Dynamics and Relativity

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Contents

1 Newtonian Dynamics -Basic Concepts

1.1 Particles

Definition. A **particle** is an object of negligible size. It has mass $m > 0$ and (perhaps) charge q.

The position of a particle is described by a position vector $\mathbf{r}(t)$ (or $\mathbf{x}(t)$) with respect to origin O. (A particle at the origin has position vector $\mathbf{0}$ - zero vector)

The Cartesian components of $\mathbf{r}(t)$ are (x, y, z) :

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

(with $\mathbf{i}, \mathbf{j}, \mathbf{k}$ orthogonal basis vectors (orthonormal)).

Notation. Sometimes $\mathbf{i}, \mathbf{j}, \mathbf{k} \to \hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$ $\mathbf{i}, \mathbf{j}, \mathbf{k}$ parallel to axes Ox, Oy, Oz

Definition. The **frame of reference**, S, is defined by the choice of coordinate axes.

Definition. The **velocity** of a particle is:

$$\mathbf{u} = \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{r}(t) = \dot{\mathbf{r}}$$

Note. Velocity is tangent to path (or trajectory)
(See Vector Calculus for more details)

Definition. The **momentum** of a particle is:

$$m\mathbf{u} = m\dot{\mathbf{r}} = \mathbf{p}$$

(this notation is often used for momentum)

Definition. The **acceleration** of a particle is:

$$\dot{\mathbf{u}} = \ddot{\mathbf{r}} = \frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathbf{r}$$

Note. Technical note:
Time derivative of $\mathbf{u}(t)$ is:

$$\dot{\mathbf{u}}(t) = \lim_{h \to 0} \left\{ \frac{\mathbf{u}(t+h) - \mathbf{u}(t)}{h} \right\}$$

With $\mathbf{u} \to \mathbf{u}_0 \iff |\mathbf{u} - \mathbf{u}_0| \to 0$

Method. With basis vectors defined as above, we can then evaluate derivative by taking derivative of each component e.g.

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{r} = \left(\frac{\mathrm{d}x}{\mathrm{d}t}, \frac{\mathrm{d}y}{\mathrm{d}t}, \frac{\mathrm{d}z}{\mathrm{d}t}\right)$$

Product rules: scalar function $f(t)$, vector functions $\mathbf{g}(t), \mathbf{h}(t)$ as follows:

$$\frac{\mathrm{d}}{\mathrm{d}t}(f\mathbf{g}) = \frac{\mathrm{d}f}{\mathrm{d}t}\mathbf{g} + f\frac{\mathrm{d}\mathbf{g}}{\mathrm{d}t}$$
$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{g}\cdot\mathbf{h}) = \frac{\mathrm{d}\mathbf{g}}{\mathrm{d}t}\cdot\mathbf{h} + \mathbf{g}\cdot\frac{\mathrm{d}\mathbf{h}}{\mathrm{d}t}$$
$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{g}\times\mathbf{h}) = \frac{\mathrm{d}\mathbf{g}}{\mathrm{d}t}\times\mathbf{h} + \mathbf{g}\times\frac{\mathrm{d}\mathbf{h}}{\mathrm{d}t}$$

Note. Prove by components

1.2 Newton's Laws of Motion

Law (Newton's 1^{st} Law)**.** There exist inertial frames of reference (or inertial frames) in which a particle remains at rest or moves in a straight line at constant speed (i.e. it moves at constant velocity) unless it is acted on by a force.
(Galileo's Law of Inertia)

Law (Newton's 2^{nd} Law)**.** In an inertial frame, the rate of change of momentum of a particle is equal to the force acting on it.

Note. This is a statement about vectors.

Law (Newton's 3^{rd} Law)**.** To every action there is an equal and opposite reaction. Forces exerted between two particles are equal in magnitude and opposite in direction.

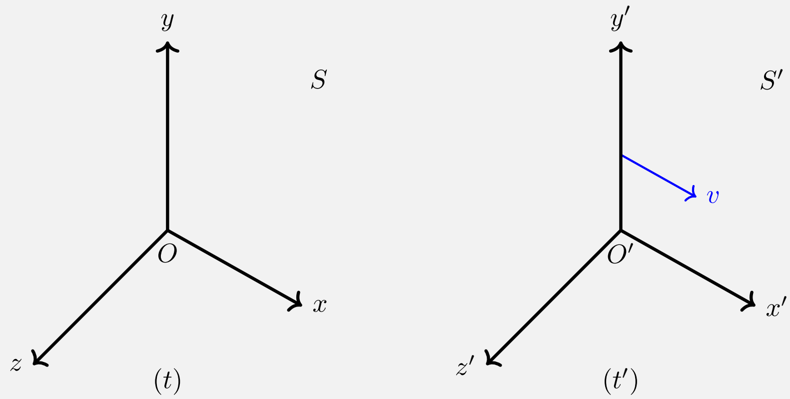
Remark. All of these statements about particles can be extended to finite bodies (comprised of many particles).

1.3 Inertial frames & Galilean Transformations

In inertial frame, acceleration is zero if force is zero.

$$\ddot{\mathbf{r}} = \mathbf{0} \iff \mathbf{F} = \mathbf{0}$$

Inertial frames are not unique,
If S is an inertial frame, then any other frame S' moving with constant velocity relative to S is also inertial.



$$x' = x - vt$$
$$y' = y$$
$$z' = z$$
$$t' = t$$

Definition. A **boost** is a transformation of form:

$$\mathbf{r}' = \mathbf{r} - \mathbf{v}t$$

Where $\mathbf{v}$ is velocity of S' relative to S.
(Generalised arbitrary direction of above)

For a particle with position $\mathbf{r}(t)$ in S and $\mathbf{r}'(t')$ in S', then velocity $\mathbf{u} = \dot{\mathbf{r}}(t)$ in S and acceleration $\mathbf{a} = \ddot{\mathbf{r}}(t)$ in S relate to values in S' by:

$$\mathbf{u}' = \mathbf{u} - \mathbf{v}, \ \mathbf{a}' = \mathbf{a}$$

Definition. The **Galilean group** (group of Galilean transformations) is generated by the set of transformations that preserve inertial frames. Boosts combined with (some of) the following:

- translation of space: $\mathbf{r}' = \mathbf{r} - \mathbf{r}_0, \ \mathbf{r}_0$ constant
- translation of time: $t' = t - t_0$
- rotations and reflections in space: $\mathbf{r}' = R\mathbf{r}$ where R is an orthogonal matrix

For any Galilean transformation we have

Note.

$$\ddot{\mathbf{r}} = \mathbf{0} \iff \ddot{\mathbf{r}}' = 0$$

S inertial $\iff$ S' inertial

1.3.1 Galilean relativity

Note. Principle of Galilean relativity is that laws of Newtonian physics are the same in all inertial frames.
i.e. laws of physics look the same:

- at any point in space
- at any time
- in whatever direction I face
- whatever constant velocity I move with

Any set of equations which describe Newtonian physics must have Galilean invariance.

Remark. Measurement of velocity is not absolute but measurement of acceleration is absolute

1.4 Newton's Second Law and Equations of Motion

From 2^{nd} Law as stated previously, for a particle subject to a force $\mathbf{F}$, the momentum $\mathbf{p}$ satisfies:

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = \mathbf{F} \text{ where } \mathbf{p} = m\mathbf{u} = m\dot{\mathbf{r}}$$

Assume m is constant. (For variable mass see later in course.)
Then

$$m\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = m\ddot{\mathbf{r}} = \mathbf{F}$$

Mass is a measure of 'reluctance to accelerate' i.e. inertia.

If $\mathbf{F}$ is specified as a function of $\mathbf{r}$, $\dot{\mathbf{r}}$, t, i.e. $\mathbf{F}(\mathbf{r}, \dot{\mathbf{r}}, t)$, then we have a 2^{nd} order differential equation for $\mathbf{r}(t)$, i.e.

$$m\ddot{\mathbf{r}} = m\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2} = \mathbf{F}\left(\mathbf{r}, \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}, t\right)$$

Need to provide initial position $\mathbf{r}(t_0)$, inertial velocity $\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{r}(t_0)$ then have unique solution.
The path/ trajectory of the particle is then determined (at all future times and at all past times).

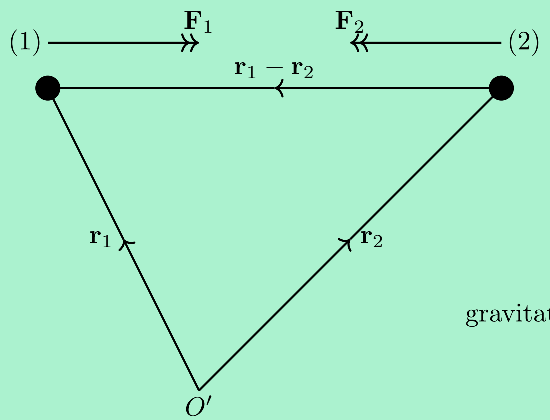
1.5 Examples of Forces

1.5.1 Gravitational Force

Equation. Newton's law of gravitation states that the gravitational force between two particles, one at $\mathbf{r}_1$, one at $\mathbf{r}_2$,is (on particle 1):

$$\mathbf{F}_1 = -\frac{Gm_1m_2}{|\mathbf{r}_2 - \mathbf{r}_2|^3}(\mathbf{r}_1 - \mathbf{r}_2) = -\mathbf{F}_2$$

where m_1 and m_2 are masses of particles and G is gravitational constant. $\mathbf{F}_i$ is force on particle i.
$\mathbf{F}_1 = -\mathbf{F}_2$ proportional to $|\mathbf{r}_1 - \mathbf{r}_2|^{-2}$ (it is an inverse square law)
Gravitational force is an attractive force.



gravitational force is 'attractive'

Note. G is a dimensional constant with dimensions $L^3 \cdot M^{-1} \cdot T^{-2}$
G called Newton's gravitational constant.
See later in course for detailed treatment of motion in presence of gravitational forces.

1.5.2 Electromagnetic Forces

Equation. Consider a particle with electric charge q, in presence of electric field $\mathbf{E}(\mathbf{r}, t)$ and magnetic field $\mathbf{B}(\mathbf{r}, t)$.

$$\mathbf{F}(\mathbf{r}, \dot{\mathbf{r}}, t) = q(\mathbf{E}(\mathbf{r}, t) + \dot{\mathbf{r}} \times \mathbf{B}(\mathbf{r}, t))$$

"Lorentz force law"
Consider $\mathbf{E}, \mathbf{B}$ as given

Example.
$\mathbf{E} = \mathbf{0}$, $\mathbf{B} =$ constant vector
Equation of motion:

$$m\ddot{\mathbf{r}} = q\dot{r} \times \mathbf{B}$$

with $\mathbf{B}$ constant. Choose axes such that $\mathbf{B} = B\hat{\mathbf{z}}$
Hence $m\ddot{z} = 0 \implies z = z_0 + ut$ (z_0 and u both constants)

$$m\ddot{x} = qB\dot{y}$$

$$m\ddot{y} = -qB\dot{x}$$

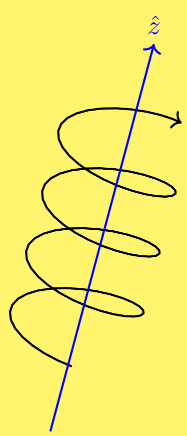
Convenient to define $\omega = \frac{qB}{m}$
Then

$$x = x_0 - \alpha\cos(\omega(t - t_0))$$

$$y = y_0 - \alpha\sin(\omega(t - t_0))$$

x_0, y_0, t_0 all constants determined by initial conditions

Note. Circular motion in x, y and constant velocity in $z \implies$ helical motion.



This is helical motion about axis in direction of magnetic field.
Motion is clockwise when viewed from direction of $\mathbf{B}$

2 Dimensional Analysis

Many problems in dynamics involve 3 basic dimensional quantities:

- L length
- M mass
- T time

dimensions of the same quantity $[x]$ can be expressed in terms of L, M, T

[density]	$M \cdot L^{-3}$
[force]	$M \cdot L \cdot T^{-2}$

Only power law functions of M, L, T are allowed e.g. don't allow $e^X = 1 + X + \frac{X^2}{2} + \dots$ with X dimensional and similarly complicated functions.

2.1 Units

Introduce units for basic dimensional quantities, L, M, T

e.g. SI Units

- m (metres) for length L
- kg (kilogrammes) for mass M
- sec (seconds) for time T

Many other physical quantities can be formed out of these basic units: e.g.
G appearing in Newton's Law of Gravitation.

$$F = \frac{Gm_1m_2}{r^2}$$

Hence dimensions of G

$$\begin{aligned} G \sim \frac{Fr^2}{m_1m_2} &= \frac{M \cdot L}{T^2} \cdot \frac{L^2}{M \cdot M} \\ &= \frac{L^3}{MT^2} \end{aligned}$$

Natural units for G: $\mathrm{m^3kg^{-1}sec^{-2}}$

$$G = 6.67 \times 10^{-11}\,\mathrm{m^3kg^{-1}sec^{-2}}$$

Note. General principle - dynamical/ physical equations must work for any consistent choice of units

2.2 Scaling

Method. Suppose that dimensional quantity Y depends on set of dimensional quantities $X_1, X_2, \ldots, X_n$
Let dimensions be $[Y] = L^a M^b T^c$

$$[X_i] = L^{a_i} M^{b_i} T^{c_i} \ (i = 1, \ldots, n)$$

If $n \leq 3$ then

$$Y = C \cdot X_1^{p_1} X_2^{p_2} X_3^{p_3}$$

and p_1, p_2 and p_3 can be determined by balancing dimensions.

$$L^a M^b T^c = (L^{a_1} M^{b_1} T^{c_1})^{p_1} \times \ldots$$

Hence

$$a = a_1 p_1 + a_2 p_2 + a_3 p_3$$
$$b = b_1 p_1 + b_2 p_2 + b_3 p_3$$
$$c = c_1 p_1 + c_2 p_2 + c_3 p_3$$

With unique solution for p_1, p_2, p_3 if dimensions X_1, X_2, X_3 are 'independent'. If $n > 3$ then $X_1, X_2, \ldots, X_n$ are not dimensionally independent - choose X_1, X_2, X_3 (assumed dimensionally independent) and $n - 3$ dimensionless quantities

$$\lambda_1 = \frac{X_4}{X_1^{q_{11}} X_2^{q_{12}} X_3^{q_{13}}}$$

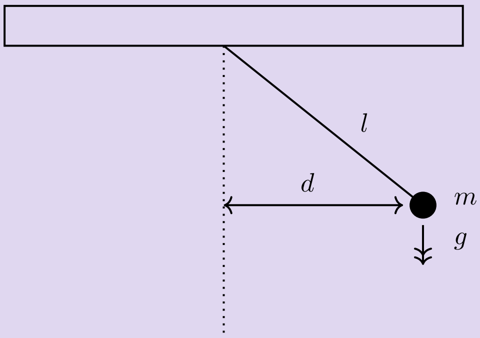
$$\lambda_1 = \frac{X_5}{X_1^{q_{21}} X_2^{q_{22}} X_3^{q_{23}}}$$

etc.
where q_{mn} are chosen to balance dimensions. Then:

$$Y = X_1^{p_1} X_2^{p_2} X_3^{p_3} \times C(\lambda_1, \lambda_2, \ldots, \lambda_{n-3})$$

[Bridgman's Theorem]

Example (simple pendulum).



How does period of oscillation P depend on m, l, d, g?

$$\begin{aligned}[P] &= T \\ [m] &= M \\ [g] &= LT^{-2} \text{ (gravity)} \\ [l] &= L \\ [d] &= L\end{aligned}$$

Form one dimensionless group ($\because n = 4$)

$$P = f\left(\frac{d}{l}\right) m^{p_1} l^{p_2} g^{p_3}$$
$$T = M^{p_1} L^{p_2} \left(\frac{L}{T^2}\right)^{p_3} \text{ (dimensions)}$$

balance dimensions:

$$\begin{aligned}M &: p_1 = 0 \\ L &: p_2 + p_3 = 0 \\ T &: 1 = -2p_3\end{aligned}$$

Hence $p_1 = 0$ $p_2 = \frac{1}{2}$ $p_3 = -\frac{1}{2}$

$$P = f\left(\frac{d}{l}\right) l^{-1/2} g^{-1/2}$$

Contains useful information e.g. $d \mapsto 2d$ and $l \mapsto 2l$ results in $P \mapsto \sqrt{2}P$
However if $d \mapsto 2d$ and $l \mapsto l$ can't say very much because P depends on precise form of f.
$f(d/l)l^{1/2}g^{1/2}$ is much more restricted than $P(m, d, l, g)$

Example (Taylor's estimate of energy of first atomic explosion).
Taylor used publicly available data on fireball-growth with time. $R(t)$ size as a function of time.
R has dimension L
Time t has dimension T
Density of air ρ has dimension $\frac{M}{L^3}$
Energy of explosion E has dimension $\frac{ML^2}{T^2}$

$$R = Ct^{\alpha}\rho^{\beta}E^{\gamma}$$

$$L = T^{\alpha}\frac{M^{\beta}}{L^{3\beta}}\frac{M^{\gamma}L^{2\gamma}}{T^{2\gamma}} \text{ (dimensions)}$$

Hence

$$M : \beta + \gamma = 0$$
$$L : -3\beta + 2\gamma = 1$$
$$T : \alpha - 2\gamma = 0$$

$$\implies \alpha = \frac{2}{5} \ \beta = -\frac{1}{5} \ \gamma = \frac{1}{5}$$

Hence

$$R(t) = Ct^{2/5}\rho^{-1/5}E^{1/5}$$

Taylor verified 2/5 power law and estimated value of E

$$E = \frac{\rho R^5}{C^5 t^2}$$

$$\frac{R^5}{t^2} \sim 6.7 \times 10^1 3 \, (m^5 s^{-2})$$

$$\rho \sim 1.25 \, \mathrm{kgm}^{-3}$$

If $C \sim 1$ then $E \sim 10^{14}$ J.
$E \sim 24 \times 10^3$ tonnes of TNT

3 Forces

3.1 Force and Potential Energy in 1 Spatial Dimension

Consider mass m moving in a straight line with position $x(t)$.
Force depends only on position x. not on velocity $\dot{x}$ or time t.
Let $F(x)$ be the force.

Definition. Define **potential energy** $V(x)$ by

$$F(x) = -\frac{\mathrm{d}V}{\mathrm{d}x}$$

Hence

$$V(x) = -\int^{x} F(x')\,\mathrm{d}x'$$

Lower limit omitted $\implies$ arbitrary constant in V.

Equation. Equation of motion determined by Newton's 2$^{\text{nd}}$ law

$$m\ddot{x} = -\frac{\mathrm{d}V}{\mathrm{d}x}$$

Definition. Define **kinetic energy**

$$T = \frac{1}{2}m\dot{x}^2$$

(will generalise to $T = \frac{1}{2}m|\dot{\mathbf{x}}|^2$ in more than 1 dimension)

Equation. Total energy

$$E = T + V = \frac{1}{2}m\dot{x}^2 + V(x)$$

Claim. Total energy is conserved i.e.

$$\frac{\mathrm{d}E}{\mathrm{d}t} = 0$$

Proof.

$$\begin{aligned}\frac{\mathrm{d}E}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}m\dot{x}^2 + V(x)\right) \\ &= m\dot{x}\ddot{x} + \frac{\mathrm{d}V}{\mathrm{d}x}\dot{x} \\ &= \dot{x}(m\ddot{x} + \frac{\mathrm{d}V}{\mathrm{d}x}) \\ &= 0\end{aligned}$$

Note. For conservation of $\frac{1}{2}m\dot{x}^2 + \Phi$ we require

$$\dot{x}F = -\frac{\mathrm{d}\Phi}{\mathrm{d}t}$$

In principle Φ may depend on $x, \dot{x}, t$.
Usually the case that there is no such Φ if F depends on $\dot{x}$ and/or t.

Example (harmonic oscillator)**.**

$$F(x) = -kx$$

(e.g. Hooke's law for elastic string)
then

$$V(x) = -\int^x (-kx')\,\mathrm{d}x' = \frac{1}{2}kx^2$$

(with appropriate choice of arbitrary constant).
Seek explicit expression for $x(t)$

$$m\ddot{x} = -kx$$

$$\begin{aligned}\implies x(t) &= A\cos\omega t + B\sin\omega t \\ \dot{x}(t) &= -\omega A\sin\omega t + \omega B\cos\omega t\end{aligned}$$

For suitable constants A, B with $\omega = \left(\frac{k}{m}\right)^{1/2}$
Check conservation of energy

$$\begin{aligned}E &= \frac{1}{2}m\dot{x}^2 + \underbrace{\frac{1}{2}kx^2}_{V(x)} \\ &= \frac{1}{2}(-\omega A\sin\omega t + \omega B\cos\omega t)^2 m + \frac{1}{2}k(A\cos\omega t + B\sin\omega t)^2 \\ &= \frac{1}{2}k(A^2 + B^2)\end{aligned}$$

Conservation of energy verified.

Method. Determining $x(t)$ given potential.
In 1-D, conservation of energy gives useful information about the motion.
Conservation of energy is a 1$^{\text{st}}$ integral of Newton's 2$^{\text{nd}}$ law.

$$E = \frac{1}{2}m\dot{x}^2 + V(x)$$

is constant. Hence,

$$\dot{x} = \pm\sqrt{\frac{2}{m}(E - V(x))}$$

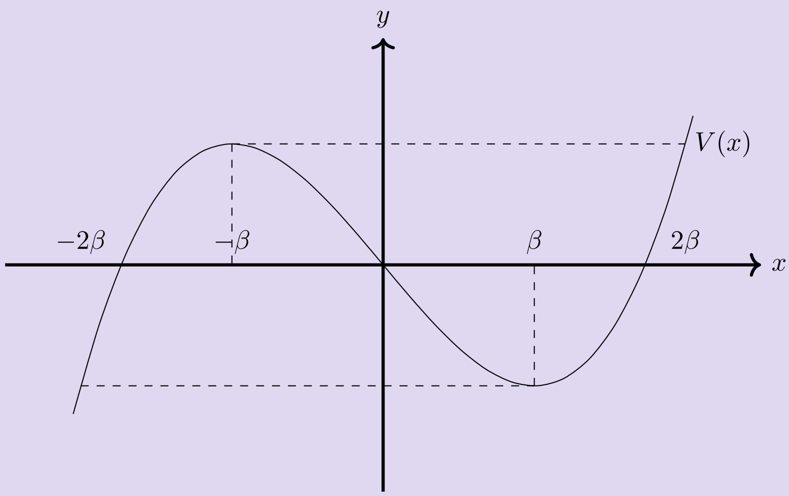
E set by initial conditions.
Hence

$$\pm\int_{x_0}^{x} \frac{\mathrm{d}x'}{\sqrt{\frac{2}{m}(E - V(x))}} = t - t_0$$

with $x = x_0$ when $t = t_0$
Implicit solution for $x(t)$
In principle can evaluate integral and find $x(t)$

Example (Qualitative insight from conservation of energy).

$$V(x) = \lambda(x^3 - 3\beta^2 x) \ \lambda, \beta \text{ positive constants}$$



What happens if we release particle from rest at $x = x_0$ for different choices of x_0?
$E = V(x_0)$ - in subsequent motion $V(x) \leq V(x_0)$ as kinetic energy positive.

- Case 1: $x_0 < -\beta$ particle moves to left with $x(t) \to -\infty$ as $t \to \infty$
- Case 2: $-\beta < x_0 < 2\beta$ particle remains confined to $-\beta < x(t) < 2\beta$
- Case 3: $2\beta < x_0$ particle moves to left, reaches $x = -\beta$ and continues with $x(t) \to -\infty$ as $t \to \infty$.
- Special case 1: $x_0 = -\beta$ particle stays at $x = -\beta$
- Special case 2: $x_0 = \beta$ particle stays at $x = \beta$
- Special case 3: $x_0 = 2\beta$ particle moves to left and comes to rest at $x = -\beta$

For special case 3, write down integral expression relating x and t.

$$\begin{aligned} t &= \int_{x(t)}^{2\beta} \frac{\mathrm{d}\tilde{x}}{\sqrt{\frac{2\lambda}{m}(2\beta^3 - \tilde{x}^3 + 3\beta^2\tilde{x})}} \\ &= \int_{x(t)}^{2\beta} \frac{\mathrm{d}\tilde{x}}{\sqrt{\frac{2\lambda}{m}(\tilde{x}+\beta)^2(2\beta - \tilde{x})}} \\ &= \int_{x(t)}^{2\beta} \frac{1}{\sqrt{\frac{2\lambda}{m}}} \cdot \frac{1}{\tilde{x}+\beta} \cdot \frac{1}{(2\beta - \tilde{x})^{1/2}} \,\mathrm{d}\tilde{x} \text{ (this diverges as } \tilde{x} \to -\beta) \end{aligned}$$

particle takes infinite time to reach $x = -\beta$. (logarithmic behaviour)

3.2 Equilibrium Points

Definition. An **equilibrium point** is a point at which a particle can stay at rest for all time. In the previous example, $x = \pm\beta$ are equilibrium points.
General condition: $V'(x) = 0$ at equilibrium.

Method. Analyse motion near equilibrium at $x = x_0$, so $V'(x_0) = 0$, assume that $x - x_0$ is small, expand $V(x)$ in Taylor Series:

$$V(x) \simeq V(x_0) + (x - x_0)V'(0) + \frac{1}{2}(x - x_0)^2 V''(x_0) + \dots$$

Equation of motion:

$$m\ddot{x} = -V'(x) \simeq -(x - x_0)V''(x_0)$$

If $V''(x_0) > 0$ local minimum of $V(x)$, harmonic oscillator equation gives:

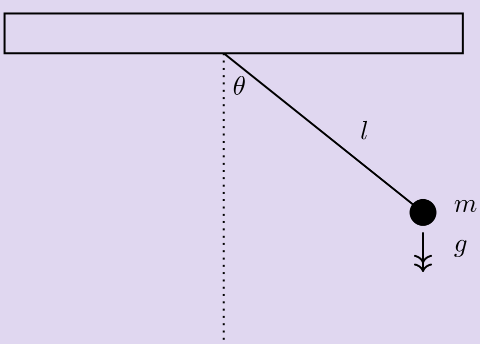
$$\text{frequency of oscillations: } \left(\frac{V''(x_0)}{m}\right)^{1/2}.$$

Stable equilibrium ('start close' $\implies$ 'stay close')
If $V''(x_0) < 0$ (local maximum of $V(x)$), exponentially increasing and exponentially decaying solutions - almost always excite exponentially increasing part - growth rate

$$\gamma = \sqrt{-\frac{V''(x_0)}{m}}$$

If $V''(x_0) = 0$ need to go to higher terms in Taylor Series to determine behaviour.
(In example $x = \beta$ stable and $x = -\beta$ unstable).

Example (Pendulum)**.**



Newtons 2$^{\text{nd}}$ Law:

$$ml\ddot{\theta} = -mg\sin\theta$$

(consider acceleration perpendicular to string)

$$ml\ddot{\theta} = -mg\sin\theta = -\frac{\mathrm{d}}{\mathrm{d}\theta}$$

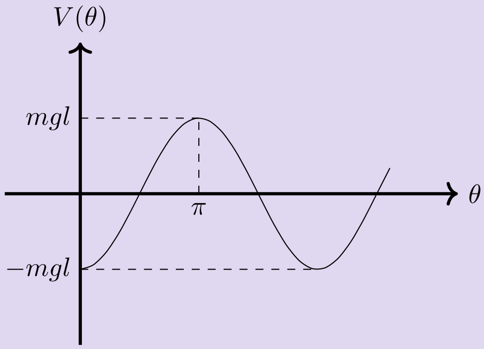
Hence

$$E = T + V = \frac{1}{2}ml\ddot{\theta}^2 - mgl\cos\theta$$

Can check

$$\frac{\mathrm{d}E}{\mathrm{d}t} = 0$$

Example (Continued)**.**



Stable equilibrium at $\theta = 0$
Unstable equilibrium at $\theta = \pi$

- $-mgl < E < mgl$: pendulum oscillates about a position of stable equilibrium
- $E > mgl$: either $\dot{\theta} > 0$ or $\dot{\theta} < 0$ for all time.

Period of oscillations?
$\dot{\theta} = 0$ at $\theta = \theta_0$ - dependence of period on θ_0?

$$\theta_0 \to 0 \to -\theta_0 \to 0 \to \theta_0$$

$$P = 4\int_0^{\theta_0} \frac{\mathrm{d}\theta}{\left(\frac{2gl(\cos\theta - \cos\theta_0)}{l^2}\right)^{1/2}}$$

Expression for $\dot{\theta}$ deduced from energy equation.

$$P = 4\left(\frac{l}{g}\right)^{1/2} \int_0^{\theta_0} \frac{\mathrm{d}\theta}{(2\cos\theta - 2\cos\theta_0)^{1/2}} = 4\left(\frac{l}{g}\right)^{1/2} F(\theta_0)$$

Recall

$$P = \left(\frac{l}{g}\right)^{1/2} H\left(\frac{d}{l}\right)$$

from dimensional analysis.
For θ_0 small,

$$F(\theta_0) \simeq \int_0^{\theta_0} \frac{\mathrm{d}\theta}{(\theta_0^2 - \theta^2)^{1/2}} = \frac{\pi}{2}$$

Then

$$P \simeq 2\pi\left(\frac{l}{g}\right)^{1/2}$$

3.3 Force and potential for motion in 3-D

Equation. Consider a particle moving in 3-D under force $\mathbf{F}$
Newtons 2$^{\text{nd}}$ Law:

$$m\ddot{\mathbf{r}} = \mathbf{F}$$

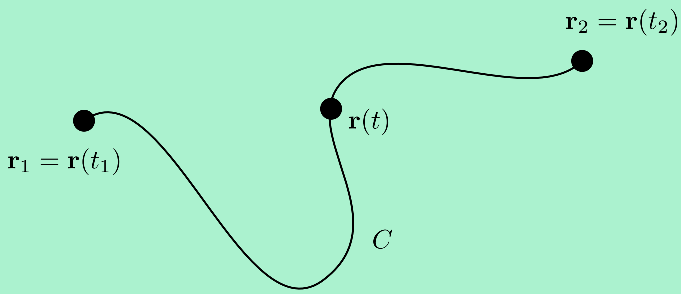
Kinetic energy

$$T = \frac{1}{2}m|\dot{\mathbf{r}}|^2 = \frac{1}{2}m|\mathbf{u}|^2$$

Then

$$\frac{\mathrm{d}T}{\mathrm{d}t} = m\dot{\mathbf{r}}\cdot\ddot{\mathbf{r}} = \mathbf{F}\cdot\dot{\mathbf{r}} = \mathbf{F}\cdot\mathbf{u} \text{ (rate of working of force on particle)}$$

Consider total work done by force on particle as it moves along a finite path:



Total work:

$$\int_{t_1}^{t_2} \mathbf{F}\cdot\mathbf{u}\,\mathrm{d}t = \int_{t_1}^{t_2} \mathbf{F}\cdot\dot{\mathbf{r}}\,\mathrm{d}t = \int_{\mathbf{r}(t_1)}^{\mathbf{r}(t_2)} \mathbf{F}\cdot\,\mathrm{d}\mathbf{r} \text{ (along } C\text{)}$$

$$\text{Total work } = \int_{t_1}^{t_2} \mathbf{F}\cdot\,\mathrm{d}\mathbf{r} = \int_{t_1}^{t_2} F_x\,\mathrm{d}x + F_y\,\mathrm{d}y + F_z\,\mathrm{d}z$$

$$\mathbf{F} = (F_x, F_y, F_z)$$

Definition. Now suppose $\mathbf{F}$ is a function of $\mathbf{r}$ only.
$\mathbf{F}(\mathbf{r})$ defines a '**force field**'.
A **conservative force field** is such that $\mathbf{F}(\mathbf{r}) = -\nabla V(\mathbf{r})$ for some function $V(\mathbf{r})$

Note. In components $F_i = \partial V/\partial x_i$

Claim. If $\mathbf{F}$ is conservative then the energy $E = T + V(\mathbf{r})$ is conserved.

Proof.

$$\begin{aligned}\frac{\mathrm{d}E}{\mathrm{d}t} &= \frac{\mathrm{d}T}{\mathrm{d}t} + \frac{\mathrm{d}}{\mathrm{d}t}V(\mathbf{r}) \\ &= m\dot{\mathbf{r}}\cdot\ddot{\mathbf{r}} + \nabla V\cdot\dot{\mathbf{r}} \\ &= \dot{\mathbf{r}}\cdot(m\ddot{\mathbf{r}} + \nabla V) \\ &= \dot{\mathbf{r}}\cdot(m\ddot{\mathbf{r}} - \mathbf{F}) \\ &= 0\end{aligned}$$

Equation. Total work done by a conservative force:

$$\begin{aligned} W &= \int_C \mathbf{F} \cdot \mathrm{d}\mathbf{r} \\ &= -\int_C \nabla V \cdot \mathrm{d}\mathbf{r} \\ &= V(\mathbf{r}_1) - V(\mathbf{r}_2) \end{aligned}$$

(follows from properties of ∇)
This is independent of the path taken between $\mathbf{r}_1$ and $\mathbf{r}_2$.

Corollary. If C is closed then no work is done by force.

Note. In general, a given $\mathbf{F}(\mathbf{r})$ is not conservative, there is no $V(\mathbf{r})$ such that $\mathbf{F} = -\nabla V$
Condition required for $\mathbf{F}(\mathbf{r})$ to be conservative is

$$\nabla \times \mathbf{F}(\mathbf{r}) = \mathbf{0}$$

see Vector Calculus curl operator.

3.4 Gravity

We have already noted the gravitational force felt by a mass m due to a mass M

$$\mathbf{F}(\mathbf{r}) = -\frac{GMm}{|\mathbf{r}|^3}\mathbf{r} = -\frac{GMm}{r^2}\hat{\mathbf{r}}$$

$\mathbf{r}$ is position vector of mass m relative to mass M

$$\mathbf{F}(\mathbf{r}) = -\nabla V$$

with

$$V(\mathbf{r}) = -\frac{GMm}{r}$$

Definition. We often define a "**gravitational potential**"

$$\Phi_g(\mathbf{r}) = -\frac{GM}{r}$$

and "**gravitational field**"

$$\mathbf{g} = -\nabla\Phi_g(\mathbf{r}) = -\frac{GM}{r^2}\hat{\mathbf{r}}$$

Equation. These are properties of mass M alone.
Effect on mass m:

$$V(\mathbf{r}) = m\Phi_g(\mathbf{r}), \ \mathbf{F}(\mathbf{r}) = m\mathbf{g}$$

Remark. We can generalise to gravitational potential associated with many masses

$$M_i \ i = 1, \dots, n$$

and position vectors

$$\mathbf{r}_i \ i = 1, \dots, n$$

Equation. Many particles:

$$\Phi_g(\mathbf{r}) = -\sum_{i=1}^{N} \frac{GM_i}{|\mathbf{r} - \mathbf{r}_i|}$$

$$\mathbf{g} = -\sum_{i=1}^{N} \frac{GM_i}{|\mathbf{r} - \mathbf{r}_i|^3}(\mathbf{r} - \mathbf{r}_i$$

Could extend to a continuous distribution of mass by generalising sums to integrals.
In particular, for a uniform spherical mass distribution centered at origin, we have

$$\Phi_g(\mathbf{r}) = -\frac{GM}{r}$$

with M the total mass – equivalent to point mass at origin (see VC)

Note. Gravitational mass vs Inertial Mass:
(Inertial mass) Newton's 2^{nd} Law

$$m\ddot{\mathbf{r}} = \mathbf{F}$$

(Gravitational mass) Newton's Law of Gravitation

$$\mathbf{F} = -\frac{GMm}{r^2}\hat{\mathbf{r}}$$

In fact, inertial mass and gravitational mass are the same ($\sim$ within 10^{-12})
(see Einstein's General Theory of Relativity)

3.4.1 Simple Results on gravity

(i) 1-D approximation:
Consider a mass m at some height z above surface of planet of mass M and radius R (assume $z \ll R$).
Potential:

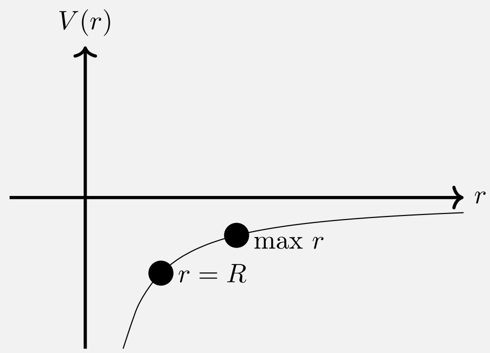
$$V(R+z) = -\frac{GMm}{R+z} = -\frac{GMm}{R} + \underbrace{\frac{GMm}{R^2}z}_{=mgz} + \dots$$

Potential energy in uniform gravitational field with

$$g = \frac{GM}{R^2} \simeq 9.8\,\mathrm{ms}^{-2}$$

(ii) Escape Velocity:
Consider leaving surface of a planet with velocity v.
Conservation of energy:

$$E = T + V = \frac{1}{2}mv^2 - \frac{GM}{r}$$



Escape if

$$v \geq \sqrt{\frac{2GM}{R}} = \text{escape velocity}$$

3.5 Electromagnetic Forces

Equation. We have that force $\mathbf{F}$ acting on a particle with charge q is

$$\mathbf{F} = q\mathbf{E} + q\mathbf{u} \times \mathbf{B} \text{ where } \mathbf{u} = \dot{\mathbf{r}}$$

(Lorentz force law)
with $\mathbf{E}(\mathbf{r}, t)$ electric field and $\mathbf{B}(\mathbf{r}, t)$ magnetic field.
Restrict to time independent fields $\mathbf{E}(\mathbf{r}), \mathbf{B}(\mathbf{r})$.
In this case we can write

$$\mathbf{E} = -\nabla\Phi_e(\mathbf{r})$$

Φ_e is the "electrostatic potential". The force $q\mathbf{E}$ is then conservative.

Claim. For time independent $\mathbf{E}(\mathbf{r})$, $\mathbf{B}(\mathbf{r})$ the energy of a particle moving under the Lorentz force law is constant

Proof.

$$\begin{aligned}
\mathbf{E} &= \frac{1}{2}m|\dot{\mathbf{r}}|^2 + q\Phi_e(\mathbf{r}) = (T+V) \\
\frac{\mathrm{d}\mathbf{E}}{\mathrm{d}t} &= m\dot{\mathbf{r}}\cdot\ddot{\mathbf{r}} + q\dot{\mathbf{r}}\cdot\nabla\Phi_e(\mathbf{r}) \\
&= \dot{\mathbf{r}}\cdot(m\ddot{\mathbf{r}} + q\nabla\Phi_e) \\
&= \dot{\mathbf{r}}\cdot(q\mathbf{E} + q\dot{\mathbf{r}}\times\mathbf{B} + q\nabla\Phi_e) \\
&= q\dot{\mathbf{r}}\cdot(\dot{\mathbf{r}}\times\mathbf{B}) \\
&= 0
\end{aligned}$$

The rate of working of a magnetic field is zero

Equation. A particle with charge A located at origin generates an electrostatic potential

$$\Phi_e(\mathbf{r}) = \frac{Q}{4\pi\varepsilon_0 r}$$

Where r is distance from origin to $\mathbf{r}$.
Electric field:

$$\mathbf{E}(\mathbf{r}) = -\nabla\Phi_e = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{\mathbf{r}}$$

ε_0 is a constant

$$8.85\times10^{-12}\,\mathrm{m}^{-3}\mathrm{kg}^{-1}\mathrm{s}^2\mathrm{C}^2$$

"electric constant" (C is the unit of charge)

Equation. Force on particle with charge q located at $\mathbf{r}$

$$\mathbf{F} = -q\nabla\Phi_e = \frac{Qq}{4\pi\varepsilon_0 r^2}\hat{\mathbf{r}}$$

'Coulomb force'

Note. Similar to gravitational force except that sign can vary
$Qq > 0$ - repulsive force (same sign)
$Qq < 0$ - attractive force (opposite signs)

3.6 Friction

Note. Friction is a contact force – e.g. between two solid bodies in contact, or between a solid body and a surrounding fluid.
Friction is a convenient description of complicated molecular phenomena.
Friction is not a "fundamental" force (ie not to be added to list of gravitational, electromagnetic, weak, strong)

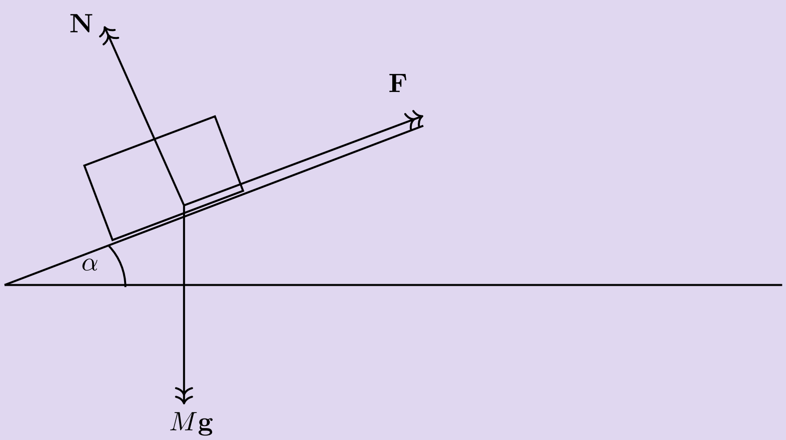
Definition. **Dry friction** – solid bodies in contact

Example. Normal force: perpendicular to contact surface - prevents intermingling of solid bodies in context

Example. Tangential force: resists tangential relative motion (e.g. sliding or slipping)

Definition. **Static friction**: if no sliding occurs

Example.



N normal force
F frictional force

$$|\mathbf{F}| \leq \mu_s |\mathbf{N}|$$

μ_s is coefficient of static friction.
Block can remain static provided

$$\alpha \leq \tan^{-1}(\mu_s)$$

Definition. **Kinetic friction**: if block starts to slide, there is a kinetic frictional force

Example.

$$|\mathbf{F}| = \mu_k |\mathbf{N}|$$

μ_k is coefficient of kinetic friction
Expect $\mu_s > \mu_k > 4$

Remark. Values of μ_s and μ_k will depend on materials in contact
e.g. rubber/ asphault has $\mu_s \simeq \mu_k \simeq 0.8$
teflon/teflon has $\mu_s \simeq \mu_k \simeq 0.04$

Definition. **Fluid drag** – solid body moving through a fluid experiences a drag force.

Two important regimes for drag:

Equation. Linear drag:

$$\mathbf{F} = -k_1 \mathbf{u}$$

where k constant and $\mathbf{u}$ is velocity of body relative to fluid,

Remark. This is relevant to 'small' objects moving through a viscous fluid (e.g. bacterium moving through body fluid)

Law (Stokes Drag Law)**.**

$$k_1 = 6\pi\eta R$$

for a moving sphere. η is viscosity of fluid, R is raius of sphere

Equation. Quadratic drag: large bodies moving through a less viscous fluid

$$\mathbf{F} = -k_2 \mathbf{u}|\mathbf{u}$$

with k_2 different to k_1. Typically,

$$k_2 = \rho_{\text{fluid}} C_D R^2$$

where ρ_{fluid} is fluid density, C_D drag coefficient, R^2 is size
Relevant to swimming of large fish, cars, aircraft, . . .

Remark. A body loses kinetic energy as a result of a drag force.
Rate of working:

$$\begin{aligned} \mathbf{F} \cdot \mathbf{u} &= -k_1 |\mathbf{u}|^2 \text{ linear drag} \\ &= -k_2 |\mathbf{u}|^3 \text{ quadratic drag} \end{aligned}$$

In the latter case, total work $\propto |\mathbf{u}|^2 \times$ distance
The fluid gains energy as a result of the drag force.

Linear drag as an example:

Example. damped oscillator met in DEs

$$m\ddot{x} = -kx - \lambda\dot{x}$$

Example. Projectile moving under uniform gravity, experiencing linear drag force

$$m\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = m\mathbf{g} - k\mathbf{u}$$

$$m\frac{\mathrm{d}^2\mathbf{x}}{\mathrm{d}t^2} = m\mathbf{g} - k\frac{\mathrm{d}x}{\mathrm{d}t}$$

with initial conditions:

$$\mathbf{x} = \mathbf{0}$$

$$\dot{\mathbf{x}} = \mathbf{u} = \mathbf{U} \text{ at } t = 0$$

Can solve for $\mathbf{u}$ in

$$m\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = m\mathbf{g} - k\mathbf{u}$$

Then solve for $\mathbf{x}$ to get:

$$\mathbf{x} = \frac{m\mathbf{g}t}{k} + \frac{m}{k}\left(\mathbf{U} - \frac{m\mathbf{g}}{k}\right)(1 - e^{-kt/m})$$

Now consider components of $\mathbf{u}$ and $\mathbf{x}$

$$\mathbf{x} = (x_1, x_2, x_3)\ \mathbf{u} = (u_1, u_2, u_3)$$

Choose

$$\mathbf{U} = (U\cos\theta, 0, U\sin\theta)\ \mathbf{g} = (0, 0, -g)$$

$$u_1 = U\cos\theta e^{-kt/m},\ u_2 = 0,\ u_3 = \left(U\sin\theta + \frac{mg}{k}\right)e^{-kt/m} - \frac{mg}{k}$$

i.e.

$$\mathbf{u} = \begin{bmatrix} U\cos\theta e^{-kt/m} \\ 0 \\ \left(U\sin\theta + \frac{mg}{k}\right)e^{-kt/m} - \frac{mg}{k} \end{bmatrix}$$

Note terminal velocity $(0, 0, -\frac{mg}{k})$ achieved on timescale m/k

Example.

$$x = \frac{mU\cos\theta}{k}(1-e^{-kt/m})\ y=0\ z = -\frac{mgt}{k} + \frac{m}{k}\left(U\sin\theta + \frac{mg}{k}\right)(1-e^{-kt/m})$$

i.e.

$$\mathbf{x} = \begin{bmatrix} \frac{mU\cos\theta}{k}(1-e^{-kt/m}) \\ 0 \\ -\frac{mgt}{k} + \frac{m}{k}\left(U\sin\theta + \frac{mg}{k}\right)(1-e^{-kt/m}) \end{bmatrix}$$

The range is the value of x when z returns to zero ($\theta > 0$)
$R(U, \theta, m, k, g)$ - range is a function of external variables.
Have dimensionless group

$$\frac{kU}{mg} = \frac{U/g}{m/k}$$

U/g is the time taken to reduce initial velocity under gravity
m/k is the time taken to achieve terminal velocity.
Dimensional analysis yields

$$R = \frac{U^2}{g}F\left(\theta, \frac{kU}{mg}\right)$$

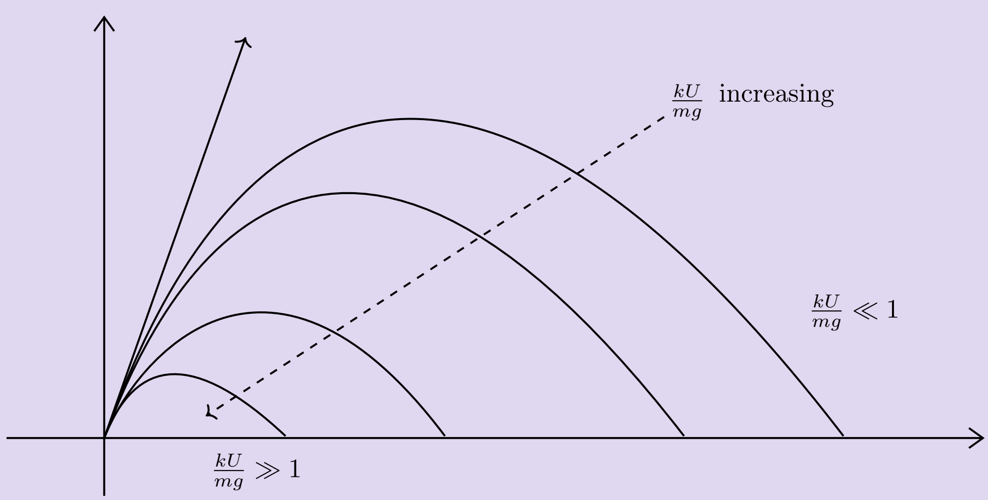
For $\frac{kU}{mg} \ll 1$, 'weak friction':

$$R = \frac{U^2}{g}\cdot 2\sin\theta\cos\theta$$

For $\frac{kU}{mg} \gg 1$, 'strong friction':

$$R = \frac{U^2}{g}\left(\frac{mg}{kU}\cos\theta\right)$$

Hence R is a decreasing function of $\frac{kU}{mg}$

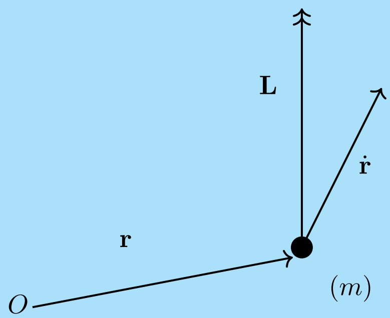


3.7 Angular Momentum

In previous 3 subsections, we focused on specific types of force – gravitational, electromagnetic, frictional.
Now to conclude section 3, we return to more general aspects of the dynamics of a single particle.

Definition. The **angular momentum** for a particle of mass m moving under influence of a force $\mathbf{F}$, position vector $\mathbf{r}(t)$, velocity $\dot{\mathbf{r}}(t)$ is

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} = \mathbf{r} \times m\dot{\mathbf{r}}$$



Equation. Then

$$\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} = m\dot{\mathbf{r}} \times \dot{\mathbf{r}} + m\mathbf{r} \times \ddot{\mathbf{r}} = \mathbf{r} \times \mathbf{F} = \mathbf{G}$$

Definition. We say $\mathbf{G}$ above is the '**torque**'

Remark. The values of $\mathbf{L}$ and $\mathbf{G}$ depend on choice of O - 'about the origin' or 'about ' any specified point.

Note. IF $\mathbf{r} \times \mathbf{F} = \mathbf{0}$ or equivalently $\mathbf{G} = \mathbf{0}$ then $\mathbf{L}$ is a constant vector i.e. angular momentum is conserved.
Angular momentum about some suitably chosen point may be constant

4 Orbits

Motivated by motion of planets, comets, etc under influence of gravitational force due to a star, planet etc – but also relevant to motion of charged particles (e.g. Rutherford Scattering)

The basic problem:

$$m\ddot{\mathbf{r}} = -\nabla V(r)$$

Particle moving in a force that is associated with potential that is only a function of radius – force is directed towards (or away from) the origin.

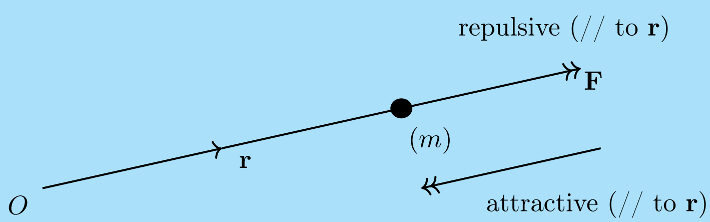
We are assuming that 'central' mass is remaining fixed – good approximation if the central mass is much larger than m (we relax this assumption later in the course)

4.1 Central Forces

Definition. **Central forces** are a special class of conservative forces with

$$V(\mathbf{r}) = V(|\mathbf{r}|)$$

$$\mathbf{F}(\mathbf{r}) = -\nabla V(|\mathbf{r}|) = -\frac{\mathrm{d}V}{\mathrm{d}r}\hat{\mathbf{r}}$$



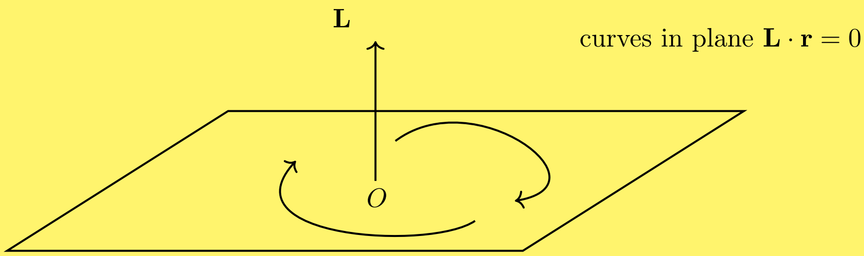
Can check:

$$\begin{aligned} r^2 &= x^2 + y^2 + z^2 \\ 2r\nabla r &= (2x, 2y, 2z) \\ &= 2\mathbf{x} \\ \implies \nabla r &= \frac{\mathbf{x}}{r} = \hat{\mathbf{r}} \end{aligned}$$

Remark. Consider the angular momentum $\mathbf{L}$ about O

$$\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} = \mathbf{r} \times = \mathbf{r} \times (-\frac{\mathrm{d}V}{\mathrm{d}r}\hat{\mathbf{r}}) = 0$$

Angular momentum about O is conserved for a central force.
Have $\mathbf{L} =$ constant, also $\mathbf{L} \cdot \mathbf{r} = 0$ (from definition of $\mathbf{L}$)
Hence motion is in a plane through the origin $O$with orientation set by value of $\mathbf{L}$
(Have reduced a 3-D problem to 2-D problem)



4.2 Polar Co-Ordinates in the Plane

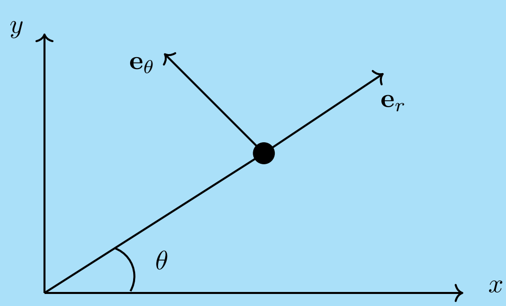
Choose z axis such that orbit lies in the plane $z = 0$ theun se polar co-ordinates in the (x, y) plane
Recall:

$$x = r\cos\theta \; y = r\sin\theta$$

Definition. We define unit vectors:
in direction of increasing r : $\mathbf{e}_r$
in direction of increasing θ : $\mathbf{e}_\theta$

$$\mathbf{e}_r = \hat{\mathbf{r}} = \begin{bmatrix}\cos\theta \\ \sin\theta\end{bmatrix} \; \mathbf{e}_\theta = \begin{bmatrix}-\sin\theta \\ \cos\theta\end{bmatrix}$$

Relative to fixed Cartesian axes
At any point $\mathbf{e}_r, \mathbf{e}_\theta$ form an orthonormal basis, but orientation depends on position.



Note.

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\mathbf{e}_r = \frac{\mathrm{d}}{\mathrm{d}\theta}\begin{bmatrix}\cos\theta \\ \sin\theta\end{bmatrix} = \begin{bmatrix}-\sin\theta \\ \cos\theta\end{bmatrix} = \mathbf{e}_\theta$$
$$\frac{\mathrm{d}}{\mathrm{d}\theta}\mathbf{e}_\theta = -\mathbf{e}_r$$

Equation. For a moving particle, θ is a function of position and hence of time.
If coordinate $(r(t), \theta(t))$ then

$$\frac{\mathrm{d}\mathbf{e}_r}{\mathrm{d}t} = \frac{\mathrm{d}\theta}{\mathrm{d}t}\frac{\mathrm{d}\mathbf{e}_r}{\mathrm{d}\theta} = \mathbf{e}_\theta\frac{\mathrm{d}\theta}{\mathrm{d}t}$$
$$\frac{\mathrm{d}\mathbf{e}_\theta}{\mathrm{d}t} = \frac{\mathrm{d}\theta}{\mathrm{d}t}\cdot\frac{\mathrm{d}\mathbf{e}_\theta}{\mathrm{d}\theta} = -\mathbf{e}_r\frac{\mathrm{d}\theta}{\mathrm{d}t}$$

Equation. Now consider the implications for expression of velocity and acceleration in terms of (r, θ) and $\mathbf{e}_r, \mathbf{e}_\theta$
We have the position vector is $\mathbf{r} = r\mathbf{e}_r$ hence

$$\dot{\mathbf{r}} = \dot{\mathbf{r}}\mathbf{e}_r + r\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{e}_r = \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta$$

$\dot{r}$ is the radial component of velocity
$r\dot{\theta}$ is the angular component of velocity
$\dot{\theta}$ has dimensions $\frac{1}{T}$, it is the 'angular velocity'
The acceleration

$$\begin{aligned}\ddot{\mathbf{r}} &= \frac{\mathrm{d}}{\mathrm{d}t} \\ &= \ddot{r}\mathbf{e}_r + \dot{r}\dot{\mathbf{e}}_r + \dot{r}\dot{\theta}\mathbf{e}_\theta + r\ddot{\theta}\mathbf{e}_\theta + r\dot{\theta}\dot{\mathbf{e}}_\theta \\ &= \underbrace{(\ddot{r} - r\dot{\theta}^2)}_{r\text{ component}}\mathbf{e}_r + \underbrace{(2\dot{r}\dot{\theta} + r\ddot{\theta})}_{\theta\text{ component}}\mathbf{e}_\theta\end{aligned}$$

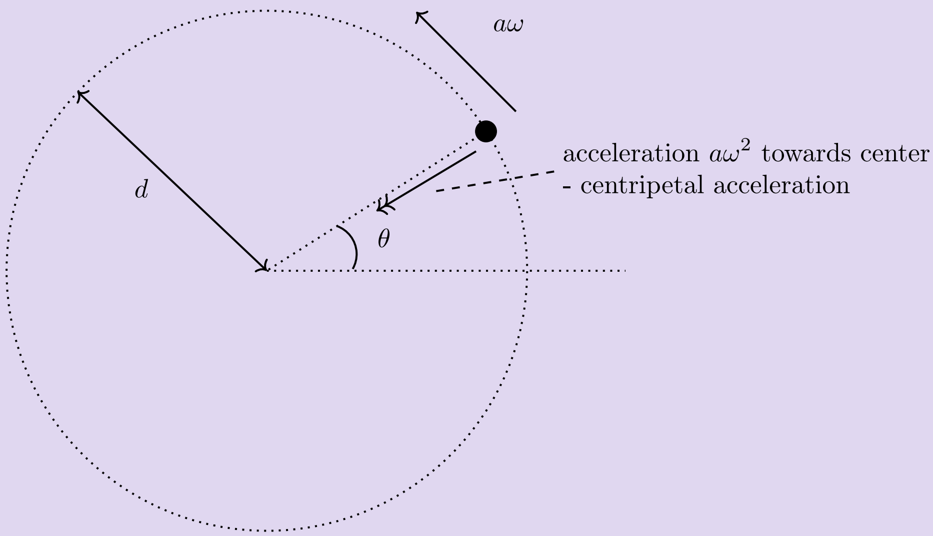
Example. circular motion with constant angular velocity

$$r = a \text{ (constant)}$$

$$\dot{\theta} = \omega \text{ (constant)}$$

$$\dot{r} = 0,\ \ddot{\theta} = 0,\ \ddot{r} = 0$$

Velocity $\dot{\mathbf{r}} = a\omega\mathbf{e}_\theta$
Acceleration $\ddot{\mathbf{r}} = -a\omega^2\mathbf{e}_r$



Newton's 2nd Law implies that a force is required to maintain circular motion (centripetal force).

4.3 Motion in a Central Force Field

Method. Newton's 2nd law

$$m\ddot{\mathbf{r}} = \mathbf{F} = -\nabla V = -\frac{\mathrm{d}V}{\mathrm{d}r}\mathbf{e}_r$$

From earlier:

$$\underbrace{m(\ddot{r} - r\dot{\theta}^2)\mathbf{e}_r}_{\text{radial}} + \underbrace{m(2\dot{r}\dot{\theta} + r\ddot{\theta})\mathbf{e}_\theta}_{\text{angular}} = \underbrace{-\frac{\mathrm{d}V}{\mathrm{d}r}\mathbf{e}_r}_{\text{radial}}$$

Angular component:

$$m(2\dot{r}\dot{\theta} + r\ddot{\theta}) = 0$$

$$\frac{m}{r}\frac{\mathrm{d}}{\mathrm{d}t}\left(r^2\dot{\theta}\right) = 0$$

equivalent to

$$mr^2\dot{\theta} = \text{constant}$$

Recall

$$\mathbf{L} = m\mathbf{r} \times \dot{\mathbf{r}} = mr\mathbf{e}_r \times (\dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta) = mr^2\dot{\theta}\mathbf{e}_z$$

i.e. magnitude of angular momentum is constant:

$$r^2\dot{\theta} = h \text{ (constant)}$$

Radial component:

$$m\ddot{r} - mr\dot{\theta}^2 = \frac{\mathrm{d}V}{\mathrm{d}r}$$

$$m\ddot{r} = -\frac{\mathrm{d}V}{\mathrm{d}r} + \frac{mh^2}{r^3} = -\frac{\mathrm{d}V_{\text{eff}}}{\mathrm{d}r}$$

With

$$V_{\text{eff}}(r) = V(r) + \frac{1}{2}\frac{mh^2}{r^2}$$

effective potential. i.e. motion particle is equivalent to 1-D motion under the influence of the effective potential.

Consider energy of particle:

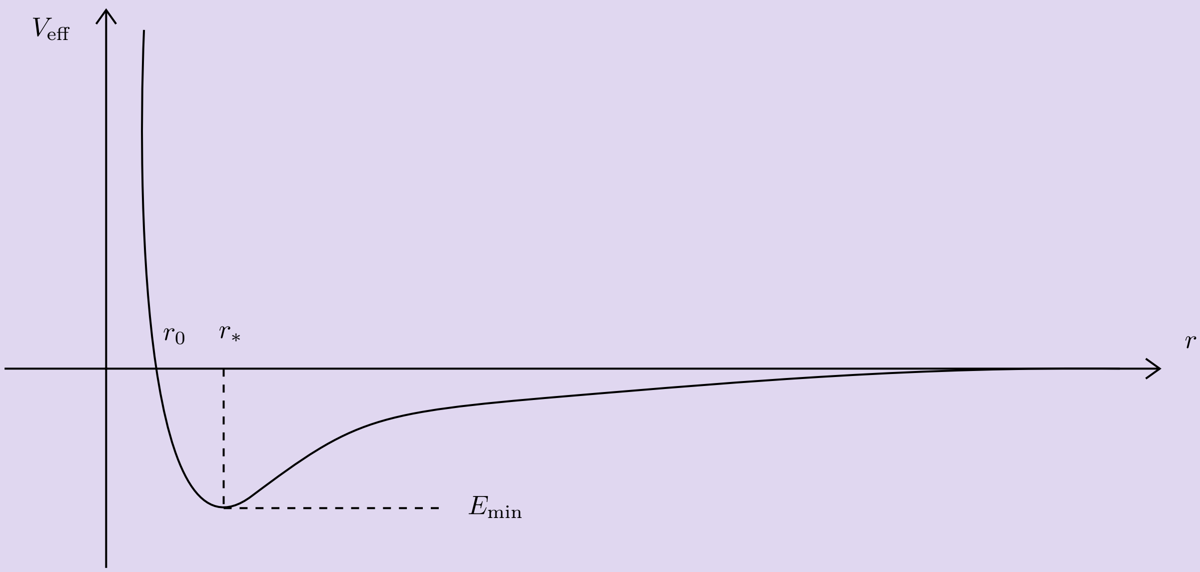
$$T + V(r) = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + V(r) = \frac{1}{2}m\dot{r}^2 + \underbrace{\frac{1}{2}m\frac{h^2}{r^2} + V(r)}_{V_{\text{eff}}(r)}$$

Example (inverse square law force).

$$V(r) = -\frac{GMm}{r}$$

$$V_{\text{eff}}(r) = -\frac{GMm}{r} + \frac{1}{2}\frac{mh^2}{r^2}$$

(given h)



$$V_{\text{eff}}(r_0) = 0,\ r_0 = \frac{h^2}{2GM}$$

$$V'_{\text{eff}}(r_*) = 0,\ r_* = \frac{h^2}{GM}$$

$$E_{\text{min}} = -m\frac{(GM)^2}{2h^2}$$

What is the possible motion of m?

$$E = E_{\text{min}}$$

$$r(t) = r_*$$

as in equilibrium,

$$\dot{\theta} = \frac{h}{r_*^2}$$

If $E_{\text{min}} < E < 0$, $r(t)$ oscillates between minimum and maximum values, $\dot{\theta}$ varies.
If $0 \leq E,\ r(t) \to \infty$ as $t \to \infty$, the particle escapes (unbound orbit)

Note. r_{min} is called the periapsis, perihelion for Sun, perigee for Earth
r_{max} is called the apoasis, aphelion for Sun, apogee for Earth

4.4 Stability of Circular Orbits

Method. Consider a general potential $V(r)$ - does a circular orbit exist and is it stable? Assume that angular momentum is given and $\neq 0$
For circular orbit

$$r(t) = r_* = \text{constant}$$

requires that $\ddot{r} = 0$ hence

$$V'_{\text{eff}}(r_*) = 0$$

condition for circular orbit.
Stable if V_{eff} has a minimum at r_*, unsstable if maximum.
Stabile if $V''_{\text{eff}} > 0$ at $r = r_*$, unstable if $V''_{\text{eff}} < 0$ at $r = r_*$.
Rewrite in terms of $V(r)$.

$$V'(r_*) - \frac{mh^2}{r_*^3} = 0$$

stable if

$$V''(r_*) + \frac{3mh^2}{r_*^4} > 0$$

i.e.

$$V''(r_*) + \frac{3V'(r_*)}{r_*} > 0 \text{ - condition for stability}$$

Example.

$$V(r) = -\frac{km}{r^p}\ p > 0,\ k > 0$$

for circular motion:

$$\frac{pkm}{r_*^{p+1}} - \frac{mh^2}{r_*^3} = 0$$

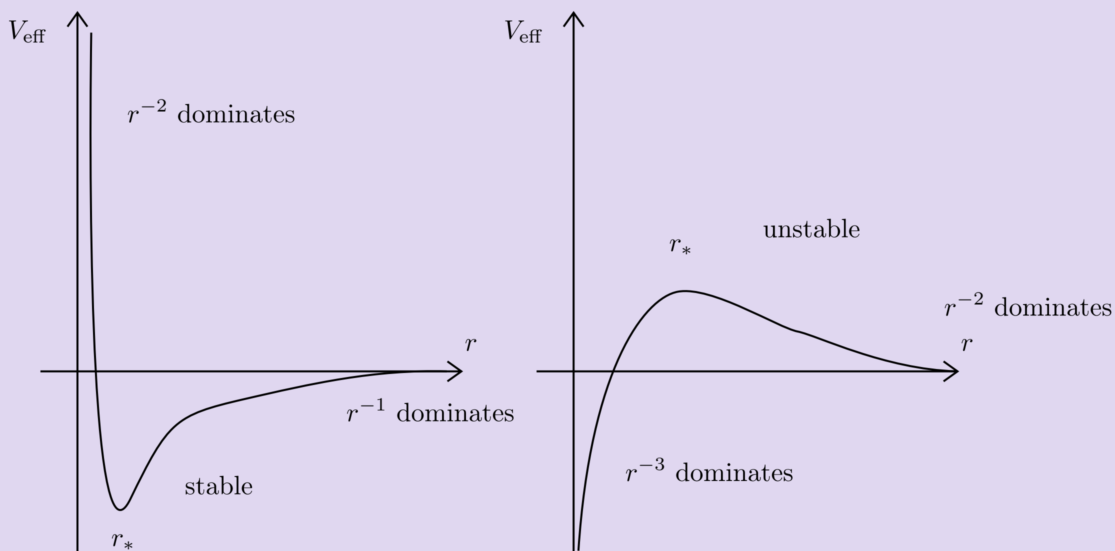
$$\implies r_*^{p-2} = \frac{pk}{h^2}$$

$$r_* = \left(\frac{pk}{h^2}\right)^{1/(p-2)}$$

There is a circular orbit for all h unless $p = 2$

$$V''(r_*) + \frac{3V'(r_*)}{r_*} = -\frac{kmp(p+1)}{r_*^{p+2}} + \frac{3kmp}{r_*^{p+2}}$$

$$= \frac{p(2-p)km}{r_*^{p-2}} \begin{cases} > 0 \text{ if } 0 < p < 2 \text{ i.e. stable} \\ < 0 \text{ if } p > 2 \text{ i.e. unstable} \end{cases}$$



$p = 1$ on left, $p = 3$ on right

4.5 The Orbit Equation

Remark. The shape of the orbit is determined by the joint variation of $r(t)$ and $\theta(t)$.
In principle we could determine $r(t)$ via the energy equation

$$E = \frac{1}{2}m\dot{r}^2 + V_{\text{eff}}(r) = \text{constant}$$

hence

$$t = \mu\frac{m}{2}\int^r \frac{\mathrm{d}r'}{\sqrt{E - V_{\text{eff}}(r')}}$$

giving $r(t)$. Then use $r(t)^2\dot{\theta} = h$ to deduce $\theta(t)$.
In practice, this is not very helpful - e.g. analytic solution only possible for a small family of $V_{\text{eff}}(r)$

Method. A better approach is to use θ as the independent variable, by writing

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\mathrm{d}\theta}{\mathrm{d}t}\cdot\frac{\mathrm{d}}{\mathrm{d}\theta} = \frac{h}{r^2}\frac{\mathrm{d}}{\mathrm{d}\theta}$$

apply to Netwon's 2$^{\text{nd}}$ Law:

$$m\frac{h}{r^2}\frac{\mathrm{d}}{\mathrm{d}\theta}\underbrace{\left(\frac{h}{r^2}\frac{\mathrm{d}}{\mathrm{d}\theta}r\right)}_{\text{sub } u=1/r} - \frac{mh^2}{r^3} = F(r)$$

Hence

$$mhu^2\frac{\mathrm{d}}{\mathrm{d}\theta}\left(-h\frac{\mathrm{d}u}{\mathrm{d}\theta}\right) - mh^2u^3 = F(u^{-1})$$

which rearranges to give

$$\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2} + u = -\frac{1}{mh^2u^2}F(u^{-1})$$

This is the orbit equation, or Binet equation.
Solve this to find $u(\theta)$, then use $\dot{\theta} = hu^2$ etc.

4.6 The Kepler Problem

Method. This is the orbit problem for the special case of a gravitational central force.

$$F(r) = -\frac{mk}{r^2}$$

Hence:

$$\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2} + u = -\frac{1}{mh^2u^2}(-mku^2) = \frac{k}{h^2}$$

Linear in u. Solution

$$u = \frac{k}{h^2} + A\cos(\theta - \theta_0)$$

Assume wlog $A \geq 0$:
if $A = 0$, $u = \frac{k}{h^2}$ circular orbit
if $A > 0$, u max when $\theta = \theta_0$, r is min.
Choose $\theta_0 = 0$

$$r = \frac{1}{u} = \frac{l}{1 + e\cos\theta}$$

$$l = \frac{h^2}{k}$$

$$e = \frac{Ah^2}{k}$$

Polar coordinate form of 'conic section'
e is the 'eccentricity' – determines shape of curve.
Rewrite in Cartesians

$$r(1 + e\cos\theta) = l$$

$$\implies r = l - ex \implies rx^2 + y^2 = (l - ex)^2$$

$$\implies (1 - e^2)x^2 + y^2 + 2elx = l^2 \tag{†}$$

If $0 \leq e < 1$: ellipse - orbit is bounded

$$\frac{l}{1+e} \leq r(\theta) \leq \frac{l}{1-e}$$

Rewrite (†)

$$\frac{(x + ea)^2}{a^2} + \frac{y^2}{b^2} = 1$$

with

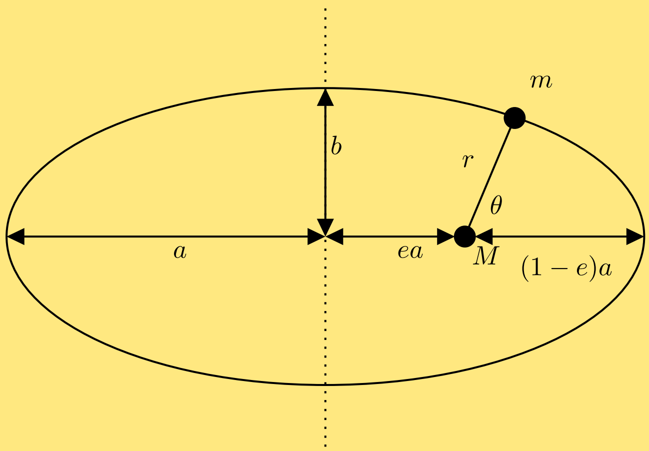
$$a = \frac{l}{1 - e^2}$$

$$b = \frac{l}{\sqrt{1 - e^2}}$$

Method. $e = 0$: circle

$$a = b = \text{radius of circle}$$

$e > 0$ - orgin lies in one of the foic of the ellipse
e, l determine a, b or vice versa



Method. $e > 1$: Hyperbola $r \to \infty$ as $\theta \to \pm\alpha$ with

$$\alpha = \cos^{-1}(-\frac{1}{e}) \in \left(\frac{\pi}{2}, \pi\right)$$

Rewrite (†) as

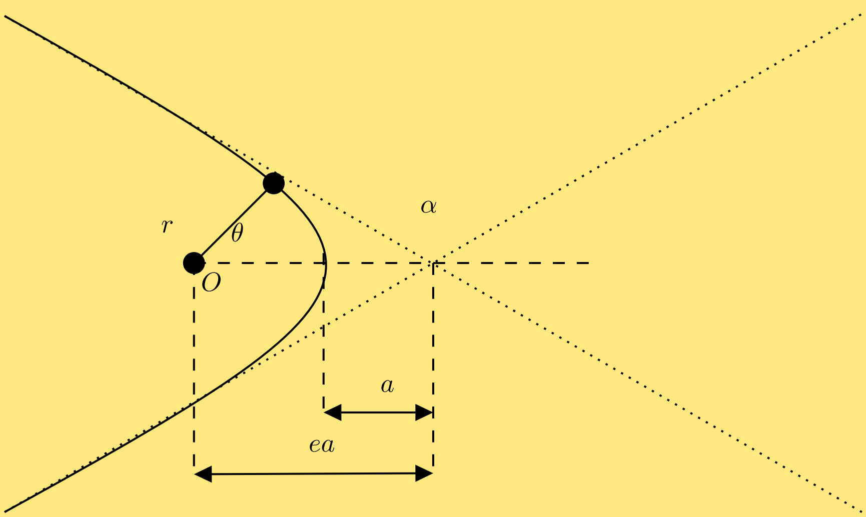
$$\frac{(x-ea)^2}{a^2} - \frac{y^2}{b^2} = 1$$

with

$$a = \frac{l}{e^2-1}$$

$$b = \frac{l}{\sqrt{e^2-1}}$$

(can check calculation)
Hyperbolic orbit represents incoming body with large velocity which is deflected by gravitational force.



Asymptotes are

$$y = \pm\frac{b}{a}(x - ea)$$

i.e.

$$bx \mp ay = eba$$

Normal vectors

$$\frac{1}{\sqrt{a^2+b^2}}\begin{bmatrix} b \\ \mp a \end{bmatrix}$$

Perpendicular distance between 'far' incoming trajectory and central mass is

$$\mathbf{r} \cdot \mathbf{n} = \begin{bmatrix} x \\ y \end{bmatrix} \cdot \frac{1}{\sqrt{a^2+b^2}}\begin{bmatrix} b \\ \mp a \end{bmatrix} = \frac{bx \mp ay}{\sqrt{a^2+b^2}} = \frac{eba}{\sqrt{a^2+b^2}} = b$$

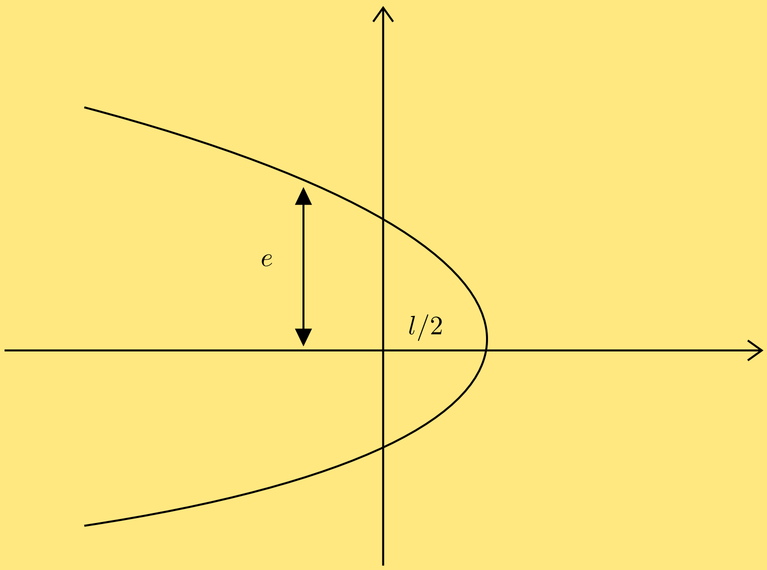
(important parameter)

Method. $e = 1$: parabola with equation

$$r = \frac{l}{1 + \cos\theta}$$

$r \to \infty$ as $\theta \to \pm\pi$
In Cartesians:

$$y^2 = l(l - 2x)$$



Marginal case between ellipse and hyperbola.

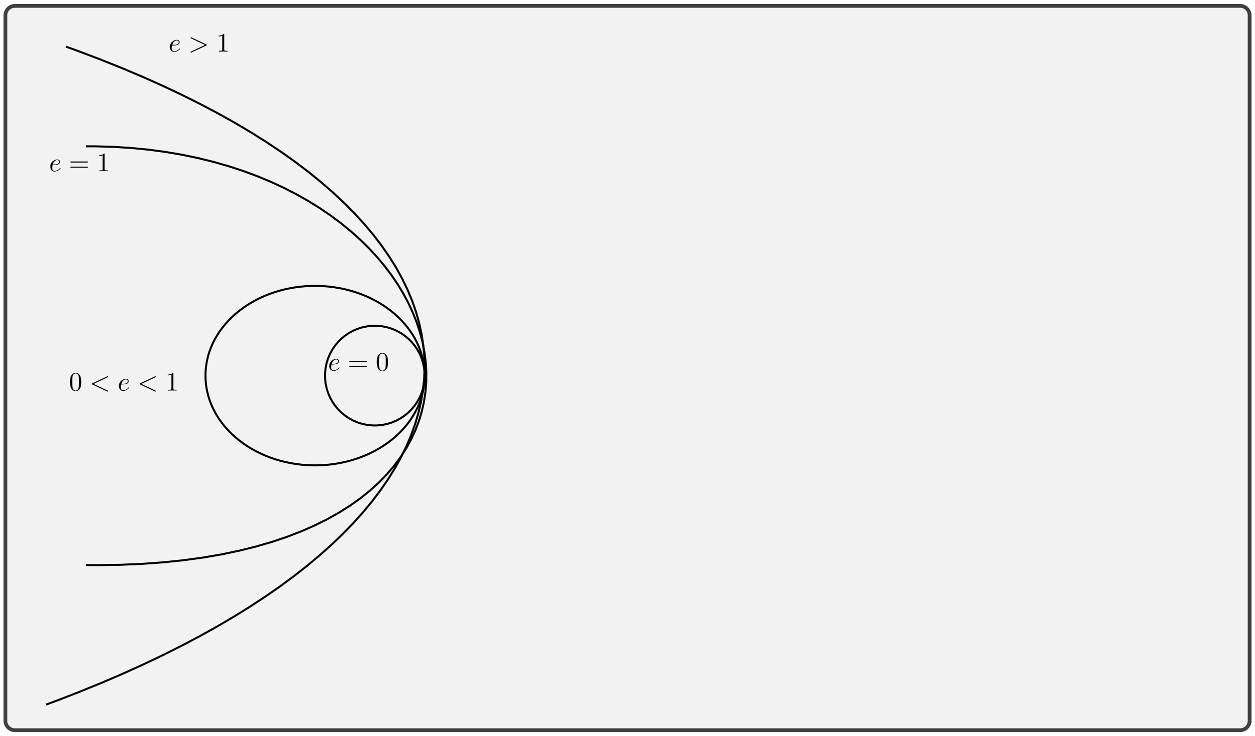
4.6.1 Energy and Eccentricity

Note. Recall:

$$\begin{aligned} E &= \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - \frac{mk}{r} \\ &= \frac{1}{2}mh^2\left(\left(\frac{\mathrm{d}u}{\mathrm{d}\theta}\right)^2 + u^2\right) - mku \\ &= \frac{1}{2}mh^2(e^2\sin^2\theta + (1 + e\cos\theta)^2)\frac{1}{e^2} - \frac{mk}{e}(1 + \cos\theta) \\ &= \frac{mk}{2l}(e^2 - 1) \end{aligned}$$

Using $\dot{r} = -h\frac{\mathrm{d}u}{\mathrm{d}\theta}$ and $l = \frac{h^2}{k}$
Bound orbits have $e < 1, E < 0$ and unbound orbits have $e > 1, E > 0$. Marginal case $e = 1, E = 0$ note also

$$e = \left(\frac{2lE}{mk} + 1\right)^{1/2}$$


$e > 1$
$e = 1$
$0 < e < 1$
$e = 0$

4.6.2 Keplers Laws of Planetary Motion

Law (Keplers Laws of Planetary Motion). (i) Orbit of planet is ellipse with Sun at focus

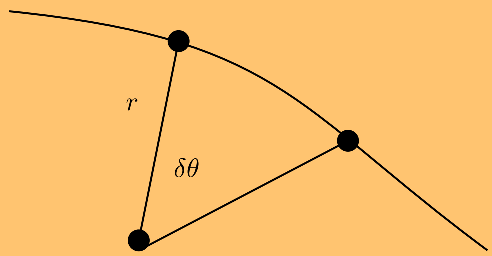
(ii) Line between planet and Sun sweeps out equal area in equal times

(iii) Square of period P is proportional to cube of semi-major axis a

$$P^2 \propto a^3$$

(i) is consistent with our solution of orbit equation (for bounded orbits)

(ii)



$\delta\theta$ = small change in θ in time δt

$$\text{area } \simeq \frac{1}{2}r^2 \mathrm{d}\theta$$

Hence rate of change is $\frac{1}{2}r^2\dot{\theta}$ = angular momentum h
i.e. follows from conservation of angular momentum

(iii) Note area of ellipse is

$$\pi ab = \frac{h}{2}P$$

recall

$$b^2 = a^2(1-e^2)$$
$$h^2 = kl = ka(1-e^2)$$

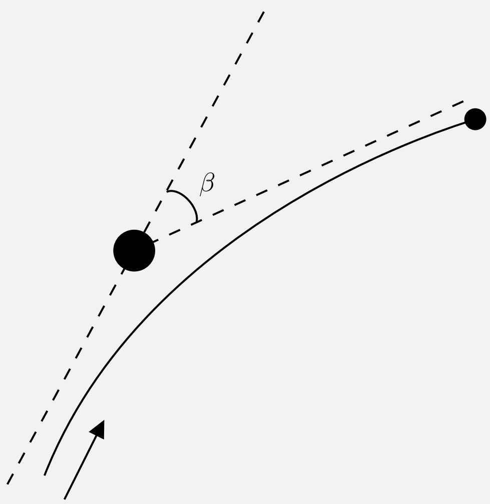
Now consider

$$\begin{aligned} P^2 &= \left(\frac{2\pi ab}{h}\right)^2 \\ &= \frac{4\pi^2 a^2 \cdot a^2(1-e^2)}{ka(1-e)^2} \\ &= \frac{4\pi^2 a^3}{k} \propto a^3 \end{aligned}$$

Note. The multiplying constant is same for all masses orbiting central mass M

4.7 Rutherford Scattering

We consider a positive charge fixed towards another fixed, positive charge.
What is the scattering angle β?



Method. Consider motion in a repulsive square law force

$$V(r) = \frac{mk}{r} \; F(r) = \frac{mk}{r^2}$$

(representing Coulomb repulsive forces between two like charges)
Solution of orbit equation

$$u = -\frac{k}{u^2} + A\cos(\theta - \theta_0)$$

w.l.o.g $\theta_0 = 0$ $A \geq 0$
Rewrite as

$$r = \frac{l}{e\cos\theta - 1}, \; l = \frac{h^2}{k}, \; e = \frac{Ah^2}{k}$$

Require $e > 1$ for $r > 0$ for some θ
Then $r \to \infty$ as $\theta \to \pm\alpha$ with

$$\alpha = \cos^{-1}\left(\frac{1}{e}\right) \in \left(0, \frac{\pi}{2}\right)$$

orbit is hyperbola.
As previously:

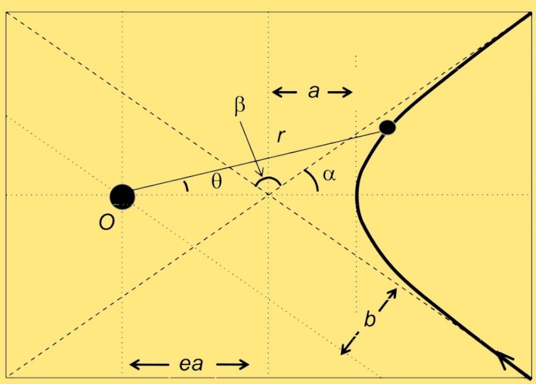
$$\frac{(x - ea)^2}{a^2} - \frac{y^2}{b^2} = 1$$

with

$$a = \frac{l}{e^2 - 1}, \; b = \frac{l}{\sqrt{e^2 - 1}}$$

(can check)
What is h?

$$h = |\mathbf{r} \times \dot{\mathbf{r}}|$$



On incoming asymptote

$$\dot{\mathbf{r}} \simeq v\mathbf{e}_{\parallel}$$

$$\mathbf{r} \simeq (\;)\mathbf{e}_{\parallel} + b\mathbf{e}_{\perp}$$

$$\implies h = bv$$

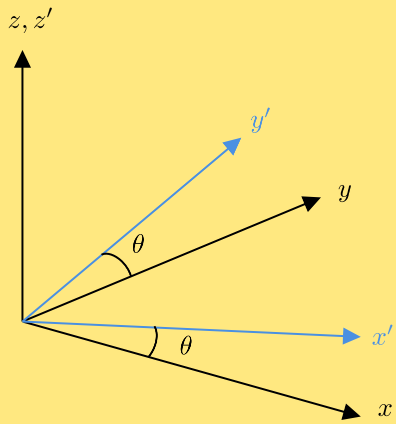
$$b = \frac{l}{\sqrt{e^2 - 1}} = \frac{l}{\tan\alpha} = \frac{l}{\tan\left(\frac{\pi}{2} - \beta\right)} = \frac{h^2}{k}\tan\frac{\beta}{2} = \frac{v^2 b^2}{k}\tan\frac{\beta}{2}$$

$$\beta = 2\tan^{-1}\left(\frac{k}{bv^2}\right)$$

5 Rotating Frames of Reference

Note. Newton's Laws valid are only in an inertial frame of reference.
A rotating frame is non-inertial and therefore the equation of motion will be modified relative to Newtons 2^{nd} Law.

Method. Let S be an inertial frame S' be a non-inertial frame – rotating about t in S – with angular velocity ω ($= \dot{\theta}$ where θ is angle between x or y axis in S and that in S')



Denote basis vectors for S

$$\mathbf{e}_i = \{\hat{\mathbf{x}},\ \hat{\mathbf{y}},\ \hat{\mathbf{z}}\} = \{\mathbf{e}_1,\ \mathbf{e}_2,\ \mathbf{e}_3\}$$

for S'

$$\mathbf{e}'_i = \{\hat{\mathbf{x}'},\ \hat{\mathbf{y}'},\ \hat{\mathbf{z}'}\} = \{\mathbf{e}'_1,\ \mathbf{e}'_2,\ \mathbf{e}'_3\}$$

Consider a particle at rest in S'. viewed in S its velocity:

$$\left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right)_S = \boldsymbol{\omega} \times \mathbf{r} \text{ where } \boldsymbol{\omega} = \omega\hat{\mathbf{z}}$$

Note. Angular velocity vector is aligned with the axis of rotation, magnitude is equal to scalar angular velocity, and viewed from the direction of the vector, the rotation is anticlockwise if $\omega > 0$

Same formula applies to any vector that is fixed in S'; in particular to the basis vectors $\mathbf{e}'_i$ i.e.

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{e}_i\right)_S = \boldsymbol{\omega} \times \mathbf{e}'_i$$

(note that $\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{e}'_z = 0$ under rotation assumed here)

Method. Consider a general time independent vector **a**

$$\mathbf{a}(t) = \sum_{i=1}^{3} a_i'(t)\mathbf{e}_i'(t)$$

expression of **a** in terms of components defined in S'
Now consider rate of change of $\mathbf{a}(t)$

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{a}(t)\right)_{S'} = \sum_{i=1}^{3}\left(\frac{\mathrm{d}}{\mathrm{d}t}a_i'(t)\right)\mathbf{e}_i(t)$$

gives rate of change observed in S'.
What about rate of change observed in S?

$$\begin{aligned}\left(\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{a}(t)\right)_{S} &= \sum_{i=1}^{3}\left(\frac{\mathrm{d}}{\mathrm{d}t}a_i'(t)\right)\mathbf{e}_i'(t) + \sum_{i=1}^{3} a_i'(t)\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{e}_i'(t)\\ &= \left(\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{a}\right)_{S'} + \boldsymbol{\omega}\times\mathbf{a}\end{aligned}$$

Remark. This is a key identity which relates rate of change of vectors seen in S' to rate of change seen in S.

Example. Apply to position vector **r**

$$\left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right)_S = \left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right)_{S'} + \boldsymbol{\omega}\times\mathbf{r} \text{ (velocity)}$$

Note that the difference depends on position.
Now apply to velocity - allow $\boldsymbol{\omega}$ to depend on time

$$\begin{aligned}\left(\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2}\right)_S &= \left(\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)_{S'} + \boldsymbol{\omega}\times\right)\left(\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)_{S'} + \boldsymbol{\omega}\times\right)\mathbf{r}\\ &= \left(\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)_{S'} + \boldsymbol{\omega}\times\right)\left(\left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right)_{S'} + \boldsymbol{\omega}\times\mathbf{r}\right)\\ &= \left(\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2}\right)_{S'} + 2\boldsymbol{\omega}\times\left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right)_{S'} + \dot{\boldsymbol{\omega}}\times\mathbf{r} + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r})\end{aligned}$$

Equation. Equation of motion in a rotating frame

$$\begin{aligned}m\left(\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2}\right)_S &= \mathbf{F}\\ &= m\left[\left(\frac{\mathrm{d}^2\mathbf{r}}{\mathrm{d}t^2}\right)_{S'} + \underbrace{2\boldsymbol{\omega}\times\left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right)_{S'} + \dot{\boldsymbol{\omega}}\times\mathbf{r} + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r})}_{\text{'fictitious forces'}}\right]\end{aligned}$$

Note. Need to take account of fictitious forces to explain motion observed in the rotating frame (or more general non-inertial)
Coriolis force:

$$-2m\dot{\boldsymbol{\omega}} \times \left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right)_{S'}$$

Euler force:

$$-m\dot{\boldsymbol{\omega}} \times \mathbf{r}$$

(in many applications take this to be zero)
Centrifugal force

$$-m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$$

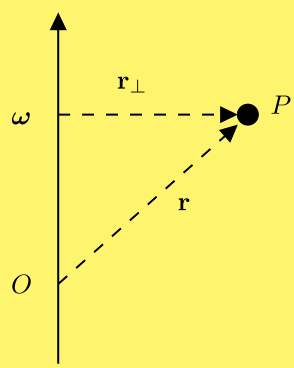
5.1 Centrifugal Force

Equation.

$$\begin{aligned} -m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r} &= -m((\boldsymbol{\omega} \cdot \mathbf{r})\boldsymbol{\omega} - \omega^2\mathbf{r}) \\ &= m\omega^2(\mathbf{r} - \hat{\boldsymbol{\omega}}(\hat{\boldsymbol{\omega}} \cdot \mathbf{r})) \\ &= m\omega^2\mathbf{r}_\perp \end{aligned}$$

$\hat{\boldsymbol{\omega}}$ is unit vector in direction of $\boldsymbol{\omega}$
$\mathbf{r}_\perp$ is part of $\mathbf{r}$ whichh is $\perp$ to $\boldsymbol{\omega}$

Note. $|\mathbf{r}_\perp|$ is perpendicular distance from point to axis of rotation.



Centrifugal force is perpendicular from the rotation axis and directed away from it, with magnitude $\propto$ distance from rotation axis

Note.

$$|r_\perp|^2 = |\mathbf{r}|^2 - (\mathbf{r}\cdot\hat{\boldsymbol{\omega}})^2 = |\hat{\boldsymbol{\omega}} \times \mathbf{r}|^2$$

$$\nabla |r_\perp|^2 = 2\mathbf{r} - 2\hat{\boldsymbol{\omega}}(\hat{\boldsymbol{\omega}}\cdot\mathbf{r}) = 2\mathbf{r}_\perp$$

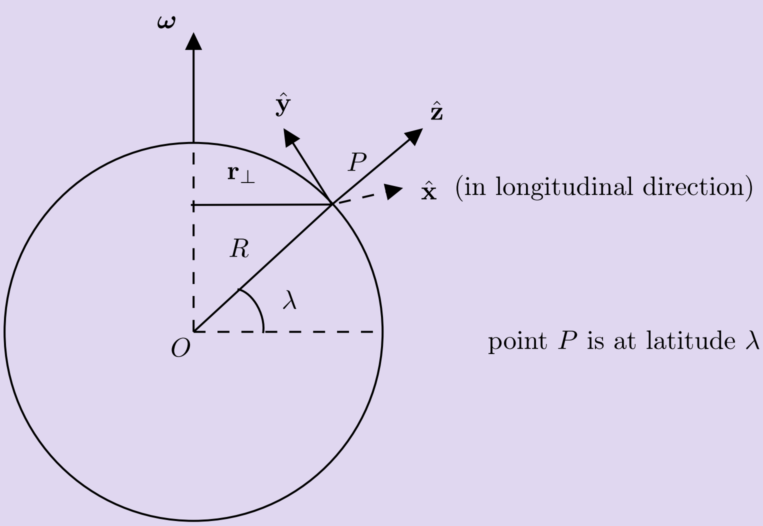
hence

$$m\omega^2\mathbf{r}_\perp = \nabla\left(\frac{1}{2}m\omega^2|\mathbf{r}_\perp|^2\right)$$

i.e. centrifugal force is a conservative force.
On a rotating planet, it is convenient to combine centrifugal force and gravitational force into 'effective gravity'

$$\mathbf{g}_{\text{eff}} = \mathbf{g} + \omega^2\mathbf{r}_\perp$$

Example. Consider rotating planet P is at latitude λ



point P is at latitude λ

$\hat{\mathbf{x}}$ horizontal, axis eastward
$\hat{\mathbf{y}}$ horizontal, axis northward
$\hat{\mathbf{z}}$ vertical

$$\mathbf{r} = R\hat{\mathbf{z}}$$

$$\boldsymbol{\omega} = \omega(\cos\lambda\hat{\mathbf{y}} + \sin\lambda\hat{\mathbf{z}})$$

$$\begin{aligned}\mathbf{g}_{\text{eff}} &= -g\hat{\mathbf{z}} + \omega^2\mathbf{r}_\perp \\ &= \hat{\mathbf{z}}(\omega^2R\cos^2\lambda - g) - \hat{\mathbf{y}}(\omega^2R\cos\lambda\sin\lambda)\end{aligned}$$

Angle between g_{eff} and $\hat{\mathbf{z}}$ is

$$\alpha = \tan^{-1}\left(\frac{\omega^2R\cos\lambda\sin\lambda}{\omega^2R\cos^2\lambda - g}\right)$$

For Earth:

$$\omega = \frac{2\pi}{86400} \sim 7.3\times10^{-5}\text{s}^{-1}$$

$$R = 6.4\times10^6\text{m}$$

$$\frac{\omega^2R}{g} \simeq 3.5\times10^{-3}$$

α is very small for Earth

5.2 Coriolis force

Method.

$$-2m\boldsymbol{\omega} \times \left(\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}\right)_{S'} = -2m\boldsymbol{\omega} \times \mathbf{v}$$

($\mathbf{v}$ shorthand for velocity observed in rotating frame)
proportional to velocity measured in the rotating frae and perpendicular to velocity - hence Coriolis force foes no work on the particle.
Consider Coriolis force on rotating planet, consider velocity tangential to surface

$$\mathbf{v} = v_x\hat{\mathbf{x}} + v_y\hat{\mathbf{y}}$$

Angular velocity is

$$\boldsymbol{\omega} = \omega(\cos\lambda\hat{\mathbf{y}} + \sin\lambda\hat{\mathbf{z}}$$

Hence

$$-2m\boldsymbol{\omega} \times \mathbf{v} = \underbrace{2m\omega\sin\lambda(v_y\hat{\mathbf{x}} - v_x\hat{\mathbf{y}})}_{\text{horizontal}} + \underbrace{2m\omega\cos\lambda v_x\hat{\mathbf{z}}}_{\text{vertical}}$$

The horizontal Coriolis force gives an acceleration to the right of the horizontal velocity on the Northern Hemisphere, to the left in the Southern Henisphere.
Can be balanced by another force e.g. pressure gradient.

Example. Ball dropped from top of tower – where does it land? (not horizontal motion) – taking account of the rotation:

$$\ddot{\mathbf{r}} = \mathbf{g} - 2\boldsymbol{\omega} \times \dot{\mathbf{r}} - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$$

Rotation is slow $\frac{\omega^2 R}{g}$ is small (about 3.5×10^{-3})
Work to first order in ω

$$\ddot{\mathbf{r}} = \mathbf{g} - 2\boldsymbol{\omega} \times \dot{\mathbf{r}} + O(\omega^2)$$

$$\implies \dot{\mathbf{r}} = \mathbf{g}t - 2\boldsymbol{\omega} \times \mathbf{r} + O(\omega^2) + 2\boldsymbol{\omega} \times \mathbf{r}(0)$$

Hence neglecting $O(\omega^2)$

$$\ddot{\mathbf{r}} = \mathbf{g} - 2\boldsymbol{\omega} \times \mathbf{g}t + O(\omega^2)$$

$$\mathbf{r} = \mathbf{g}\frac{t^2}{2} - \frac{1}{3}\boldsymbol{\omega} \times \mathbf{g}t^3 + \mathbf{r}(0) + O(\omega^2)$$

Now consider

$$\mathbf{g} = \begin{bmatrix} 0 \\ 0 \\ -g \end{bmatrix}$$

$$\boldsymbol{\omega} = \begin{bmatrix} 0 \\ \omega \\ 0 \end{bmatrix}$$

$$\mathbf{r}(0) = \begin{bmatrix} 0 \\ 0 \\ R+h \end{bmatrix}$$

Hence

$$\mathbf{r}(t) = \begin{bmatrix} 0 \\ 0 \\ -\frac{gt^2}{2} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ R+h \end{bmatrix} + \frac{1}{3}\omega g \begin{bmatrix} t^3 \\ 0 \\ 0 \end{bmatrix}$$

Particle hits ground when $h = \frac{gt^2}{2}$, $t = (\frac{2h}{g})^{1/2}$
Horizontal displacement

$$\frac{1}{3}\omega g \left(\frac{2h}{g}\right)^{3/2}$$

Hits ground east of base of tower
(consistent with conservation of angular momentum)

6 Systems of Particles

We have considered the dynamics of particles (i.e. masses concentrated at a single point) – now we move on to consider systems of particles.

Method. Consider N particles, mass m_i, position $\mathbf{r}_i(t)$, momentum $\mathbf{p}_i(t) = m_r\dot{\mathbf{r}}_i$
Newton's 2$^{\text{nd}}$ Law applies to the i-th particle individually

$$m_r\ddot{\mathbf{r}}_i = \dot{\mathbf{p}}_i = \mathbf{F}_i$$

$\mathbf{F}_i$ is the total force applied onto the particle.
We distinguish between external and internal forces

$$\mathbf{F}_i = \mathbf{F}_i^{\text{ext}} + \sum_{\substack{j=1\\ j\neq i}}^{N}\mathbf{F}_{ij}$$

$\mathbf{F}_{ij}$ is the force exerted on i-th particle by j-th particle.
($\mathbf{F}_{ii} = \mathbf{0}$)
$\mathbf{F}_i^{\text{ext}}$ is the external force exerted on the i-th particle.
Newtons 3$^{\text{rd}}$ Law implies

$$\mathbf{F}_{ij} = -\mathbf{F}_{ji}$$

e.g. gravitation

$$\mathbf{F}_{ij} = -\frac{GM_i m_j(\mathbf{r}_i - \mathbf{r}_j)}{|\mathbf{r}_i - \mathbf{r}_j|^3}$$

$$\mathbf{F}_{ji} = -\frac{GM_i m_j(\mathbf{r}_j - \mathbf{r}_i)}{|\mathbf{r}_j - \mathbf{r}_i|^3}$$

6.1 Center of Mass

Note. Total mass

$$M = \sum_{i=1}^{N} m_i$$

Equation. Center of mass located at

$$\mathbf{R} = \frac{1}{M}\sum_{i=1}^{N} m_i\mathbf{r}_i$$

Equation. Total linear mommentum

$$\mathbf{P} = \sum_{i=1}^{N} m_i\dot{\mathbf{r}}_i = \sum_{i=1}^{N}\mathbf{p}_i = M\dot{\mathbf{R}}$$

i.e. total linear momentum is equivalent to that of point mass M located at $\mathbf{R}$.

Equation. Then

$$\begin{aligned}\dot{\mathbf{P}} &= M\ddot{\mathbf{R}} \\ &= \sum_{i=1}^{N} \mathbf{p}_i \\ &= \sum_{i=1}^{N} \mathbf{F}_i^{\text{ext}} + \underbrace{\sum_{i=1}^{N}\sum_{j=1}^{n} \mathbf{F}_{ij}}_{=0 \text{ (pairwise sum)}} \\ &= \mathbf{F}^{\text{ext}}\end{aligned}$$

Center of mass moves as if it is the position of mass M under the influence of force

$$\mathbf{F}^{\text{ext}} = \sum_{i=1}^{N} \mathbf{F}_i^{\text{ext}}$$

Note. If $\mathbf{F}^{\text{ext}} = 0$ then $\dot{\mathbf{P}} = \mathbf{0}$ so total momentum is conserved.
There will be a 'center of mass' frame with origin at center of mass – which is inertial. In this frame $\dot{\mathbf{R}} = \mathbf{0}$, e.g. take $\mathbf{R} = 0$

6.2 Motion Relative to Centre of Mass

Equation. Let

$$\mathbf{r}_i = \mathbf{R} + \mathbf{s}_i$$

where $\mathbf{s}_i$ is position vector relative to the centre of mass

$$\begin{aligned}\sum_{i=1}^{N} m_i \mathbf{s}_i &= \sum_{i=1}^{N} m_i(\mathbf{r}_i = \mathbf{R}) \\ &= \sum_{i=1}^{N} m_i \mathbf{r}_i - \sum_{i=1}^{N} m_i \mathbf{R} \\ &= M\mathbf{R} - M\mathbf{R} \\ &= \mathbf{0}\end{aligned}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \sum_{i=1}^{N} m_i \mathbf{s}_i = \mathbf{0}$$

Equation. Total linear momentum

$$\mathbf{P} = \sum_{i=1}^{N} m_i(\dot{\mathbf{R}} + \dot{\mathbf{s}}_i) = \sum_{i=1}^{N} m_i \dot{\mathbf{R}}_i = M\dot{\mathbf{R}}$$

6.3 Angular Momentum

Equation. Total angular momentum

$$\mathbf{L} = \sum_{i=1}^{N} \mathbf{r}_i \times \mathbf{p}_i \text{ (about } O\text{)}$$

$$\begin{aligned}
\dot{\mathbf{L}} &= \underbrace{\sum_{i=1}^{N} \dot{\mathbf{r}}_i}_{=\mathbf{0}} + \mathbf{p}_i + \sum_{i=1}^{N} \mathbf{r}_i \times \dot{\mathbf{p}}_i \\
&= \sum_{i=1}^{N} \mathbf{r}_i \times \mathbf{F}_i^{\text{ext}} + \sum_{i=1}^{N} \mathbf{r}_i \times \sum_{j=1}^{n} \mathbf{F}_{ij} \\
&= \sum_{i=1}^{N} \mathbf{r}_i \times \mathbf{F}_i^{\text{ext}} + \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{n} (\mathbf{r}_i \times \mathbf{F}_{ij} + \mathbf{r}_j \times \mathbf{F}_{ji}) \\
&= \sum_{i=1}^{N} \mathbf{r}_i \times \mathbf{F}_i^{\text{ext}} + \frac{1}{2} \sum_{i=1}^{N} \sum_{j=1}^{n} (\mathbf{r}_i - \mathbf{r}_j) \times \mathbf{F}_{ij} \\
&(= 0 \text{ if } \mathbf{F}_{ij} \parallel \mathbf{r}_i - \mathbf{r}_j) \qquad (*)
\end{aligned}$$

If (*) is satisfied

$$\dot{\mathbf{L}} = \sum_{i=1}^{N} \mathbf{r}_i \times \mathbf{F}_i^{\text{ext}} = \mathbf{G}^{\text{ext}}$$

This is the total external torque acting on system

Equation. Now return to motion, position relative to center of mass
Total angular momentum

$$\begin{aligned}
\mathbf{L} &= \sum_{i=1}^{N} m_i (\mathbf{R} + \mathbf{s}_i) \times (\dot{\mathbf{R}} + \dot{\mathbf{s}}_i) \\
&= \sum_{i=1}^{N} m_i (\mathbf{R} \times \dot{\mathbf{R}}) + \underbrace{\sum_{i=1}^{N} m_i \mathbf{R} \times \dot{\mathbf{s}}_i}_{=\mathbf{0}} + \underbrace{\sum_{i=1}^{N} m_i \mathbf{s}_i \times \dot{\mathbf{R}}}_{=\mathbf{0}} + \sum_{i=1}^{N} m_i \mathbf{s}_i \times \dot{\mathbf{s}}_i \\
&= M(\mathbf{R} \times \dot{\mathbf{R}}) + \sum_{i=1}^{N} m_i \mathbf{s}_i \times \dot{\mathbf{s}}_i
\end{aligned}$$

Terms equal to 0 due to $\sum m_i \mathbf{s}_i = \mathbf{0}$
So $\mathbf{L}$ = angular momentum of a particle of mass M at $\mathbf{R}$ moving with velocity $\dot{\mathbf{R}}$ + angular momentum associated with motion of particles relative to the centre of mass

6.4 Energy

Equation. Total kinetic energy

$$
\begin{aligned}
T &= \sum_{i=1}^{N} \frac{1}{2} m_i \dot{\mathbf{r}}_i^2 \\
&= \sum_{i=1}^{N} \frac{1}{2} m_i (\dot{\mathbf{R}} + \dot{\mathbf{s}}_i)^2 \\
&= \frac{1}{2} \dot{\mathbf{R}}^2 \sum_{i=1}^{n} m_i + \underbrace{\sum_{i=1}^{n} m_i \dot{\mathbf{s}}_i \cdot \mathbf{R}}_{=0} + \sum_{i=1}^{N} \frac{1}{2} m \dot{\mathbf{s}}_i^2 \\
&= \frac{1}{2} \dot{\mathbf{R}}^2 \sum_{i=1}^{n} m_i + \sum_{i=1}^{N} \frac{1}{2} m \dot{\mathbf{s}}_i^2
\end{aligned}
$$

So $T =$ KE of particle mass M moving with velocity $\dot{R}+$ KE associated with paritcle motion relative to the centre of mass

Now ask is energy conserved?

Method. Consider

$$
\begin{aligned}
\frac{\mathrm{d}T}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t} \sum_{i=1}^{N} \frac{1}{2} m_i \mathbf{r}_i^2 \\
&= \sum_{i=1}^{N} m_i \dot{\mathbf{r}}_i \cdot \ddot{\mathbf{r}}_i \\
&= \sum_{i=1}^{N} \dot{\mathbf{r}}_i \cdot \mathbf{F}_i^{\text{ext}} + \sum_{i=1}^{N} \dot{\mathbf{r}}_i \cdot \sum_{j=1}^{N} \mathbf{F}_{ij} \\
&= \sum_{i=1}^{N} \dot{\mathbf{r}}_i \cdot \mathbf{F}_i^{\text{ext}} + \sum_{i=1}^{N} \sum_{j>i}^{N} (\dot{\mathbf{r}}_i - \dot{\mathbf{r}}_j) \cdot \mathbf{F}_{ij}
\end{aligned}
$$

If external forces are defined by a potential

$$\mathbf{F}_i^{\text{ext}} = -\nabla_{\mathbf{r}_i} V_i^{\text{ext}}$$

and internal forces are defined by a potential

$$\mathbf{F}_{ij} = -\nabla_{\mathbf{r}_i} V(\mathbf{r}_i - \mathbf{r}_j)$$

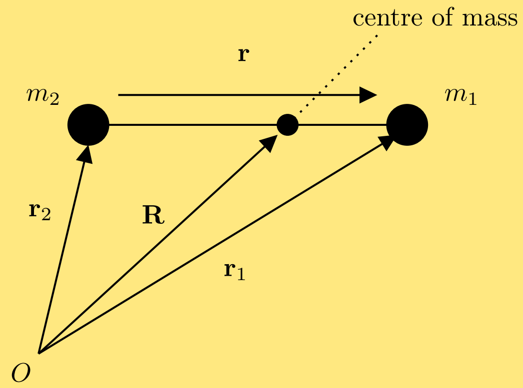
then

$$\frac{\mathrm{d}T}{\mathrm{d}t} = -\frac{\mathrm{d}}{\mathrm{d}t} \sum_{i=1}^{N} V^{\text{ext}}(\mathbf{r}_i) - \frac{\mathrm{d}}{\mathrm{d}t} \sum_{i=1}^{N} \sum_{j>i} V(\mathbf{r}_i - \mathbf{r}_j)$$

i.e. we have a conservation of energy (move RHS terms to LHS)

6.5 Two-Body Problem

Method.



Consider two particles, with no external force acting. The center of mass is at

$$\mathbf{R} = \frac{1}{M}(m_1\mathbf{r}_1 + m_2\mathbf{r}_2)$$

$$M = m_1 + m_2$$

Separation vector

$$\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$$

Since

$$\mathbf{F}^{\text{ext}} = \mathbf{0}$$

have

$$\ddot{\mathbf{R}} = \mathbf{0}$$

so the centre of mass moves with constant velocity.
Consider $\ddot{\mathbf{r}}$:

$$\ddot{\mathbf{r}} = \ddot{\mathbf{r}}_1 - \ddot{\mathbf{r}}_2 = \frac{\mathbf{F}_{12}}{m_1} - \frac{\mathbf{F}_{21}}{m_2} = \left(\frac{1}{m_1} + \frac{1}{m_2}\right)\mathbf{F}_{12}$$

Equivalently,

$$\mu\ddot{\mathbf{r}} = \mathbf{F}_{12}$$

where

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

μ is the 'reduced mass'
(note $\mu < m_1, \mu < m_2$) i.e.standard from of Newton's 2^{nd} Law but for particle mass μ

Method. Now consider gravitational force:

$$\mu\ddot{\mathbf{r}} = -\frac{Gm_1m_2}{|\mathbf{r}|^3}\mathbf{r}$$

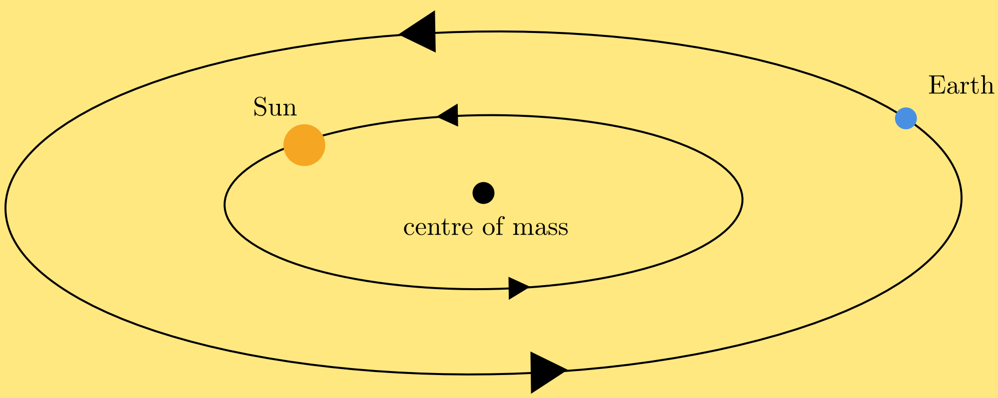
hence

$$\ddot{\mathbf{r}} = -G(m_1+m_2)\frac{\mathbf{r}}{|\mathbf{r}|^3}$$

this is motion of particle under effect of gravitational force due to mass $m = m_1 + m_2$ fixed at origin. Consider Earth-Sun orbit – both move about the center of mass, both orbits ave the same shape, but size of orbit is different for each.
Ratio of masses

$$\text{Earth/Sun } \simeq 3 \times 10^{-4}$$

Earth-sun orbit is about 1.5×10^7 km
Hence sun displacement ~ 450km



Equation. Kinetic energy

$$T = \frac{1}{2}M\dot{\mathbf{R}}^2 + \frac{1}{2}\mu\dot{\mathbf{r}}^2$$

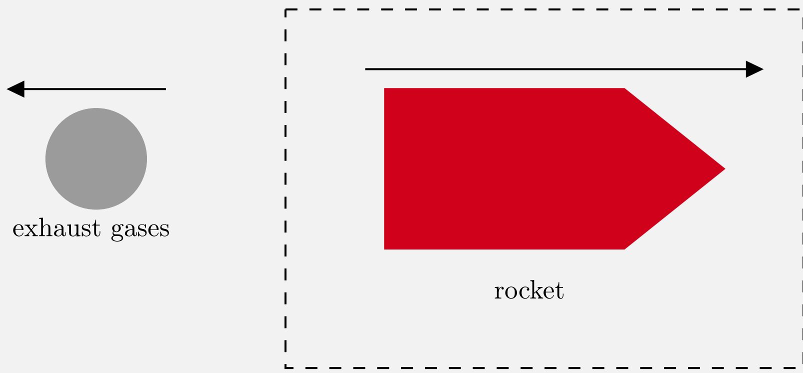
Angular momentum

$$\mathbf{L} = M\mathbf{R} \times \dot{\mathbf{R}} + \mu\mathbf{r} \times \dot{\mathbf{r}}$$

(special forms for 2-body problem of general expressions derived earlier)

6.6 Variable Mass Problems

6.6.1 Rocket Problem



Have a rocket moving in 1-dimension with speed $v(t)$ and mass $m(t)$. Rocket propels itself by expelling mass at velocity u relative to rocket
Total momentum is conserved because there are no external forces acting.

Equation. Total momentum at $t + \delta t$:

$$m(t+\delta t)v(t+\delta t) + (m(t) - m(t+\delta t))(v(t) - u + O(\delta t))$$

Change in momentum from t to $t + \delta t$:

$$m(t+\delta t)v(t+\delta t) + (m(t) - m(t+\delta t))(v(t) - u + O(\delta t)) - m(t)v(t)$$

$$\simeq \left(\frac{\mathrm{d}m}{\mathrm{d}t}u + m\frac{\mathrm{d}v}{\mathrm{d}t}\right)\delta t + O(\delta t^2) = 0$$

$$\implies \frac{\mathrm{d}m}{\mathrm{d}t}u + m\frac{\mathrm{d}v}{\mathrm{d}t} = 0$$

Generalise to

$$\frac{\mathrm{d}m}{\mathrm{d}t}u + m\frac{\mathrm{d}v}{\mathrm{d}t} = F_{\text{ext}}$$

in the presence of external forces.
In the absence of external forces:

$$m\frac{\mathrm{d}v}{\mathrm{d}t} = -\frac{\mathrm{d}m}{\mathrm{d}t}u$$

hence

$$v(t) = v(0) + u\log\left(\frac{m(0)}{m(t)}\right)$$

with $m(0),\ v(0)$ initial mass and velocity

7 Rigid Bodies

Definition. A **rigid body** is an extended object that can be considered as a multi-particle system such that the distance between any two particles in the body remains constant, i.e.

$$|\mathbf{r}_i - \mathbf{r}_j| = \text{constant } \forall i, j$$

The possible motion of a rigid body is a superposition of the two basis transformations that are isometries of Euclidean Space, i.e. they preserve distance - i.e. translations and rotations.

7.1 Angular Velocity

Equation. Recall from section 5, if a particle is rotating about an axis through O, with angular velocity $\boldsymbol{\omega}$ then the velocity is

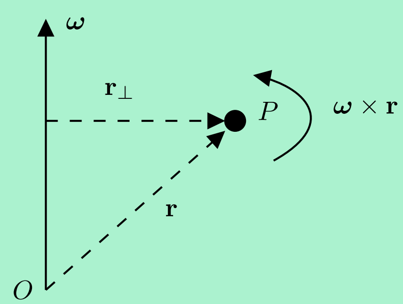
$$\dot{\mathbf{r}} = \boldsymbol{\omega} \times \mathbf{r}$$

with

$$|\dot{\mathbf{r}}| = \omega r_\perp$$

where $r_\perp$ is the perpendicular distance to the axis of rotation.
If the particle has mass m then the kinetic energy

$$\begin{aligned} T &= \frac{1}{2} m \dot{\mathbf{r}}^2 \\ &= \frac{1}{2} (\boldsymbol{\omega} \times \mathbf{r}) \cdot (\boldsymbol{\omega} \times \mathbf{r}) \\ &= \frac{1}{2} m \omega^2 r_\perp^2 \end{aligned}$$



Note that if $\boldsymbol{\omega} = \omega \mathbf{n}$, then $r_\perp = |\mathbf{r} \times \mathbf{r}|$

$$T = \frac{1}{2} m r_\perp^2 \omega^2 = \frac{1}{2} I \omega^2$$

I is moment of inertia of particle about axis of rotation.

7.2 Moment of Inertia for a Rigid Body

Equation. Consider a rigid body to be mad up of N particles
Consider the body to be rotating about an axis through the origin with angular velocity $\boldsymbol{\omega}$
Then for each particle:

$$\dot{\mathbf{r}}_i = \boldsymbol{\omega} \times \mathbf{r}_i$$

Note that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}|\mathbf{r}_i - \mathbf{r}_j|^2 &= 2(\mathbf{r}_i - \mathbf{r}_j) \cdot (\dot{\mathbf{r}}_i - \dot{\mathbf{r}}_j) \\ &= 2(\mathbf{r}_i - \mathbf{r}_j) \cdot (\boldsymbol{\omega} \times (\mathbf{r}_i - \mathbf{r}_j)) \\ &= 0\end{aligned}$$

Constant with properties of a rigid body, i.e. $|\mathbf{r}_i - \mathbf{r}_j|$ does not vary in time.
Now consider the kinetic energy of body

$$\begin{aligned}T &= \sum_{i=1}^{N} \frac{1}{2} m_i \mathbf{r}_i^2 \\ &= \sum_{i=1}^{N} \frac{1}{2} m_i |\boldsymbol{\omega} \times \mathbf{r}_i|^2 \\ &= \sum_{i=1}^{N} \frac{1}{2} m_i \omega^2 |\mathbf{n} \times \mathbf{r}_i|^2 \\ &= \frac{1}{2} I \omega^2\end{aligned}$$

with I the moment of inertia of the body for rotation about axis $\mathbf{n}$ through origin.
Now consider angular momentum:

$$\mathbf{L} = \sum_{i=1}^{N} m_i \mathbf{r}_i \times \underbrace{(\boldsymbol{\omega} \times \mathbf{r}_i)}_{= \dot{\mathbf{r}}_i}$$

Consider $\boldsymbol{\omega} = \omega \mathbf{n}$. Then

$$\mathbf{L} = \omega \sum_{i=1}^{N} m_i \mathbf{r}_i \times (\mathbf{n} \times \mathbf{r}_i)$$

Equation. Now consider part of $\mathbf{L}$ which is $\parallel$ to rotation axis

$$\begin{aligned}\mathbf{L}\cdot\mathbf{n} &= \omega\sum_{i=1}^{N} m_i\mathbf{n}\cdot(\mathbf{r}_i\times(\mathbf{n}\times\mathbf{r}_i)) \\ &= \omega\sum_{i=1}^{N} m_i|\mathbf{n}\times r_i|^2 \\ &= \omega\sum_{i=1}^{N} m_i(r_i)_\perp^2 \\ &= I\omega\end{aligned}$$

Component of angular momentum in direction of rotation axis is $I\omega$
In general $\mathbf{L}$ is not $\parallel$ to rotation axis.
But we have:

$$\mathbf{L} = \sum_{i=1}^{N} m_i(|\mathbf{r}_i|^2\boldsymbol{\omega} - (\mathbf{r}_i\cdot\boldsymbol{\omega})\mathbf{r}_i)$$

(which is a linear function of the vector $\boldsymbol{\omega}$)
I.e.

$$\mathbf{L} = I\boldsymbol{\omega}$$

where I is a matrix like object
e.g. in suffix notation

$$L_\alpha = I_{\alpha\beta}\omega_\beta$$

(using summation convention.)
$I_{\alpha\beta}$ is a symmetric tensor:

$$I_{\alpha\beta} = \sum_{i=1}^{N} m_i\{|\mathbf{r}_i|^2\delta_{\alpha\beta} - (\mathbf{r}_i)_\alpha(\mathbf{r}_i)_\beta\}$$

In general there are 3 directions such that $I\boldsymbol{\omega}\parallel\boldsymbol{\omega}$ corresponding to principal axes of tensor.
If body is rotated about one of the principal axis, then

$$\mathbf{L}\parallel\boldsymbol{\omega}$$

holds for any shape of body

Remark. The simple case: if we choose to rotate in some direction such that $\mathbf{L}\parallel\boldsymbol{\omega}$ then

$$\mathbf{L} = I(\mathbf{n})\boldsymbol{\omega}$$

where $I(\mathbf{n})$ is the moment of inertia bout axis of rotation.
We often consider bodies that are symmetric about a perpendicular axis – rotating about that axis guarantees above property.

$$T = \frac{1}{2}I\omega^2$$

$$\mathbf{L} = I\boldsymbol{\omega}$$

Recall results for linear momentum:

$$T = \frac{1}{2}Mv^2$$

$$\mathbf{p} = M\mathbf{v}$$

7.3 Calculations of Moments of Inertia

Equation. For a solid body we replace mass-weighted sums over particles by mass-weighted integrals. Consider a body occupying a volume V with mass density $\rho(\mathbf{r})$

Total mass:

$$M = \int_V \rho(\mathbf{r})\, \mathrm{d}V$$

Center of mass position vector

$$\mathbf{R} = \frac{1}{M} \int_{\mathbf{r}} \rho(\mathbf{r})\, \mathrm{d}V$$

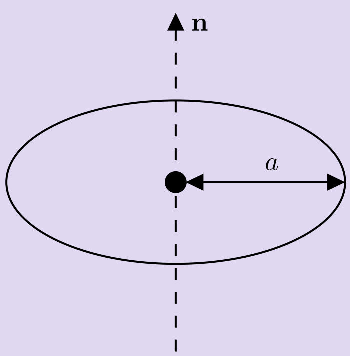
Moment of inertia about axis $\mathbf{n}$

$$I = \int_V \rho(\mathbf{r}) |\mathbf{r}_\perp|^2\, \mathrm{d}V = \int_V \rho(\mathbf{r}) |\mathbf{n} \times \mathbf{r}|^2\, \mathrm{d}V$$

Note. Use the obvious modifications of these formulae for bodies that correspond to mass distributed over a surface or along a curve, as surface(or area) integrals or as line integrals.

We now calculate I for some very simple examples

Example. Uniform thin ring of mass M, radius a, about axis through center of ring and $\perp$ plane of ring.



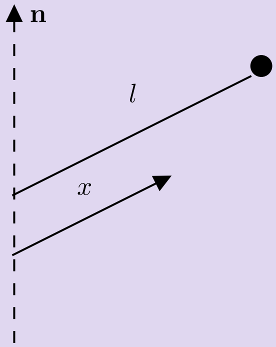
For this case we reduce volume integral to a line integral

$$\rho = \text{mass per unit length } = \frac{M}{2\pi a}$$

$$I = \int_0^{2\pi} \underbrace{\left(\frac{M}{2\pi a}\right)}_{\rho} \cdot \underbrace{a^2}_{r_\perp^2} \cdot \underbrace{a\,\mathrm{d}\theta}_{\mathrm{d}l} = Ma^2$$

Every point in the body has $r_\perp = a$

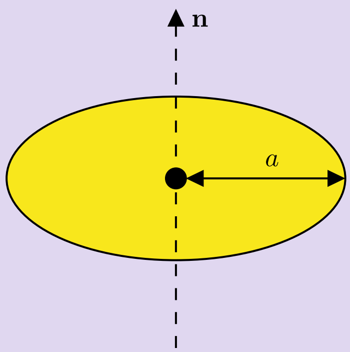
Example. Uniform thin rod, mass M, length l, with axis of rotation through one end and $\perp$ length



$$\rho = \frac{M}{l}$$

$$I = \int_0^l \left(\frac{M}{l}\right) x^2 \, \mathrm{d}x = \frac{1}{3} M l^2$$

Example. Uniform this disc of mass M, radius a with axis of rotation through the centre of the disc and $\perp$ to plane of disc



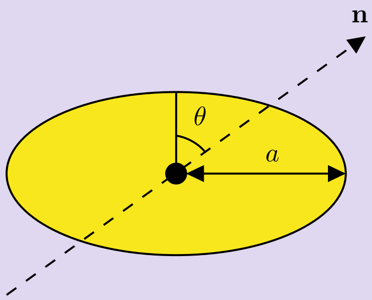
Use area integral $\mathrm{d}V \to \mathrm{d}A$,

$$\rho = \frac{M}{\pi a^2} \text{ mass per unit area}$$

$$I = \int_{r=0}^{a} \int_{\theta=0}^{2\pi} \left(\frac{M}{\pi a^2}\right) r^2 \, r \, \mathrm{d}r \, \mathrm{d}\theta$$

$$I = \frac{M}{\pi a^2} \int_0^a r^2 \cdot r \, \mathrm{d}r \int_0^{2\pi} \mathrm{d}\theta = \frac{M}{\pi a^2} \cdot \frac{a^4}{4} \cdot 2\pi = \frac{1}{2} M a^2$$

Example. Uniform thin disc, mass M, radius a, axis of rotation is through centre, but in plane of the disc

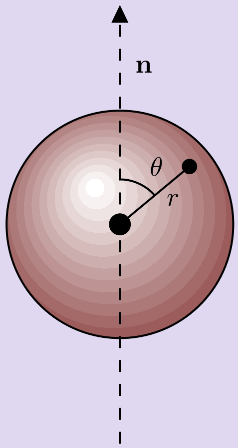


Again use 2-D polars
Perpendicular distance to axis for point with co-ordinates (r, θ) is $r \sin\theta$

$$\begin{aligned} I &= \int_{r=0}^{a} \int_{\theta=0}^{2\pi} \left(\frac{M}{\pi a^2}\right) r^2 \sin\theta \, r \, \mathrm{d}r \, \mathrm{d}\theta \\ &= \frac{M}{\pi a^2} \int_0^a r^3 \, \mathrm{d}r \int_{\theta=0}^{2\pi} \sin^2\theta \, \mathrm{d}\theta \\ &= \frac{M}{\pi a^2} \cdot \frac{a^4}{4} \cdot \pi \end{aligned}$$

$$\frac{1}{4} M a^2$$

Example. Uniform sphere mass M, radius a, axis of rotation through centre



Uee spherical polar co-ordinates r, θ, ϕ and choose $\theta = 0$ to correspond to the axis of rotation. Density $\frac{M}{4\pi a^3/3}$

$$\begin{aligned} I &= \int_{r=0}^{a} \int_{\theta=0}^{2\pi} \int_{\phi=0}^{2\pi} \left(\frac{M}{4\pi a^3/3} \right) r^2 \sin\theta \, \sin\theta r^2 \, \mathrm{d}r \, \mathrm{d}\theta \, \mathrm{d}\phi \\ &= \frac{3M}{4\pi a^3} \int_0^a r^4 \, \mathrm{d}r \int_0^{\pi} \sin^3\theta \, \mathrm{d}\theta \int_0^{2\pi} \, \mathrm{d}\phi \\ &= \frac{3M}{4\pi a^3} \times \frac{a^5}{5} \times \frac{4}{3} \times 2\pi \\ &= \frac{2ma^2}{5} \end{aligned}$$

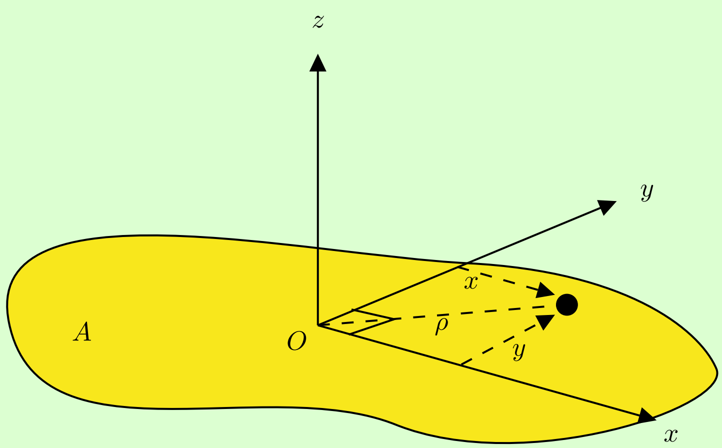
Method. The following are simple general results that are useful when calculating moments of inertia:

Theorem (Perpendicuar Axes Theorem)**.** For a 2-dimensional body (lamina) then

$$I_z = I_x + I_y$$

Where I_z moment of inertia about z axis $\perp$ lamina
$I_x,\ I_y$ are moments of inertia about respectively the x, y axes which lie in the plane

Proof.



Let A be a lamina as shown
Then

$$I_x = \int_A \rho y^2 \,\mathrm{d}A$$

$$I_y = \int_A \rho x^2 \,\mathrm{d}A$$

$$I_z = \int_a \rho r^2 \,\mathrm{d}A = \int_A \rho(x^2 + y^2) \,\mathrm{d}A = I_x + I_y$$

as required

Note. Sometimes we have a symmetry such that $I_x = I_y$. In this case,

$$I_z = 2I_x = 2I_y$$

Warning. Applies only to 2-D object

Example. Symmetric case: disc

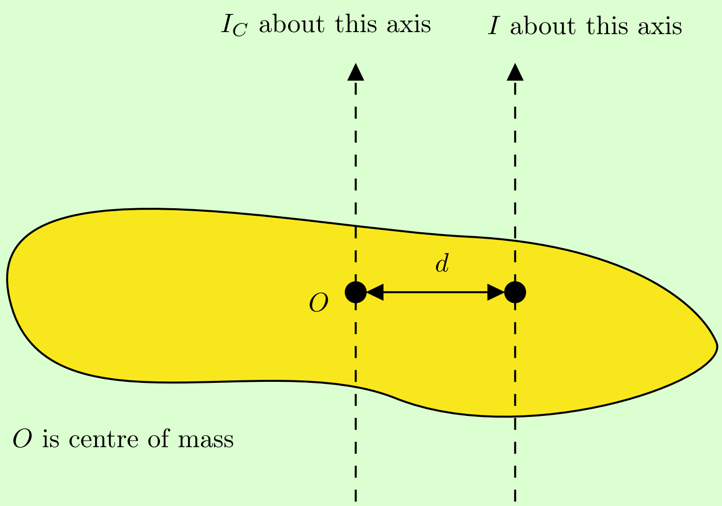
$$I_z = \frac{1}{2}Ma^2 = 2 \cdot \left(\frac{1}{4}Ma^2\right) = 2I_x = 2I_y$$

I_z has axis of rotation $\perp$ plane of disc. $I_x,\ I_y$ have axis of rotation in plane of disc

Theorem (Parallel Axes Theorem)**.** If a rigid body of mass M has moment of inertia I_c about axis through the centre of mass, then the moment of inertia about a parallel axis a distance d from the axis through the centre of mass is

$$I = I_C + Md^2$$

Proof.



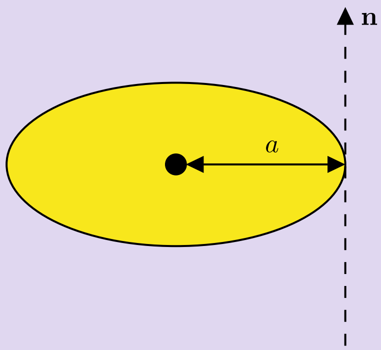
Choose Cartesian axes with origin O at the centre of mass and z direction $\parallel$ axus if rotation. Let second axis pas through the point $(d, 0, 0)$. Denote the volume of the body by V.
Then

$$I_c = \int_V \rho(x^2 + y^2)\,\mathrm{d}V$$

$$\begin{aligned} I &= \int \rho((x-d)^2) + y^2)\,\mathrm{d}V \\ &= \int \rho(x^2 + y^2)\,\mathrm{d}V - 2\underbrace{\int \rho x d\,\mathrm{d}V}_{=0} + d^2 \int \rho\,\mathrm{d}V \\ &= I_c + Md^2 \end{aligned}$$

We have second integral 0 as origin id center of mass

Example. Uniform thin disc, mass M, radius a.
Want to find moment of inertia about axis $\perp$ plane of disc, through a point on circumference:



Exploit previous theorem:

$$I = I_c + Ma^2$$

(a^2 is d^2 with d the distance between axes)

$$I = \frac{3}{2}Ma^2$$

7.4 Motion of a Rigid Body

Remark. General motion of a rigid body can be described by translation of the centre of mass, following a trajectory $\mathbf{R}(t)$, together with a rotation about an axis through the centre of mass. Following section 6.2, we specify the points fixed in the body relative to the mass by writing:

$$\mathbf{r}_i = \mathbf{R} + \mathbf{s}_i$$

$$\dot{\mathbf{r}}_i = \dot{\mathbf{R}} + \dot{\mathbf{s}}_i$$

and noting:

$$\sum_{i=1}^{N} m_i \mathbf{r}_i = M\mathbf{R} \implies \sum_{i=1}^{N} m_i \mathbf{s}_i = \mathbf{0}$$

Method. If body is rotating about centre of mass with angular velocity $\boldsymbol{\omega}$, then

$$\dot{\mathbf{s}}_i = \boldsymbol{\omega} \times \mathbf{s}_i$$

and

$$\dot{\mathbf{r}}_i = \dot{\mathbf{R}} + \boldsymbol{\omega} \times \mathbf{s}_i$$

Then kinetic energy (recalling section 6.2) is

$$\begin{aligned} T &= \frac{1}{2} M \dot{\mathbf{R}}^2 + \frac{1}{2} \sum_{i=1}^{N} m_i \dot{\mathbf{s}}_i^2 \\ &= \frac{1}{2} M \dot{\mathbf{R}}^2 + \frac{1}{2} I_C \omega^2 \end{aligned}$$

where I_C is the moment about axis $\parallel \boldsymbol{\omega}$ through the centre of mass

$$T = \text{translation KE} \ + \ \text{rotational KE}$$

Shown in section 6.1, for general particle system, we have linear momentum and angular momentum satisfy:

$$\dot{\mathbf{P}} = \mathbf{F}$$

($\mathbf{F}$ is total external force) and

$$\dot{\mathbf{L}} = \mathbf{G}$$

($\mathbf{G}$ is total external torque)
i.e. for a rigid body there are two equations that determine the translational and rotational motion

Note. Can sometimes determine motion by exploiting conservation of energy

Method. $\mathbf{L}$ and $\mathbf{G}$ depend on choice of origin: can take origin to be any point fixed in an inertial frame (considered previously in section 6.1) or we can define $\mathbf{L}$ and $\mathbf{G}$ with respect to the center of mass – the equation relating them still holds.
Demonstration of this result:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(M\mathbf{R}\times\dot{\mathbf{R}}+\sum_{i=1}^{N}m_i\mathbf{s}_i\times\dot{\mathbf{s}}_i\right) &= \mathbf{G}\\ &= M\dot{\mathbf{R}}\times\dot{\mathbf{R}}+m\mathbf{R}\times\ddot{\mathbf{R}}+\frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{n}m_i\mathbf{s}_i\times\dot{\mathbf{s}}_i\right)\\ &= \mathbf{R}\times\mathbf{F}^{\mathrm{ext}}+\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{i=1}^{N}m_i\mathbf{s}_i\times\dot{\mathbf{s}}_i\right)\end{aligned}$$

($\mathbf{G}$ is external torque about $\mathbf{0}$)
Hence

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{i=1}^{N}m_i\mathbf{s}_i\times\dot{\mathbf{s}}_i\right) &= \mathbf{G}-\mathbf{R}\times\mathbf{F}^{\mathrm{ext}}\\ &= \sum_{i=1}^{N}\mathbf{r}_i\times\mathbf{F}_i^{\mathrm{ext}}-\mathbf{R}\times\mathbf{F}^{\mathrm{ext}}\\ &= \sum_{i=1}^{N}(\mathbf{r}_i-\mathbf{R})\times\mathbf{F}_i^{\mathrm{ext}}\\ &= \text{ external torque about center of mass}\end{aligned}$$

as claimed

Example. Consider motion in a uniform gravitational field, acceleration $\mathbf{g}$, which is constant. The total gravitational force and toque acting on a rigid body are the same as those that would act on a particle of mass M located at the centre of mass
(often referred to as the centre of gravity)
To confirm:

$$\mathbf{F} = \sum_{i=1}^{N} \mathbf{F}_i^{\text{ext}} = \sum_{i=1}^{N} m_i \mathbf{g} = M\mathbf{g}$$

$$\begin{aligned}\mathbf{G} &= \sum_{i=1}^{N} \mathbf{G}_i^{\text{ext}} \\ &= \sum_{i=1}^{N} \mathbf{r}_i \times m_i \mathbf{g} \\ &= \left(\sum_{i=1}^{N} m_i \mathbf{r}_i\right) \times \mathbf{g} \\ &= M\mathbf{R} \times \mathbf{g}\end{aligned}$$

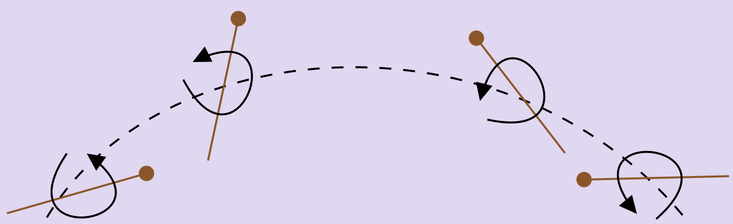
Note. The gravitational toque about the centre of mass is zero

$$\sum_{i=1}^{N} \mathbf{s}_i \times m_i \mathbf{g} = \left(\sum_{i=1}^{N} m_i \mathbf{s}_i\right) \times \mathbf{g} = \mathbf{0}$$

Consider the gravitational potential

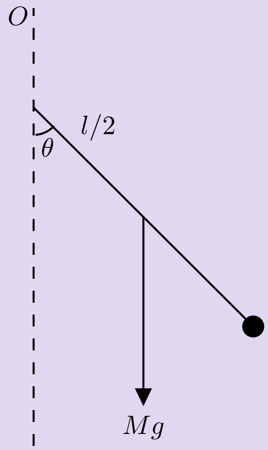
$$V^{\text{ext}} = \sum_{i=1}^{N} V_i^{\text{ext}} = -\sum_{i=1}^{N} m_i \mathbf{r}_i \cdot \mathbf{g} = -M\mathbf{R} \cdot \mathbf{g}$$

Example. Stick thrown into air



The centre of mass follows a parabola.
Angular velocity of stick about centre of mass is constant because gravitational toque about the centre of mass is zero.

Example. Rod of length l and mass M, fixed at a pivot point O. Density of rod is uniform.



Consider angular momentum about pivot point O.

$$\omega = \dot{\theta}$$

$$L = I\dot{\theta} = \frac{1}{3}Ml^2\dot{\theta}$$

(using result from previous example)
Gravitational torque about O:

$$G = -Mg \times \frac{l}{2}\sin\theta$$

Note. torque associated with force at pivot is **0**

$$\dot{L} = G \implies I\ddot{\theta} = -Mg\frac{l}{2}\sin\theta$$
$$\implies \ddot{\theta} = -\frac{3g}{2l}\sin\theta$$

equivalent to a simple pendulum of length $\frac{2l}{3}$, angular frequency of small oscillations $\sqrt{\frac{3g}{2l}}$ and period of small oscillations $2\pi\sqrt{\frac{2l}{3g}}$
Alternative approach using energy:

$$T + V = \frac{1}{2}I\dot{\theta}^2 - \frac{Mgl}{2}\cos\theta = E$$

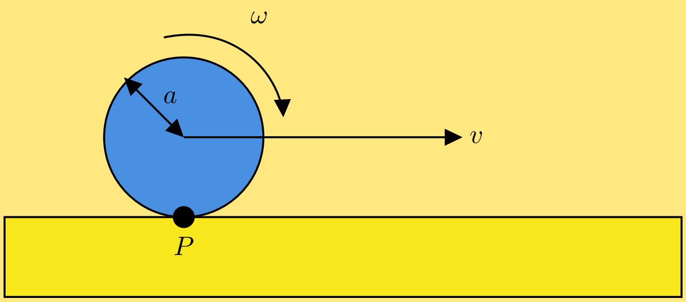
E constant:

$$\begin{aligned}\frac{\mathrm{d}E}{\mathrm{d}t} &= I\dot{\theta}\ddot{\theta} + \frac{Mgl\dot{\theta}\sin\theta}{2} \\ &= 0\end{aligned}$$

due to equation of motion derived previously

7.5 Sliding vs Rolling

Method. Consider cylinder or sphere, radius a, moving along a stationary horizontal surface.



General motion is combination of rotation about centre of mass with angular velocity ω and translation of center of mass with velocity v.
P is instantaneous point of contact between body and surface.
The horizontal velocity of point on sphere

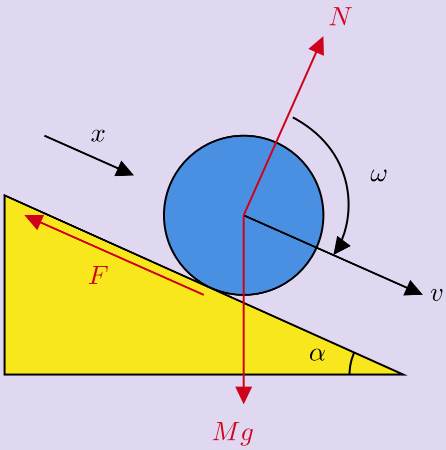
$$v_{\text{slip}} = v - a\omega$$

Point of contact P slips and there may be a kinetic frictional force.

- A pure sliding motion is $\omega = 0$ and
$$v_{\text{slip}} = v \neq 0$$
- A pure rolling motion $v \neq 0, \ \omega \neq 0$ and
$$v_{\text{slip}} = v - a\omega = 0$$
no slip condition. The point of contact P is stationary.

A rolling body can be described instantaneously as rotating about the point of contact with angular velocity ω

Example. Rolling downhill



Consider a cylinder or sphere of radius a, mass m, rolling down an inclined plane at angle α to the horizontal.
x is distance down slope travelled by the centre of mass, $v = \dot{x}$.
Rolling or no slip condition

$$v - a\omega = 0$$

Analyse using energy:
Kinetic energy

$$\begin{aligned} T &= \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2 \\ &= \frac{1}{2}\left(M + \frac{I}{a^2}\right)v^2 \end{aligned}$$

Normal and frictional forces do not work since they act at the point P and $v_{\text{slip}} = 0$
Hence, energy is conserved, with potential energy

$$V = -Mgx\sin\alpha$$

$$T + V = \frac{1}{2}\left(M + \frac{I}{a^2}\right)v^2 - Mgx\sin\alpha$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(T + V) = \left(M + \frac{I}{a^2}\right)\dot{x}\ddot{x} - Mg\dot{x}\sin\alpha = 0$$

Hence

$$\left(\frac{I}{a^2}\right)\ddot{x} = Mg\sin\alpha$$

rotational contribution implies that acceleration is smaller than it would be for a friction-less particle.
e.g. for cylinder

$$I = \frac{1}{2}Ma^2$$

$$\ddot{x} = \frac{2}{3}g\sin\alpha$$

Example (cont.)**.** Alternatively, analyse using forces and torques:
Rate of change of linear momentum along plane:

$$M\dot{v} = Mg\sin\alpha - F$$

Rate of change of angular momentum about centre of mass:

$$I\dot{\omega} = aF$$

This is the torque due to F.

$$\text{Rolling condition} \implies I\dot{v} = a\dot{\omega} \implies \frac{I\dot{v}}{a} = aF$$

hence

$$M\dot{v} = Mg\sin\alpha - \frac{I\dot{v}}{a^2} \implies \left(M + \frac{I}{a^2}\right)\dot{v} = Mg\sin\alpha$$

i.e. same conclusion as using energy.
Another alternative: consider toque about P

$$I_P = IMa^2$$

($\|$ axes theorem)

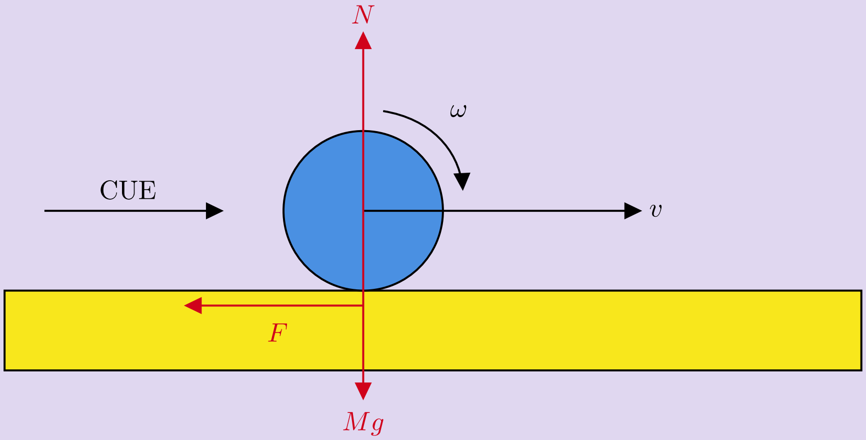
$$I_P\dot{\omega} = Mga\sin\alpha \; v = a\omega$$

hence

$$\left(Ma^2 + I\right)\frac{\dot{v}}{a} = Mga\sin\alpha$$

(dividing by a on both sides shows we have yielded the same equation)

Example. Sliding to rolling transition



Snooker ball is hit centrally by the cue:

$$v = v_0,\ \omega = 0 \text{ at } t = 0$$

(note no torque applied to ball in initial impact)
Kinetic frictional force:

$$F = \mu_k N = \mu_k M g$$

with μ_k = coefficient of kinetic friction
Linear motion:

$$M\dot{v} = -F$$

angular motion

$$I\dot{\omega} = aF$$

recall that

$$I = \frac{2}{5}Ma^2$$

for sphere.
Hence:

$$v = v_0 - \mu_k g t$$

$$\omega = \frac{5}{2a}\mu_k g t$$

matching the initial conditions
hence:

$$v_{\text{slip}} = v - a\omega = v_0 - \frac{7}{2}\mu_k g t$$

slipping when

$$v_{\text{slip}} > 0,\ \text{for } 0 \le t < \frac{2v_0}{7\mu_k g}$$

rolling begins when $t = \frac{2v_0}{7\mu_k g} = t_{\text{roll}}$

Example (cont.). Note that at $t = t_{\text{roll}}$,

$$\begin{aligned} T &= \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2 \\ &= \frac{1}{2}M\left(1 + \frac{2}{5}\right)v_{\text{roll}}^2 \\ &= \frac{5}{7}\left(\frac{1}{2}Mv_0^2\right) \end{aligned}$$

Check loss of KE due to friction:

$$\begin{aligned} \int_0^{t_{\text{roll}}} Fv_{\text{slip}}\,\mathrm{d}t &= \int_0^{t_{\text{roll}}} F(v_0 - \frac{7}{2}\mu_k gt)\,\mathrm{d}t \\ &= \frac{1}{7}Mv_0^2 \\ &= \frac{2}{7}\left(\frac{1}{2}Mv_0^2\right) \end{aligned}$$

which is in agreement with our previous result

8 Special Relativity

For very large velocities, the Newtonian theory of dynamics is not applicable and must be refined. The necessary refinement is Einstein's Special Theory of Relativity (1905).
The differences between the two theories are significant only when velocities are comparable with the speed of light c.

$$c = 299,792,458 \text{ ms}^{-1} \simeq 3 \times 10^8 \text{ ms}^{-1}$$

8.1 Postulates of Special Relativity

Note. Special Relativity (SR) is based on two postulates.
Postulate 1: The laws of physics are the same in all inertial frames.
Postulate 2: The speed of light in a vacuum in the same in all inertial frames.

The need for postulate 2 arises from many experiments that fail to detect any dependence of the speed of light on the relative velocities of inertial frames.
Addition of postulate 2 requires a radical revision of our understanding of space and time as well as the relationships between energy, momentum and mass.
Illustration: consider S and S' – inertial frames – related by Galilean transformation

$$x' = x - vt, \; y' = y, \; z' = z, \; t' = t$$

Path of light ray in S:

$$x = ct$$

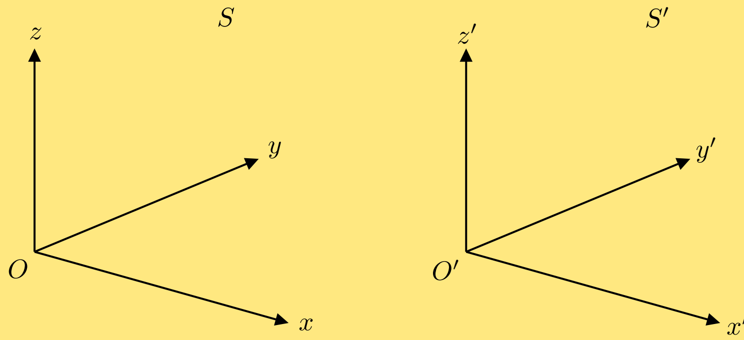
In S':

$$x' = x - vt = ct - vt = (c - v)t'$$

Speed of light in S' is $c - v$, contradicting postulate 2.
Need to construct a new sort of transformation for which postulate 2 is satisfied.
We have to treat space and time on equal footing.

8.2 The Lorentz Transformation

Method. Consider S and S', both inertial, such that origins of each coincide at $t = t' = 0$.



S' is moving at sped v in x direction relative to S
Assume for the present $y' = y$ and $z' = z$.
Consider the relation between (x, t) and (x', t).
Postulate 1 implies that constant velocty paths in S are constant velocity paths in S'.
Constant velocity paths are straight lines in (x, t) and (x', t').
Therefore the transformation from (x, t) to (x', t') must be linear.
O' moves with speed v in S, hence

$$x' = \gamma(x - vt) \text{ for } \gamma = \gamma(|v|)$$

($|v|$ because no preferred direction).
O moves with speed $-v$ in S', hence

$$x = \gamma(|v|)(x' + ct')$$

Consider light ray passing through O and O' at $t = t' = 0$
Equation for light ray:

$$x = ct \text{ (in } S)$$

$$x' = ct' \text{ (in } S')$$

Follows from postulate 2
Hence

$$\begin{aligned} x = ct &= \gamma(x' + vt') \\ &= \gamma(c + v)t' \end{aligned}$$

$$x' = ct' = \gamma(x - vt) = \gamma(x - v)t$$

For consistency:

$$\gamma^2 \left(1 - \frac{v}{c}\right)\left(1 + \frac{v}{c}\right) = 1$$

hence

$$\gamma(v) = \left(1 - \frac{v^2}{c^2}\right)^{-1/2}$$

γ is the Lorentz factor.
We have

$$x = \gamma(x' + vt') \tag{1}$$

$$x' = \gamma(x - vt) \tag{2}$$

Equation. From (1) we have:

$$\begin{aligned} vt' &= \frac{x}{\gamma} - x' \\ &= \frac{x}{\gamma} - \gamma(x - vt) \\ &= \gamma \underbrace{\left(\frac{1}{\gamma^2} - 1\right)}_{v^2/c^2} x + \gamma vt \\ &= \gamma \left(vt - \frac{v^2}{c^2} x\right) \end{aligned}$$

Hence:

$$t' = \gamma \left(t - \frac{vx}{c^2}\right) \tag{3}$$

(2) and (3) define the Lorentz transformation (or 'Lorentz Boost')
Straightforward to show from (2) and (3) that:

$$x = \gamma(x' + vt')$$

$$t = \gamma \left(t' + \frac{vx}{c^2}\right)$$

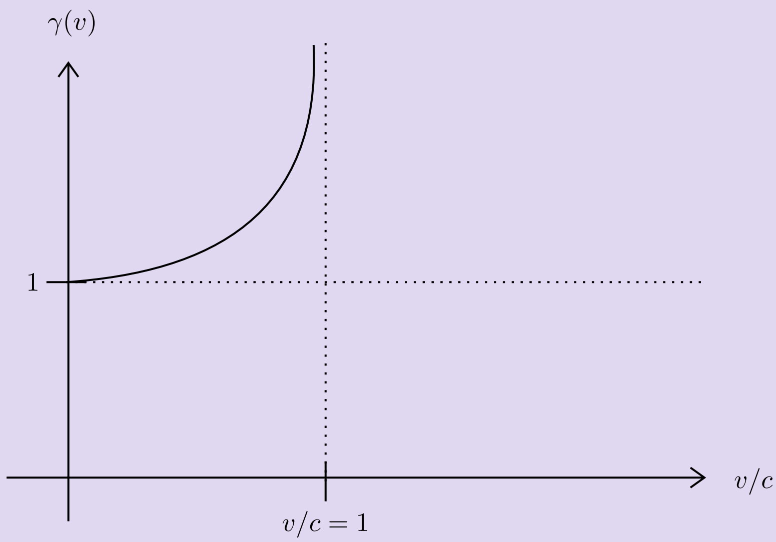
(note that the sign of v has simply been swapped)
Note that y, z do not change if the velocity of S' is in the x direction relative to S.
Consider the Lorentz factor $\gamma(v)$. $\gamma(v) \geq 1$ and $\gamma \to \infty$ as $|v| \to c$.
We recover the Galilean limit for small v/c

Example.

$$\gamma = 2 \text{ has } \frac{v}{c} = 0.866 = \frac{\sqrt{3}}{2}$$

$$\gamma = 5 \text{ has } \frac{v}{c} = 0.98$$

$$\gamma = 10 \text{ has } \frac{v}{c} = 0.995$$



$$\gamma \simeq \frac{1}{\sqrt{2}} \cdot \frac{1}{(1 - v/c)^{1/2}}$$

Check speed of light.

(i) Light ray travelling in x direction.

In S:

$$x = ct,\ y = 0,\ z = 0$$

In S':

$$x' = \gamma(x - vt) = \gamma(c - v)t$$
$$t' = \gamma(t - \frac{vx}{c^2}) = \gamma(1 - \frac{v}{c}t)$$
$$y' = 0,\ z' = 0\ \frac{x'}{t'} = \frac{\gamma(c-v)}{\gamma(1 - v/c)} = c$$

as required

(ii) Light ray travelling in y direction in S In S:

$$y = ct,\ x = 0,\ z = 0$$

In S':

$$x' = \gamma(x - vt),\ t' = \gamma(t - vx/c^2)$$
$$y' = y,\ z' = z$$

Hence:

$$x' = -\gamma vt,\ t' = \gamma t,\ y' = ct,\ z = 0$$

Consider speed

$$\begin{aligned}(\text{speed})^2 &= (x\text{-component})^2 + (y\text{-component})^2 \\ &= v^2 + \frac{c^2}{\gamma^2} \\ &= v^2 + c^2(1 - v^2/c^2) \\ &= c^2\end{aligned}$$

Note. a general property of Lorentz transformations is:

$$\begin{aligned}c^2t'^2 - r'^2 &= c^2t'^2 - (x'^2 + y'^2 + z'^2) \\ &= c^2\gamma^2\left(t - \frac{vx}{c^2}\right)^2 - (x - vt)^2\gamma^2 - y^2 - z^2 \\ &= c^2\gamma^2(t^2 - \frac{2vxt}{c^2} + \frac{v^2x^2}{c^2}) - \gamma^2(x^2 - 2vxt + v^2t^2) - y^2 - z^2 \\ &= c^2t^2 - x^2 - y^2 - z^2 = c^2t^2 - r^2\end{aligned}$$

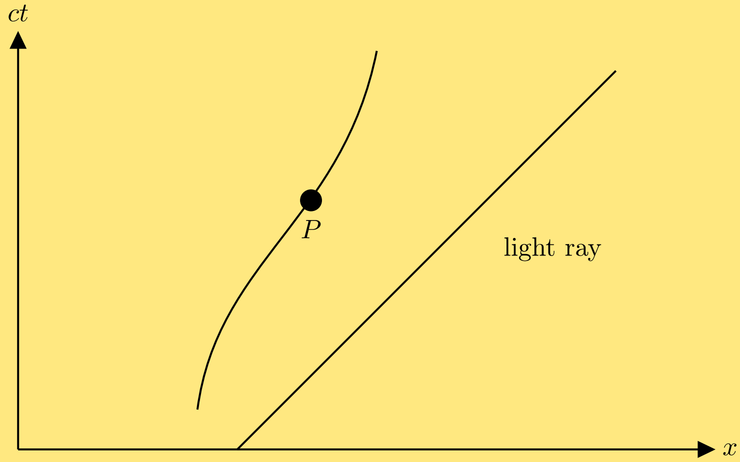
So

$$r' = ct \iff r = ct$$

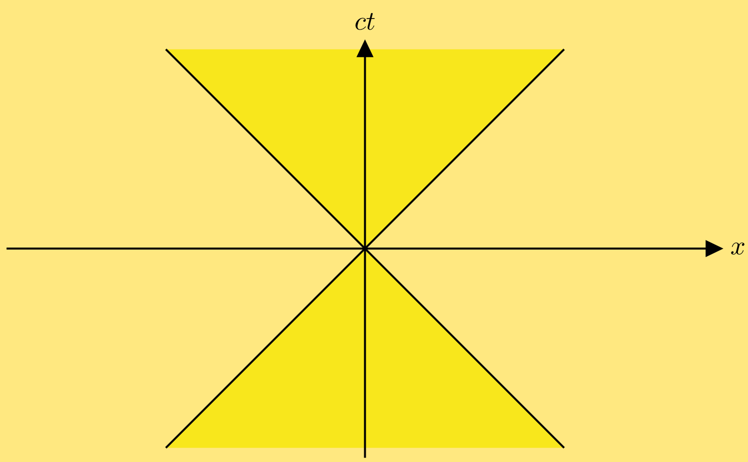
Note quantity is invariant under Lorentz transformations

8.3 Space-time Diagrams

Method. Consider 1 space dimension (x) and time (t) in inertial frame S.
Plot x on horizontal axis and ct on vertical axis



This convention for containing x and t is called Minkowski Space-time
Each point P in space time represents an event labelled by coordinates (x, ct).
Moving particle traces out a curve in (x, ct) space - world line - which would be a straight line if the particle was moving with uniform velocity.
Light rays in x direction would have world lines at 45° to axes.
Particles cannot travel with speed $v > c$, therefore world lines of particles are restricted to certain regions of the space time plane



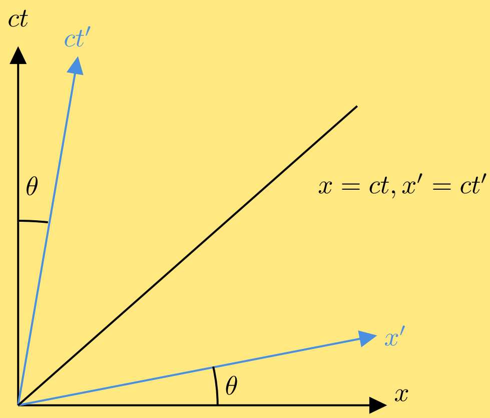
Allowed regions for particle at $x = 0$ at $t = 0$

Method. We can also draw axes of frame S' which is moving at speed v relative to S.
t' axis: corresponds to $x' = 0$ hence to $x = vt$ or

$$x = \frac{v}{c} \cdot ct$$

x'acis: corresponds to $t' = 0$ hence

$$ct = \frac{vx}{c}$$



x', xt' axes are symmetric about diagonal ($x = ct$), so

$$x = ct \iff x' = ct'$$

velocity of light is the same in S and S'

Equation. Comparison of velocities.
Consider a particle moving with constant velocity u' in S', with S' is travelling at velocity v, relative to S.
What is the velocity u measured in S?
World line of particle in S':

$$x' = u't'$$

World line of particle in S:

$$x' = ut$$

From Lorentz transformation:

$$\begin{aligned} x &= \gamma(x' + vt;) \\ &= \gamma(u' + v)t' \end{aligned}$$

$$t = \gamma(t' + \frac{vx'}{c}) = \gamma(1 + \frac{vu'}{c^2})t'$$

Hence

$$u = \frac{u' + v}{1 + \frac{u'v}{c^2}}$$

Formula for composition of velocities
(inverse by swapping sign of v)
In the limit $u', v \ll c$, we recover Galilean transformation result

Note.

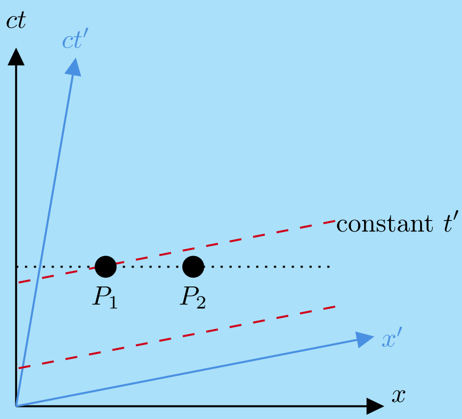
$$\begin{aligned} c - u &= c - \frac{u' + v}{1 + \frac{u'v}{c^2}} \\ &= \frac{(c-u')(c-v)}{1+\frac{u'v}{c^2}} \\ &> 0 \end{aligned}$$

If $u' < c$ and $v' < c$, we cannot exceed speed of light through successive boost transformations.

8.4 Simultaneity, Causality and Consequences

8.4.1 Simultaneity

Definition. Two events P_1 and P_2 are **simultaneous** in S if $t_1 = t_2$



Remark. Events that are simultaneous in S' have the same value of t' i.e. they lie on lines $t - vx/c$ constant.
event P_2 occurs before P_1 in 'S'.

Note.

$$P_1(x_1, ct), P_2(x_2, ct) \implies t_1' = \gamma\left(t - \frac{vx_1}{c^2}\right), t_2' = \gamma\left(t - \frac{vx_2}{c^2}\right)$$

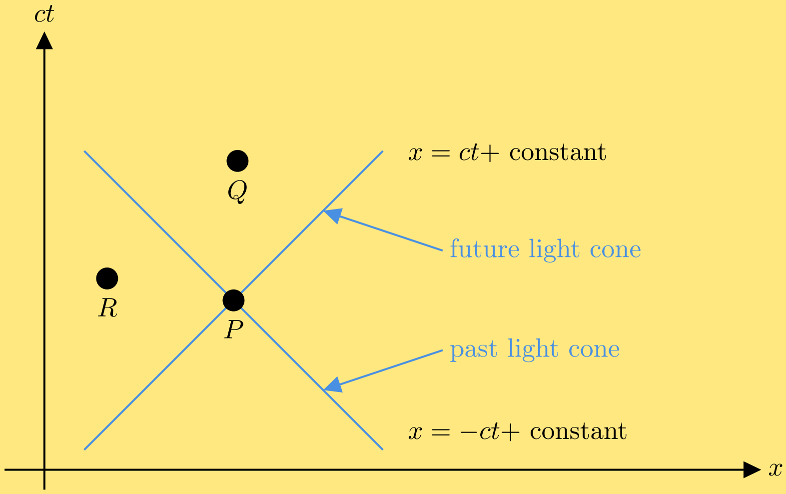
Hence $t_2' < t_1'$ is $x_2 > x_1$

Warning. Simultaneity IS frame dependent

8.4.2 Causality

Different observers may disagree on the time ordering of events, but we can construct a viewpoint where cause and effect are consistent.
Lines of simultaneity cannot be inclined at more than 45° to x axis (because $|v| < c$)

Method. Lines/ surfaces emerging from an event P at angle 45° to x axis form a light cone (a past light cone and a future light cone).



All observers agree that Q occurs after P, but ordering of P and R in time is frame dependent.
If nothing travels faster than c then P and R cannot influence each other.
P can ONLY influence events inside its future light cone.
P can ONLY be influenced by events inside its past light cone.

8.4.3 Time Dilation

Method. A clock that is stationary in S' ticks at time intervals $\delta t'$. What is the time interval between ticks in S?
Lorentz transformation:

$$t = \gamma(t' + \frac{vx'}{c^2})$$

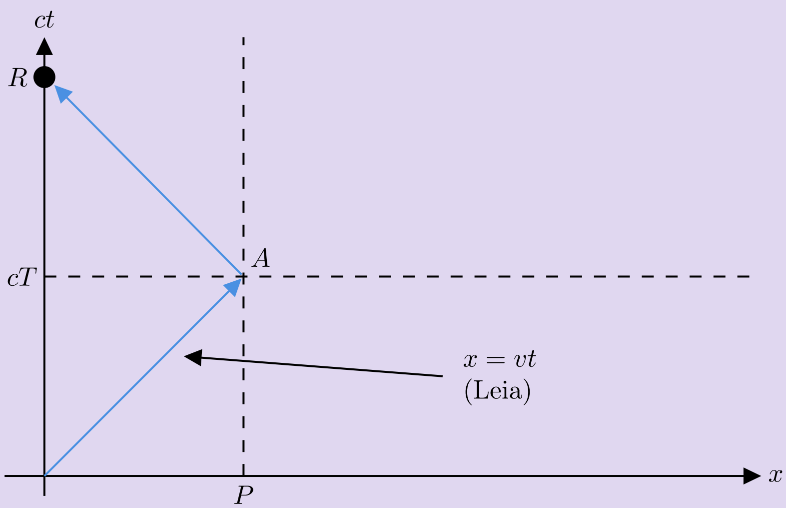
hence

$$\delta t = \gamma \delta t'$$

moving clocks run slowly.

Definition. **Proper time** is time measured in the rest frame of a moving object

Example (Time Dilation - the Twin Paradox)**.** Consider two twins Luke and Leia.
Luke stays at home.
Leia travels at a constant speed v to a distant planet P. Turns around and returns home.
In Luke's frame of reference:



In Luke's frame of reference Leia's arrived at P - event A has

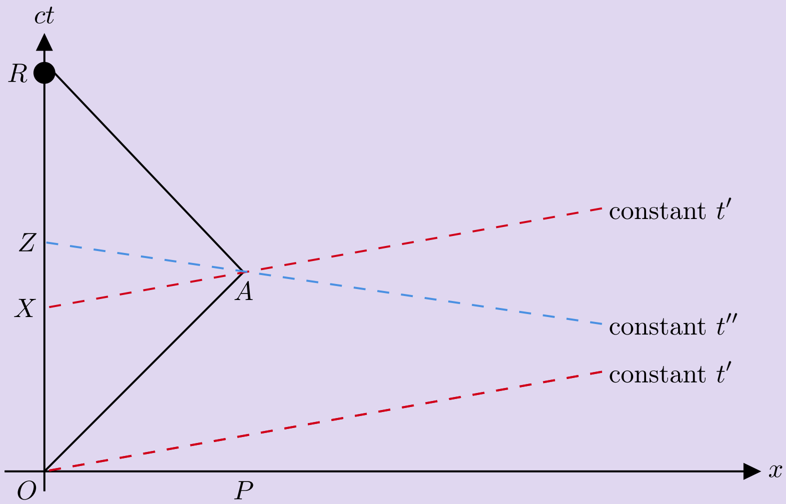
$$(x, ct) = (vT, cT)$$

Time experience by Leia on her outward journey:

$$T' = \gamma(T - \frac{v}{c} \cdot vT) = \frac{T}{\gamma}$$

On Leia's return (R) Luke has aged by $2T$, Leia has aged by $\frac{2T}{\gamma} < 2T$, so Leia is younger than Luke when she returns - by time dilation.
From Leia's perspective, Luke has travelled away at speed v and then returned at speed v, hence he should be younger. Paradox.

Example (cont.)**.** Why is the problem not symmetric?



In the frame of reference of Leia's outward journey:

$$A : x' = 0,\ t' = \frac{T}{\gamma}$$

$$X : x = 0,\ t' = \frac{T}{\gamma}$$

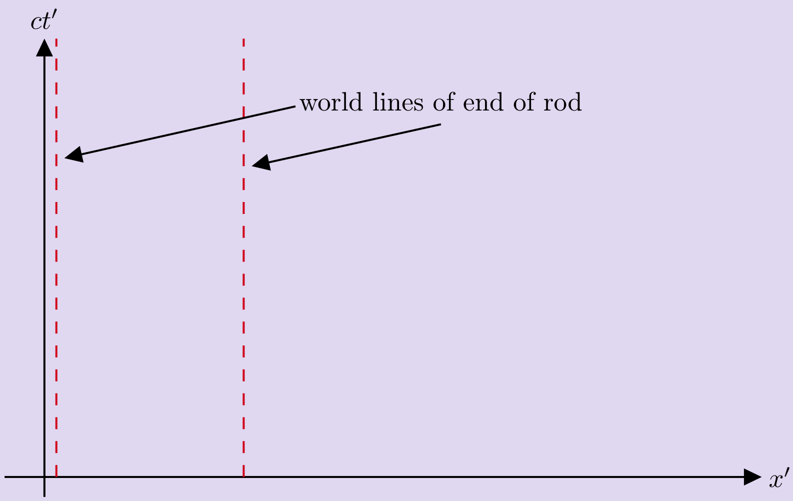
$$t' = \gamma\left(t - \frac{vx}{c^2}\right)$$

$$\implies t' = t\gamma \implies t = \frac{t'}{\gamma} = \frac{T}{\gamma^2}$$

Each thinks that the other has aged less, by factor $1/\gamma$.
Return journey: Luke sees Leia aging from $A \to R$. Leia sees Luke aging from $Z \to R$.
(Z is simultaneous with A in frame moving with Leia on her return journey, i.e. moving speed $-v$ relative to Luke's frame)
Leia's change of direction implies that Luke has aged instantaneously from X to Z.
(Lea's frame is not inertial as she changes direction.)

8.4.4 Length Contraction

Example. A rod of length L' is stationary in S'.
What is the length of the rod in S?



Length is distance between the two end at inthe same instant in time

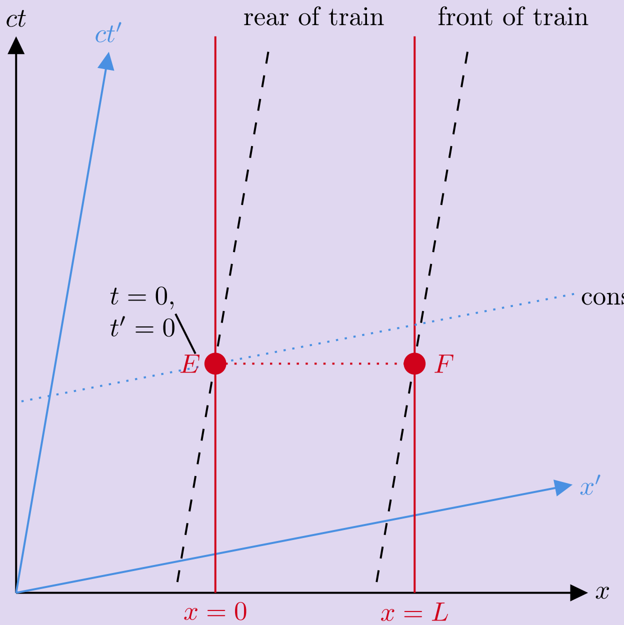
$$x' = 0 \implies \gamma(x - vt) = 0$$

$$x' = L \implies \gamma(x - vt) = L'$$

Hence L (length of rod in S') $= \frac{L'}{\gamma} < L'$
Moving objects appear to be contracted in length in the direction in which they are travelling

Definition. **Proper length** is the length measured in the rest frame of the object

Example. Does a train of length $2L$ fit alongside a platform of length L if it is travelling at a speed v such that $\gamma = 2$?
For observers on the platform, the train contracts to length L – it fits! For observers on the train, the platform contracts from L to $L/2$ - it doesn't



S: platform is stationary
S' : train stationary
Train is defined $x' = 0$ (rear) and $x' = 2L$ (front)
E is event when rear of train is coincident with rear of platform - assume this occurs at $t = t' = 0$
Front of train is $x' = 2L$, and front of platform is $x = L$.
E and F are simultaneous in S:

$$x' = \gamma(x - vt),\ 2L = \gamma(L - vt) \implies t = 0\ (\gamma = 2)$$

$$x = \gamma(x' + vt'),\ L = \gamma(2L + vt')$$

$$\implies t' = \frac{L - 2\gamma L}{v} = \frac{-3L}{V} < 0$$

i.e. rear of train coincident with rear of platform after front of train coincident with front of platform
i.e. doesn't fit

8.5 Geometry of Space-time

8.5.1 Invarianvce Interval

Equation. Consider two events $P,\ Q$ with space time co-ordinates $(ct_1, x_1),\ (ct_2, x_2)$
Time separation

$$\Delta t = t_1 - t_2$$

Space separation

$$\Delta x = x_t - x_2$$

Invariance interval between P and Q is defined as

$$(\Delta s)^2 = c^2 \Delta t^t = \Delta x^2$$

All observers in inertial frames agree on the value of $(\Delta s)^2$.
(Exercise: show above from Lorentz transformation)
In 3 space dimensions, define the invariant interval as

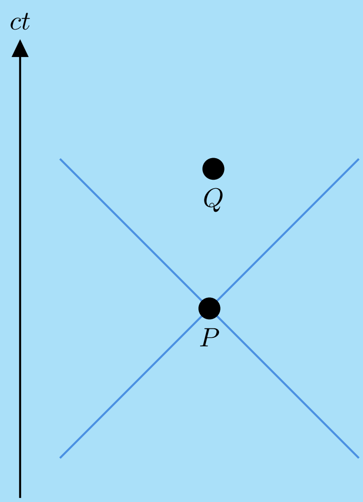
$$\Delta s^2 = c^2 \Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2$$

If the separation between P and Q is very small, then we have an expression for the infinitesival invariant interval.

$$\mathrm{d}s^2 = c^2\,\mathrm{d}t^2 - (\,\mathrm{d}x^2 + \,\mathrm{d}y^2 + \,\mathrm{d}z^2)$$

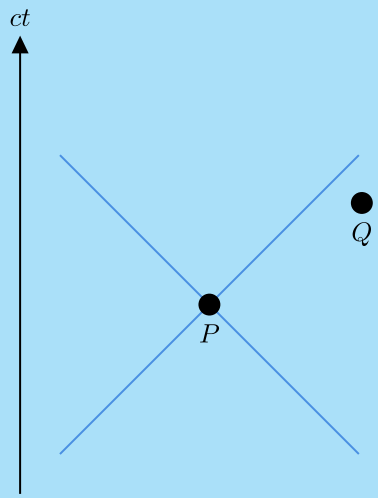
Space-time is topologically equivalent to $\mathbb{R}^4$, when endowed with a distance measure $\mathrm{d}s^2$.
(though $\mathrm{d}s^2$ is not positive definite.)
This is Minkowski space time - sometimes described as a space of $1+3$ dimensions.

Definition. Events with $\mathrm{d}s^2 > 0$ are **time-like separated**
There is a frame of reference such that the two events occur at the same space position (but at different times).



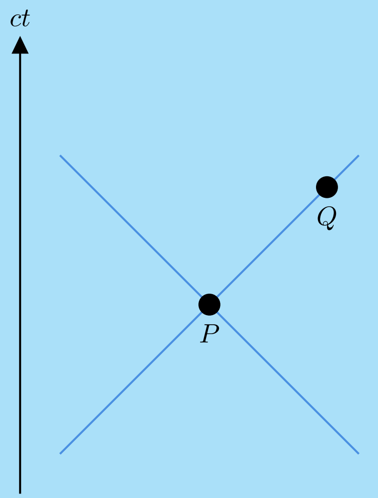
The event Q lies in either the forward or the backward light cone of P.
(The time-ordering of P and Q is agreed by all observers)

Definition. Events with $\mathrm{d}s^2 < 0$ are **space-like separated**
We can find a frame of reference such that they occur at the same time



Q lies outside the forward and backward light cone of P. (Which of P or Q occurs at earlier time is observer dependent.)

Definition. If $\mathrm{d}s^2 = 0$ then the events are **light-like separated** and one lies on the light cone of the other.



(on light cone either in forward or backward direction)

Warning. $\mathrm{d}s^2 = 0$ does NOT imply that P and Q are the same event.
(Our metric does not have the 'usual' metric properties.)

8.5.2 The Lorentz Group

Method. The co-ordinates of an event P in frame S can be written as a 4-vector (i.e. a 1 component vector) - X.

$$X = \begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix}$$

we use superscripts to denote index (write X^μ

$$X^1 = ct,\ X^2 = x,\ X^3 = y,\ X^3 = z$$

Equation. INvariance interval between P and origin O can be written as an inner product

$$X \cdot X = X^T \eta X = X^\mu \eta_{\mu\nu} X^\nu$$

where

$$\eta = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

the Minkowski metric.
We have

$$X \cdot X = c^2t^2 - x^2 - y^2 - z^2$$

Note. 4 vectors with $X \cdot X > 0$ are time like 4 vectors with $X \cdot X < 0$ are space like 4 vectors with $X \cdot X = 0$ are light like

Remark. The Lorentz transformation is a linear transformation that takes the components X in S to be components X' in S'

Method. The Lorentz transformation can be represented by a 4×4 matrix Λ.

$$X' = \Lambda X$$

X are the components in S
X' are the components in S'
Lorentz transformations can be defined as the set of Λ that leave the inner product $X \cdot X$ unchanged, i.e.

$$X \cdot X = X' \cdot X' \text{ for all } \Lambda$$

implying

$$\Lambda^T \eta \Lambda = \eta \qquad (*)$$

Example. Two classes of possible Λ are

$$\Lambda = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & R & \\ 0 & & & \end{bmatrix}$$

with

$$R^T R = I$$

i.e. R is spacial reflection/ rotation. And:

$$\Lambda = \begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

with

$$\beta = \frac{v}{c}$$

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2}$$

Lorentz boost, with velocity in x direction (in this example).

Definition. The set of all Λ satisfying (*) is called the **Lorentz group** $O(1,3)$. It includes time reversals and spatial reflections.
The subgroup with $\det \Lambda = 1$ is the **proper Lorentz group** $SO(1,3)$, whcih includes composition of time reversal and spacial reflection.
The subgroup that preserves the direction of time and spatial circulation is the **restricted Lorentz group** $SO^+(1,3)$ - generated by rotations and boosts (in arbitrary directions).

8.5.3 Rapidity

This is a way of labelling Lorentz transformations

Method. Focus on 2×2 submatrix operations on (ct, x).
Write:

$$\Lambda[\beta] = \begin{bmatrix} \gamma & -\gamma\beta \\ -\gamma\beta & \gamma \end{bmatrix} \; \gamma = (1-\beta^2)^{-1/2}$$

(boost in x-direction).
Consider combining two boosts:

$$\begin{aligned} \Lambda[\beta_1]\Lambda[\beta_2] &= \begin{bmatrix} \gamma_1 & -\gamma_1\beta_1 \\ -\gamma_1\beta_1 & \gamma_1 \end{bmatrix} \begin{bmatrix} \gamma_2 & -\gamma_2\beta_2 \\ -\gamma_2\beta_2 & \gamma_2 \end{bmatrix} \\ &= \begin{bmatrix} \gamma_1\gamma_2(1+\beta_1\beta_2) & -\gamma_1\gamma_1(\beta_1+\beta_2) \\ -\gamma_1\gamma_2(\beta_1+\beta_2) & \gamma_1\gamma_2(1+\beta_1\beta_2) \end{bmatrix} \\ &= \Lambda\left[\frac{\beta_1+\beta_2}{1+\beta_1\beta_2}\right] \end{aligned}$$

Check - note relation to velocity transformation law.
Recall that for spacial rotations

$$R(\theta) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

with

$$R(\theta_1)R(\theta_2) = R(\theta_1+\theta_2) \text{ -- composition law}$$

For Lorentz boosts, define φ such that $\beta - \tanh\varphi$

$$(\implies \gamma = \cosh\varphi,\; \gamma\beta = \sinh\varphi)$$

Then the composition law can be

$$\Lambda(\varphi_1)\Lambda(\varphi_2) = \Lambda(\varphi_1+\varphi_2)$$

Suggests that Lorentz boosts can be regarded as 'hyperbolic rotations' in space-time

8.6 Relativistic Kinematics

Method. A particle moves along a trajectory $\mathbf{x}(t)$. Its velocity is

$$\frac{\mathrm{d}}{\mathrm{d}\mathbf{x}} = \mathbf{u}(t)$$

Path in space-time is parametrised by t. But both $\mathbf{x}$ and t vary under Lorentz transformation – describing the path in a different frame will be difficult.
Proper time: First consider a particle at rest in S' with

$$\mathbf{x}' = \mathbf{0}$$

The invariant interval between points on its world line is

$$\Delta s^2 = c^2 \Delta t'^2$$

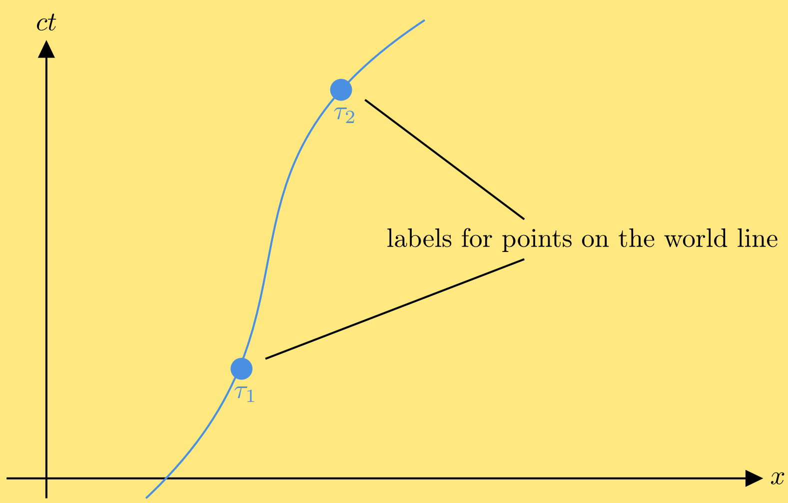
Define proper time τ such that

$$\Delta\tau = \frac{1}{c}\Delta s$$

This is the change in time experienced in the rest frame of the particle. But the equation

$$\Delta\tau = \frac{1}{c}\Delta s$$

holds in all frames because Δs is Lorentz invariant. Note that Δs is real in all frames - the interval is timelike.
The world line of the particle can now be parametrised in terms of τ



Method (continued)**.** Infinitesimal changes are related by:

$$\begin{aligned} \mathrm{d}\tau &= \frac{\mathrm{d}s}{c} \\ &= \frac{1}{c}\sqrt{c^2\,\mathrm{d}t^2 - |\mathrm{d}\mathbf{x}|^2} \\ &= \frac{1}{c}\sqrt{c^2\,\mathrm{d}t^2 - |\mathbf{u}|^2} \\ &= \left(1 - \frac{u^2}{c^2}\right)^{1/2} \mathrm{d}t \end{aligned}$$

Hence

$$\frac{\mathrm{d}t}{\mathrm{d}\tau} = \gamma_{\mathbf{u}} \text{ with } \gamma_{\mathbf{u}} = \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}}$$

Total time observed by particle moving along its world line

$$T = \int \mathrm{d}\tau = \int \frac{\mathrm{d}t}{\gamma_{\mathbf{u}}}$$

Definition (4-velocity)**.** **Position 4-vector** of a particle is

$$X(\tau) = \begin{bmatrix} ct(\tau) \\ \mathbf{x}(\tau) \end{bmatrix}$$

The **4-velocity** is defined by

$$U = \frac{\mathrm{d}X}{\mathrm{d}\tau} = \begin{bmatrix} c\frac{\mathrm{d}t}{\mathrm{d}\tau} \\ \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}\tau} \end{bmatrix} = \frac{\mathrm{d}t}{\mathrm{d}\tau}\begin{bmatrix} c \\ \mathbf{u} \end{bmatrix} = \gamma_{\mathbf{u}}\begin{bmatrix} c \\ \mathbf{u} \end{bmatrix}$$

where

$$\mathbf{u} = \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}$$

Notation. Another possible notation:

$$\underset{\text{position 4-vector}}{(ct, \mathbf{x})}, \underset{\text{4-velocity}}{(\gamma_{\mathbf{u}} c, \gamma_{\mathbf{u}} \mathbf{u})}$$

Equation. If frames S and S' are such that X and X' are corresponding components of position vector, then

$$X' = \Lambda X$$

Correspondingly

$$U = \Lambda U$$

relates the components of the 4-velocity
(Any quantity whose components transform according to this rule is a 4-vector.)
U is a 4-vector because X is a 4-vector and τ is an invariant (i.e. $\frac{\mathrm{d}X}{\mathrm{d}\tau}$ is a 4-vector).
But e.g. $\frac{\mathrm{d}X}{\mathrm{d}t}$ is not a 4-vector because t is not an invariant.
The scalar product $U \cdot U$ is invariant under Lorentz transformations, i.e.

$$U \cdot U = U' \cdot U'$$

In the rest frame of a particle moving with 4-velocity U,

$$U = \begin{bmatrix} c \\ \mathbf{0} \end{bmatrix}$$

hence

$$U \cdot U = c^2$$

In any other frame

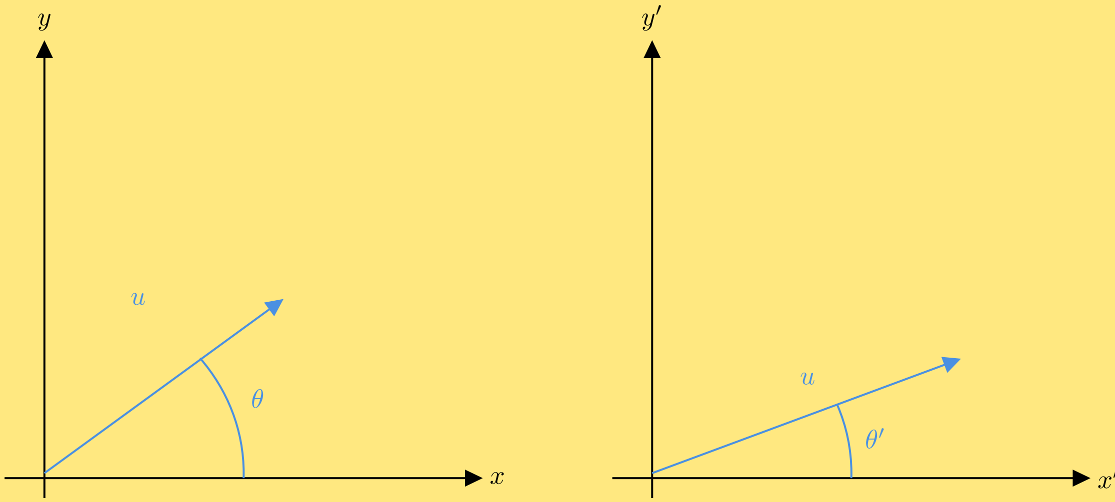
$$U \cdot U = c^2 = \gamma^2(c^2 - \mathbf{u}^2) - c^2$$

as expected

8.6.1 Transformations of Velocities as 4-Vectors

Note. We have previously seen that the transformation law for velocities in SR is not simply the addition rule implied by Galilean transformations.

Method. Now consider transformation $S \to S'$ with S' moving at speed v in the x-direction relative to S.



$$U' = \begin{bmatrix} \gamma_{\mathbf{u}'} c \\ \gamma_{\mathbf{u}'} u' \cos\theta' \\ \gamma_{\mathbf{u}'} u' \sin\theta' \\ 0 \end{bmatrix}, \text{ with } U' = \Lambda U$$

with

$$\Lambda = \begin{bmatrix} \gamma_v & -\gamma_v v/c & 0 & 0 \\ -\gamma_v v/c & \gamma_v & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

In frame S: 4 velocity

$$U = \begin{bmatrix} \gamma_{\mathbf{u}} c \\ \gamma_{\mathbf{u}} u \cos\theta \\ \gamma_{\mathbf{u}} u \sin\theta \\ 0 \end{bmatrix}$$

Deduce

$$u' \cos\theta' = \frac{u\cos\theta - v}{1 - \frac{uv\cos\theta}{c^2}}$$

$$\tan\theta' = \frac{u\sin\theta}{\gamma_u(u\cos\theta - v)}$$

(Exercise: check)

Definition. There is change in perceived angle of velocity, – '**aberration**' (due to the motion of the observer).
applies when $u = c$, i.e. aberration applies to light rays. (Apparent direction of stars changes as a result of the Earth's orbital motion.)

Note. Orbital velocity is $\sim 3 \times 10^4\,\mathrm{ms}^{-1}$, c is $\sim 3 \times 10^8\,\mathrm{ms}^{-1}$ i.e. $v/c \sim 10^{-4}$, change in angle is small, but observable.

8.7 Relativistic Physics

Definition. **4-momentum of a particle** with mass m and with 4-velocity U is

$$P = mU = m\gamma_{\mathbf{u}} \begin{bmatrix} c \\ \mathbf{u} \end{bmatrix}$$

(has 4 components $P^0,\ P^1,\ P^2,\ P^3$)
For P to be a 4-vector, then m must be an invariant.
m is the rest mass, i.e. mass measured in the rest frame of the particle.
4 momentum of a system of particles is the sum of the 4-momenta of the individual particles – conserved in absence of external forces.

Equation. Spatial components of P ($\mu = 1, 2, 3$) are relativistic 3 momentum

$$\mathbf{p} = \gamma_{\mathbf{u}} m \mathbf{u}$$

Same as for Newtonian physics except that mass m is modified to $\gamma_{\mathbf{u}} m$.

Definition. We interpret $\gamma_{\mathbf{u}} m$ as the '**apparent mass**'. Note that $|\mathbf{p}|$ and the apparent mass $\to \infty$ as $|\mathbf{u}| \to c$.

Moral. What does P^0 represent?

$$P^0 = \gamma_{\mathbf{u}} mc = \frac{mc}{\sqrt{1 - \mathbf{u}^2/c^2}} = \frac{1}{c}\left(mc^2 + \frac{1}{2}m\mathbf{u}^2 + \dots\right)$$

Natural interpretation of P^0 is an energy.

$$P = \begin{bmatrix} E/c \\ \mathbf{p} \end{bmatrix} \text{ with } E = \gamma_{\mathbf{u}} mc^2 = mc^2 + \frac{1}{2}m\mathbf{u}^2 + \dots$$

Note that $E \to \infty$ as $|\mathbf{u}| \to c$ (if $m \neq 0$).
(P is sometimes called the energy-momentum 4-vector)

Equation. For a stationary particle with rest mass m, we have that

$$E = mc^2$$

Implication is that mass if a form og energy. Energy of a moving particle

$$E = mc^2 + \underbrace{(\gamma_{\mathbf{u}} - 1)mc^2}_{\text{Relativistic K.E.}}$$

Relativistic K.E. is a generalisation of the Newtonian formula.
Since

$$P \cdot P = \frac{E^2}{c^2} - |\mathbf{p}|^2$$

is Lorentz invariant, we have

$$P \cdot P = m^2c^2$$

hence

$$\frac{E^2}{c^2} = p^2 + m^2c^2$$

(important relation between momentum and energy)

Moral. In Newtonian physics mass and energy are separately conserved. In Special Relativity, mass is not conserved, it is a form of energy. Mass can be converted into kinetic energy and vice-versa.

Note. SUMMARY:

- 4-momentum
$$P = \begin{bmatrix} E/c \\ \mathbf{p} \end{bmatrix}$$
- E is relativistic energy
- $P \cdot P$ is invariant
$$P \cdot P = E^2/c^2 - |\mathbf{p}|^2 = m^2c^2$$
(m is rest mass)
- Mass is a form of energy – can convert into KE and vice versa.

8.7.1 Massless Particles

Definition. **Massless particles** are particles with zero rest mass ($m = 0$), e.g. photons.

Equation. These particles can have non-zero momentum and non-zero energy, because they travel at the speed of light.
($\gamma_{\mathbf{u}} = \infty$).
In this case $P \cdot P = 0$.
These particles are travelling along light-like trajectories and therefore they have no proper time (can't transform to rest frame).
Energy and (3-)momentum for such particles satisfies:

$$\frac{E^2}{c^2} = |\mathbf{p}|^2$$

i.e.

$$E = |\mathbf{p}|c$$

Then

$$P = \frac{E}{c}\begin{bmatrix} 1 \\ \mathbf{n} \end{bmatrix}$$

with $\mathbf{n}$ unit vector in direction of travel.

8.7.2 Newton's 2$^{\text{nd}}$ Law in Special Relativity

Equation. Law has the form

$$\frac{\mathrm{d}P}{\mathrm{d}\tau} = F$$

where F is a 4-vector – the 4-force

Equation. The relation between 4-force F and 3-force $\mathbf{F}$ is

$$F = \gamma_{\mathbf{u}}\begin{bmatrix} \mathbf{F} \cdot \mathbf{u}/c \\ \mathbf{F} \end{bmatrix}$$

Thus

$$\frac{\mathrm{d}E}{\mathrm{d}\tau} = \gamma_{\mathbf{u}}\frac{\mathbf{F} \cdot \mathbf{u}}{c}, \ \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}\tau} = \gamma_{\mathbf{u}}\mathbf{F}$$

hence

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{\mathbf{F} \cdot \mathbf{u}}{c}, \ \frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = \mathbf{F}$$

Equivalently, for particle of rest mass m

$$F = mA \text{ with } A = \frac{\mathrm{d}U}{\mathrm{d}\tau}, \text{ the 4-acceleration}$$

8.8 Applications of Special Relativity to Particle Physics

Method. Many physical problems can be solved using conservation of 4-momentum – i.e. consider

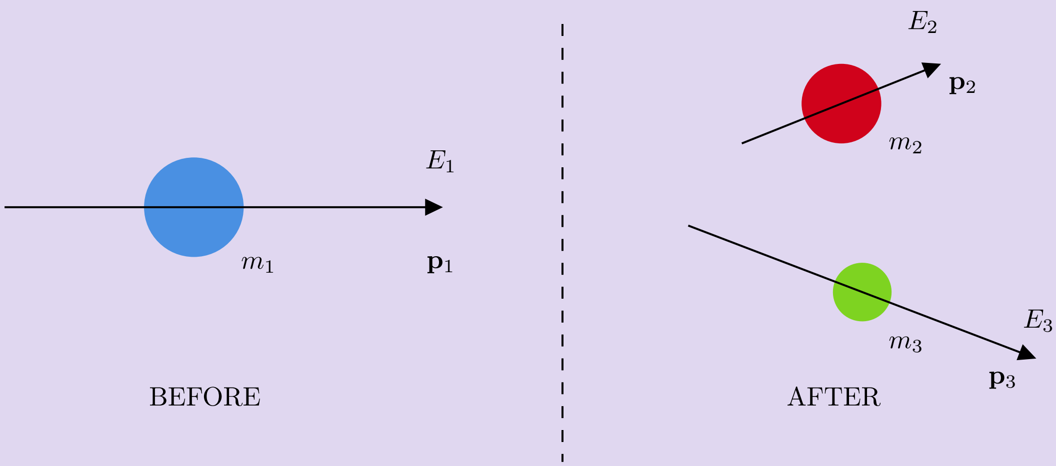
$$P = \begin{bmatrix} E/c \\ \mathbf{p} \end{bmatrix}$$

for a system of particles.
Often convenient to write down conservation of 4-momentum in the centre of momentum frame (CM frame) i.e. frame in which

$$P^1 = P^2 = P^3 = 0$$

(may be problematic for massless particles)

Example (Particle Decay)**.** A particle of mass m, decays into two particles of mass m_2 and m_3



We have that

$$P_1 = P_2 + P_3$$

0 component:

$$E_1 = E_2 + E_3$$

1, 2, 3 components:

$$\mathbf{p}_1 = \mathbf{p}_2 + \mathbf{p}_3$$

Center of momentum frame:

$$\mathbf{p}_1 = \mathbf{0} \implies \mathbf{p}_1 = -\mathbf{p}_3$$

$$\frac{E_1}{c} = m_1 c = \frac{E_2}{c} + \frac{E_3}{c}$$

$$\frac{E_2}{c} = \sqrt{\mathbf{p}_2^2 + m_2^2 c^2}$$

$$\frac{E_3}{c} = \sqrt{\mathbf{p}_3^2 + m_3^2 c^2}$$

so

$$\frac{E_1}{c} = \sqrt{\mathbf{p}_2^2 + m_2^2 c^2} + \sqrt{\mathbf{p}_3^2 + m_3^2 c^2} \geq m_2 c + m_3 c$$

i.e. decay is possible only if $m_1 \geq m_2 + m_3$.
(Recall that mass may not be conserved.)

Example (Higgs to photon decay).

$$h \to \gamma\gamma \text{ (2 photons)}$$

Conservation of 4-momentum

$$P_h = P_{\gamma_1} + P_{\gamma_2}$$

In Higgs rest frame:

$$P_h = \begin{bmatrix} m_h c \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} E_{\gamma_1}/c \\ \mathbf{p}\gamma_1 \end{bmatrix} + \begin{bmatrix} E_{\gamma_2}/c \\ \mathbf{p}\gamma_1 \end{bmatrix}$$

(1, 2, 3 components):

$$\mathbf{0} = \mathbf{p}_{\gamma_1} + \mathbf{p}_{\gamma_2}$$

(0 component):

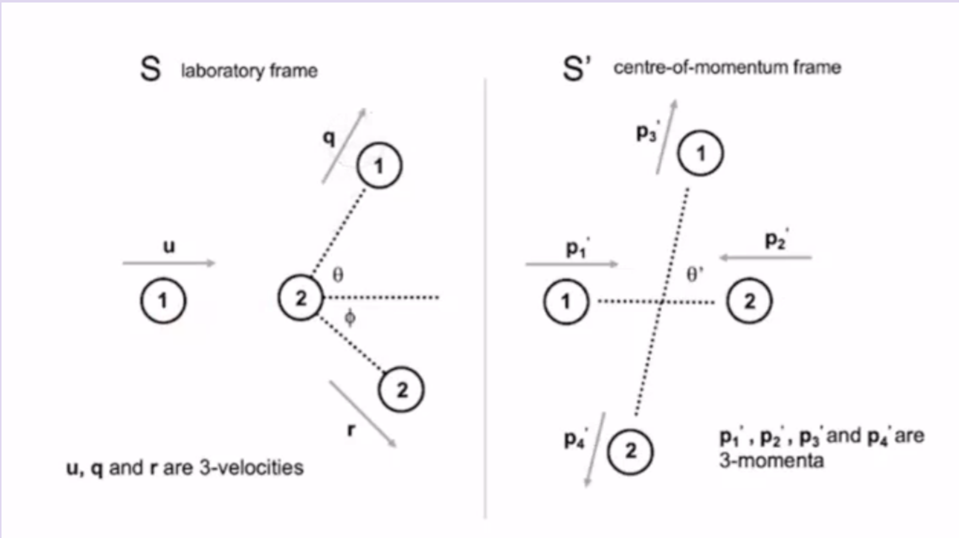
$$\frac{E_{\gamma_1}}{c} = |\mathbf{p}\gamma_1|, \ \frac{E_{\gamma_2}}{c} = |\mathbf{p}\gamma_2|$$

(we recall $E^2/c^2 = \mathbf{p}^2 + m^2c^2$, follows from zero rest mass of photon)

$$\implies E_{\gamma_1} = E_{\gamma_2} \implies E_{\gamma_1} = E_{\gamma_2} = \frac{1}{2} m_h c^2$$

Note that mass is not conserved.

Example (Particle scattering)**.** 2 identical particles collide and retain their identities



Conservation of 4 momentum:

$$P_1 + P_2 = P_3 + P_4$$

(from latter collision)
What is the relation between θ and ϕ?
Because 1 and 2 have the same mass, their speeds are equal both before and after the collision.
Let speeds before and after collision be v and w

In CM frame:

$$P_1' + P_2' = P_3' + P_4'$$

$$P_1' = \begin{bmatrix} m\gamma_v c \\ m\gamma_v v \\ 0 \\ 0 \end{bmatrix} \quad P_2' = \begin{bmatrix} m\gamma_v c \\ -m\gamma_v v \\ 0 \\ 0 \end{bmatrix} \text{ (BEFORE)}$$

$$P_3' = \begin{bmatrix} m\gamma_w c \\ m\gamma_w w \cos\theta' \\ m\gamma_w w \sin\theta' \\ 0 \end{bmatrix}$$

$$P_4' = \begin{bmatrix} m\gamma_w c \\ -m\gamma_w w \cos\theta' \\ -m\gamma_w w \sin\theta' \\ 0 \end{bmatrix}$$

0 component:

$$2m\gamma_v c = 2m\gamma_w c$$

hence $v = w$

Example (continued)**.** Now apply Lorentz transformation $S' \to S$.

$$\Lambda = \begin{bmatrix} \gamma v & \gamma v v/c & 0 & 0 \\ \gamma v v/c & \gamma_v & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(determined by requirement that particle 2 is at rest before the collision)

$$P_1 = \Lambda P_1' = \begin{bmatrix} m\gamma_v^2(c + v^2/c) \\ m\gamma_v^2(v+v) \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} m\gamma_u c \\ m\gamma_u u \\ 0 \\ 0 \end{bmatrix}$$

(using u is initial speed of particle 1)
Particle 1 after collision (as seen in S)

$$P_3 = \underbrace{\begin{bmatrix} m\gamma_1 c \\ m\gamma_q q\cos\theta \\ m\gamma_q q \sin\theta \\ 0 \end{bmatrix}}_{\text{Specification of problem in } S} = \underbrace{\begin{bmatrix} m\gamma_v^2 c + \frac{v^2}{c}\cos\theta' \\ m\gamma_v^2(v + v\cos\theta') \\ m\gamma_v v \sin\theta' \\ 0 \end{bmatrix}}_{\text{Lorentz transformation applied to } P_3'}$$

Hence (considering 1 and 2 components)

$$\tan\theta = \frac{m\gamma_v v \sin\theta'}{m\gamma_v^2 v(1+\cos\theta')} = \frac{1}{\gamma_v}\tan\left(\frac{\theta'}{2}\right)$$

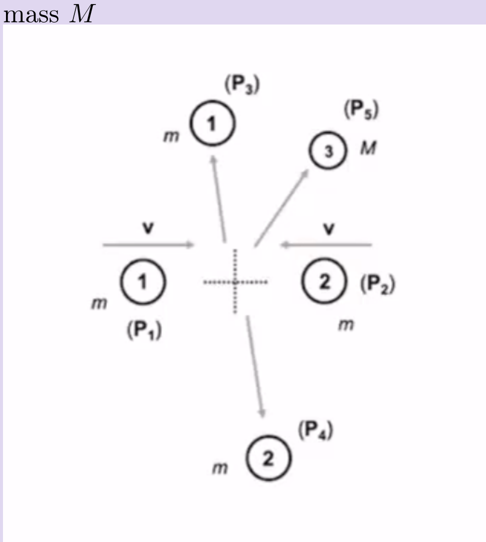
$$\tan\phi = \frac{m\gamma_v v \sin\theta'}{m\gamma_v^2 v(1-\cos\theta')} = \frac{1}{\gamma_v}\cot\left(\frac{\theta'}{2}\right)$$

Hence

$$\tan\theta \cdot \tan\phi = \frac{1}{\gamma_v^2} = \frac{2}{1+\gamma_u} \leq 1$$

(In Newtonian limit, $\tan\theta \cdot \tan\phi = 1 \implies$ angle between trajectories is $\pi/2$. In SR, angle is $\leq \pi/2$)

Example (Particle creation)**.** Collide two particles of rest mass m - create third particle with rest mass M



Center of momentum frame:
Conservation of 4-momentum:

$$P_1 + P_2 = \underset{(1)}{P_3} + \underset{(2)}{P_4} + \underset{(3)}{P_5}$$

(RHS are after)
Let $v, -v$ be velocities of 1 and 2 before collision; hence

$$P_1 + P_2 = \begin{bmatrix} 2m\gamma_v c \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} E_3/c + E_4/c + E_5/c \\ \mathbf{0} \end{bmatrix}$$

(total 3-momentum is $\mathbf{0}$ after the collision)

$$2m\gamma_v c^2 = E_3 + E_4 + E_5 \geq (m + m + M)c^2 = (2m + M)c^2$$

so

$$\gamma_v \geq \left(1 + \frac{M}{2m}\right)$$

for particle creation to be possible.
I.e. initial KE in CM frame must satisfy

$$2m(\gamma_v - 1)c^2 \geq Mc^2$$

Example (continued). Now transform to a frame where one particle is moving with velocity u and the other is at rest. Then

$$u = \frac{2v}{1 + \frac{v^2}{c^2}}$$

(e.g. by velocity composition rule)
hence

$$\gamma_u = 2\gamma_v^2 - 1 \text{(recall previous example)}$$

and

$$\gamma_u = 2\gamma_v^2 - 1 \geq 2\left(1 + \frac{M}{2m}\right)^2 - 1 = 1 + \frac{2M}{m} + \frac{M^2}{2m^2}$$

Hence in this frame then KE,

$$mc^2(\gamma_u - 1) \geq mc^2\left(\frac{2M}{m} + \frac{M^2}{2m^2}\right) \geq 2Mc^2 + \frac{M^2c^2}{2m} \geq Mc^2$$

i.e. greater, perhaps significantly greater than that required in COM frame.
Hence it is much easier to create new particles by colliding particle beams than by colliding one beam with a fixed target. (Used in design of particle accelerators)