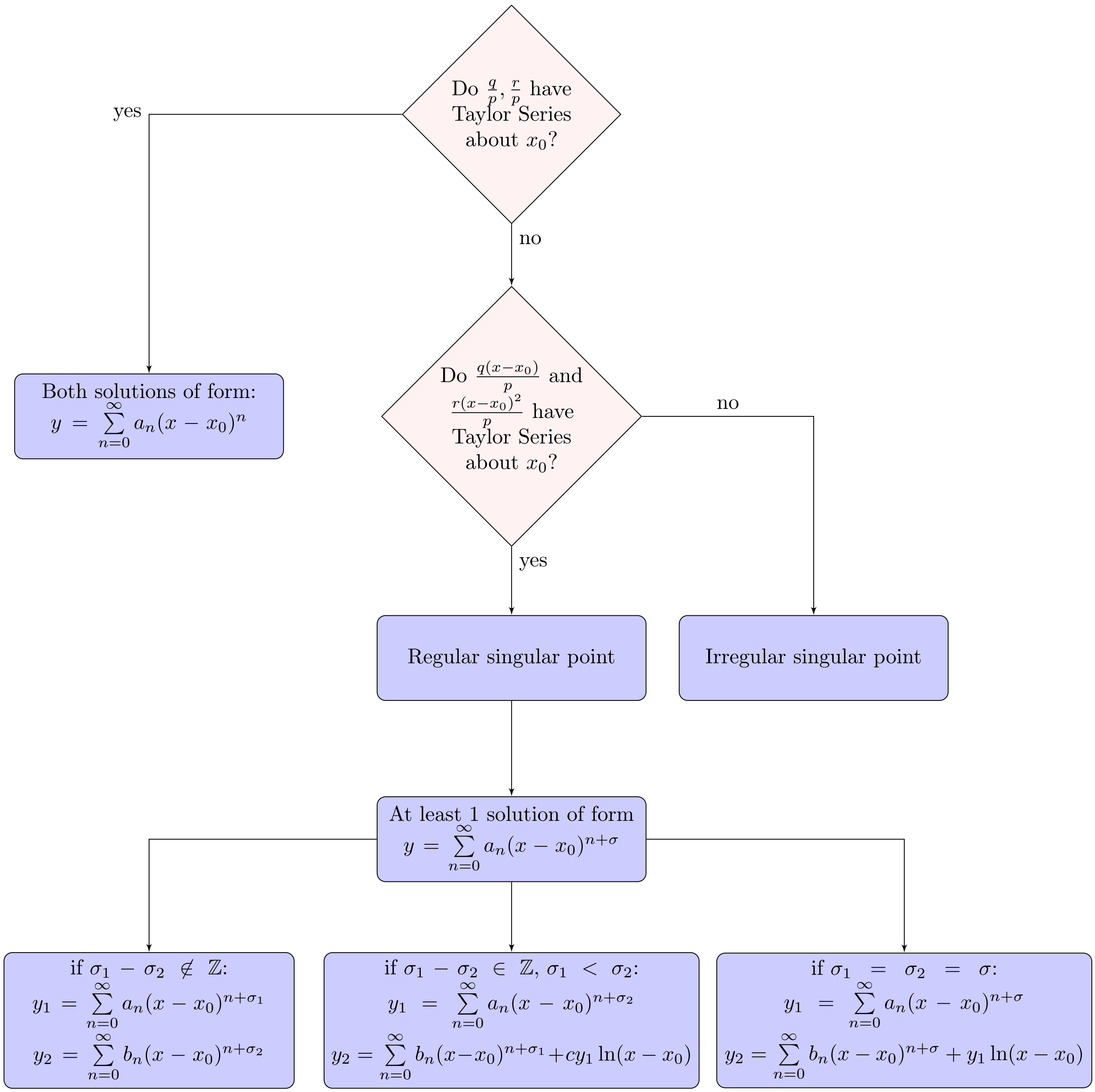
3.7 Discrete Equations

Method. Finding particular integral:

Form of f_n	Particular Integral
k^n	Ak^n if $k \neq k_1, k_2$
k_1^n, k_2^n	$Ank_1^n + Bnk_2^n$
n^p	$An^p + Bn^{p-1} + \cdots + Cn + D$

3.8 Method of Frobenius

Method. To solve DE $py'' + qy' + r = 0$, use flowchart below: (Sub $n = 0$ to determine value of σ)



4 Multivariate Functions: Applications

4.1 Directional Derivative

Definition.

$$\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix}$$

Method. To find directional derivative in a given direction:

$$\frac{\mathrm{d}f}{\mathrm{d}s} = \hat{\mathbf{s}} \cdot \nabla f$$

Where $\hat{\mathbf{s}}$ is unit vec in direction desired.

Note. ∇f is perpendicular to contours of $f(x, y)$

4.2 Taylor Series for Multivariate Functions

Equation. Multivariate Taylor Series:

$$\begin{aligned} f(x,y) = & f(x_0, y_0) \\ & + (x - x_0)\ f_x|_{x_0,y_0} + (y - y_0)\ f_y|_{x_0,y_0} \\ & + \frac{1}{2}[(x - x_0)^2\ f_{xx}|_{x_0,y_0} + (y - y_0)^2\ f_{yy}|_{x_0,y_0} + 2(x - x_0)(y - y_0)\ f_{xy}|_{x_0,y_0}] \\ & + \dots \end{aligned}$$

$$f(\mathbf{x}) = f(\mathbf{x}_0) + \delta\mathbf{x} \cdot \nabla f(\mathbf{x}_0) + \frac{1}{2}\delta\mathbf{x} H \delta\mathbf{x}^T + \dots$$

Where H,defined below, evaluated at $\mathbf{x}_0$

Definition. **Hessian matrix** for a function f:

$$H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$$

4.3 Stationary Points

- Near min/max contours of f are elliptical
- Near saddle, contours of f are hyperbolic
- Contours of f can only cross at saddle points

4.3.1 Classifying Stationary points

Definition. Since H symmetric, it can be diagonalised wrt principal axes with evals on diagonal. **Signature** is sequence of determinants:

$$\left|f_{x_1x_1}\right|, \begin{vmatrix} f_{x_1x_1} & f_{x_1x_2} \\ f_{x_2x_1} & f_{x_2x_2} \end{vmatrix}, \dots, \begin{vmatrix} f_{x_1x_1} & \dots & f_{x_1x_n} \\ \vdots & \ddots & \vdots \\ f_{x_nx_1} & \dots & f_{x_nx_n} \end{vmatrix}$$

Method. Classifying stationary points:
- Minimum ($\lambda_i > 0$) $\iff$ signature $+, +, +, +, \dots$
- Maximum ($\lambda_i < 0$) $\iff$ signature $-, +, -, +, \dots$
- Otherwise saddle

5 Systems of ODEs

5.1 Systems of Linear ODEs

Method. Solving system of equations:

$$\dot{y_1} = ay_1 + by_2 + f_1(t)$$

$$\dot{y_2} = cy_1 + dy_2 + f_2(t)$$

Or more generally:

$$\dot{\mathbf{Y}}_1 = M\mathbf{Y} + \mathbf{F}$$

(i) Write $\dot{\mathbf{Y}} = \mathbf{Y}_c + \mathbf{Y}_p$

$$\mathbf{Y}_c = A\mathbf{v}_1 e^{\lambda_1 t} + B\mathbf{v}_2 e^{\lambda_2 t}$$

Where $\mathbf{v}_i$ evecs and λ_i evals

(ii) For $\mathbf{Y}_p$ try same guess as with only 1 equation but with vector in front e.g. if you see $\begin{bmatrix} 4 \\ 1 \end{bmatrix} e^t$ then try $\mathbf{u}e^t$.
if you see $\begin{bmatrix} 2 \\ 3 \end{bmatrix} t^2$ then try $\mathbf{u}_1 t^2 + \mathbf{u}_2 t + \mathbf{u}_3$ etc.
Remember can have component zero and can sum for different terms.
($\mathbf{u}$ is constant vector)
If forcing term matches, put a t in front as usual.

5.2 Phase Portraits

Method. Drawing phase portraits:

(i) Eigenvectors give direction of straight lines, eigenvalue tells you whether the line points towards/ away

(ii) In between straight lines, fill as appropriate. In $\lambda_1\lambda_2 > 0$, consider which has greater modulus to determine which influences more.

Note. Types of phase portrait (near fixed points):

(i) Saddle Node $\lambda_1, \lambda_2 \in \mathbb{R}$ and $\lambda_1\lambda_2 < 0$
(ii) Stable Node $\lambda_1, \lambda_2 \in \mathbb{R}$, $\lambda_1\lambda_2 > 0$ and $\lambda_1, \lambda_2 < 0$
(iii) Unstable Node $\lambda_1, \lambda_2 \in \mathbb{R}$, $\lambda_1\lambda_2 > 0$ and $\lambda_1, \lambda_2 > 0$
(iv) Stable spiral λ_1, λ_2 complex conjugate pair, $\text{Re}(\lambda_1, \lambda_2) < 0$
(v) Unstable spiral λ_1, λ_2 complex conjugate pair, $\text{Re}(\lambda_1, \lambda_2) > 0$
(vi) Center $\text{Re}(\lambda_1, \lambda_2) = 0$ (determine direction of rotation by evaluating system near point to find sign of $\dot{y}_1, \dot{y}_2$)

5.3 Non linear system of ODEs

Method. To determine the stability of stationary points and behaviour around:

$$\dot{x} = f(x, y)$$

$$\dot{y} = g(x, y)$$

$$(x, y) = (x_0 + \xi(t), y_0 + \eta(t))$$

$$\implies \begin{bmatrix} \dot{\xi} \\ \dot{\eta} \end{bmatrix} = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix}\Bigg|_{x_0, y_0} \begin{bmatrix} \xi \\ \eta \end{bmatrix}$$

Evals of matrix above determine stability and behaviour accordingly to note above.

6 PDEs

6.1 1st Order Wave Equation

Method. To solve PDE of form:

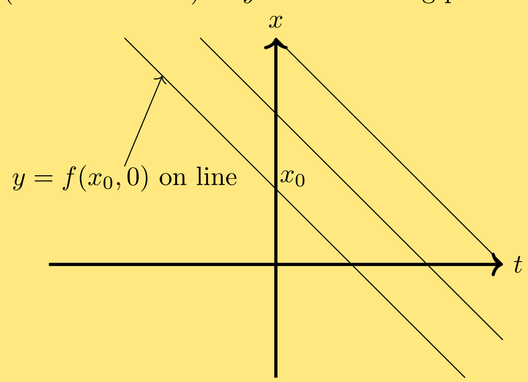
$$\frac{\partial y}{\partial t} - c\frac{\partial y}{\partial x} = 0$$

with c constant:
Use method of characteristics:

(i) Consider sampling y along path $x(t)$ where:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -c \implies \frac{\mathrm{d}y}{\mathrm{d}t} = 0$$

(from chain rule) so $y = \text{const.}$ along paths $x = x_0 - ct$



(ii) This gives general solution $y = f(x + ct)$. Use boundary condition to find $f(x_0)$.

Note. If forcing term $g(t)$ on RHS, solve $\frac{\mathrm{d}y}{\mathrm{d}t} = g(t)$

6.2 2nd Order Wave Equation

Method. To solve PDE of form:

$$\frac{\partial^2 y}{\partial t^2} - c^2\frac{\partial^2 y}{\partial x^2} = 0$$

Have:

$$\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right) y = 0$$

So:

$$\frac{\partial y}{\partial t} - c\frac{\partial y}{\partial x} = 0 \text{ or } \frac{\partial y}{\partial t} + c\frac{\partial y}{\partial x} = 0$$

So $y = f(x + ct) + g(x - ct)$.

Note. If forcing term $g(t)$on RHS, solve $\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} = g(t)$

6.3 Diffusion Equation

Method. To solve PDE of form:

$$\frac{\partial y}{\partial t} = \kappa \frac{\partial^2 y}{\partial x^2}$$

Define

$$\eta = \frac{x^2}{4\kappa t}$$

Seek solutions of form $y = t^{-\alpha} f(\eta)$
After subbing into PDE and trivial algebra:

$$\alpha f + f'\eta + f''\eta + \frac{f'}{2} = 0$$

Which simplifies to:

$$\eta \frac{\mathrm{d}}{\mathrm{d}\eta} + \frac{1}{2}(f' + 2\alpha f) = 0$$

Let $\alpha = \frac{1}{2}$ as it is still arbitrary at this stage, yielding:

$$\eta \frac{F}{\eta} + \frac{F}{2} = 0$$

Where $F = f + f'$. Giving one solution $F = 0 \, \forall \eta$.

$$\implies f = Ae^{-\eta}$$

Hence

$$y = At^{-\frac{1}{2}} e^{\frac{-x^2}{4\kappa t}}$$

And we can set A from ICs.