Geometry

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Contents

0 Overview

Definition. A **topological surface** is a topological space Σ s.t.

(i) $\forall p \in \Sigma$ there is an open neighbourhood $p \in U \subset \Sigma$ s.t. U is homeomorphic to $\mathbb{R}^2$, or a disc $D^2 \subset \mathbb{R}^2$, with its usual Euclidean topology

(ii) Σ is Housdorff and second countable

Remarks.

(i) $\mathbb{R}^2 \cong D(0,1) = \{x \in \mathbb{R}^2 : \|x\| < 1\}$ (homeomorphic to)

(ii) A space X is **Housdorff** if for $p \neq q$ in $X \exists$ disjoint open sets $p \in U$ and $q \in V$ in X.
A space X is **second countable** if it has a countable base, i.e. $\exists \{U_i\}_{i \in \mathbb{N}}$ open sets s.t. every open set is a union of some of the U

(iii) If X is Housdorff/ second countable, so are subspaces of X. Euclidean space has these properties. (For second countable, consider the open sets $B(c,r)$ with $c \in \mathbb{Q}^n \subset \mathbb{R}^n$ and $r \in \mathbb{Q}_+ \subset \mathbb{R}_+$)

(i) is the point. (ii) is for technical honesty.

0.1 Examples of Topological Spaces

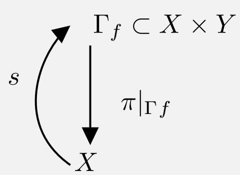
Examples. (i) $\mathbb{R}^2$ the plane

(ii) Any open subset of $\mathbb{R}^2$, i.e. $\mathbb{R}^2 \backslash z$ where z is closed.
- $z = \{0\}$. $\mathbb{R}^2 \backslash \{0\}$ is a surface
- $z = \{(0,0)\} \cup \{(0, 1/n) : n = 1, 2, 3, \dots\}$

(iii) Graphs: let $f : \mathbb{R}^2 \to \mathbb{R}$ be a continuous function. The fraph

$$\begin{aligned}\Gamma_f &= \{(x, y, f(x,y)) : (x,y) \in \mathbb{R}^2\} \\ &\subset \mathbb{R}^2\end{aligned}$$

Recall if X, Y are spaces, the product topology on $X \times Y$ has basic open sets $U \times V$ with $U \subset X$ and $V \subset Y$ open. It has the feature that $f : Z \to X \times Y$ is continuous $\iff$ $\pi_X \circ f : Z \to X$ and $\pi_Y \circ f : Z \to Y$ are continuous, where π_X is the projection to X and π_Y is the projetcion to Y.
Application: $\Gamma_f \subset X \times Y$, if $f : X \to Y$ is continuous, is homeomorphic to X.



$s : x \mapsto (x, f(x))$

(continuous by above)

So $\pi|_{\Gamma_f}$ and s are inverse homeomorphisms. So

$$\Gamma_f \cong \mathbb{R}^2$$

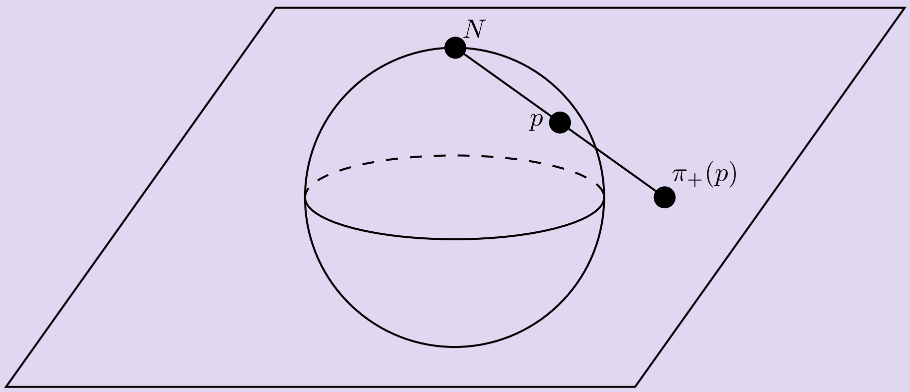
for any $f : \mathbb{R}^2 \to \mathbb{R}$ so Γ_f is a topological surface

Note. As a topological surface, Γ_f is independent of f. Later, as a geometric object, it will reflect features of f

Examples. (iv) The sphere:

$$S^2 = \{(x, y, z) \in \mathbb{R}^2 : x^2 + y^2 + z^2 = 1\}$$

(subspace topology)

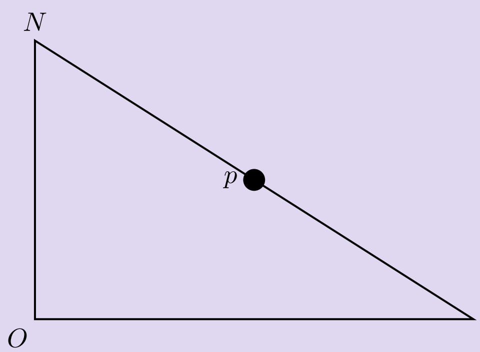


Stereographic projection

$$\pi_+ : S^2 \backslash \{(0, 0, 1) \to \mathbb{R}^2 \quad (z = 0) \subset \mathbb{R}^3$$

$$(x, y, z) \mapsto \left(\frac{x}{1-z}, \frac{y}{1-z}\right)$$

We can check:

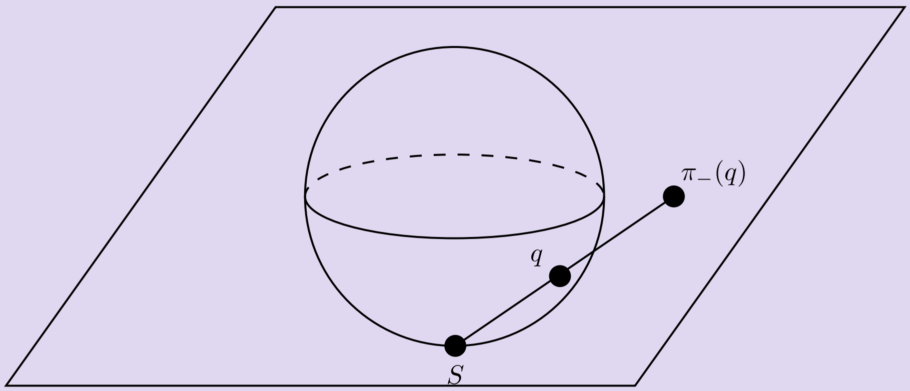


$\pi_+(p)$
$= (x_0, y_0, 0)$
$= (x, y, z) + \lambda(x, y, z - 1)$

Examples (continued)**.** (iv) We note that π_+ is continuous and has inverse

$$(u,v) \mapsto \left(\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \frac{u^2+v^2-1}{u^2+v^2+1}\right)$$

So π_+ is a continuous bijection with continuous inverse and hence a homeomorphism.



Stereographic projection

$$\pi_- : S^2 \backslash \{(0,0,-1) \to \mathbb{R}^2 \quad (z=0) \subset \mathbb{R}^3$$

$$(x,y,z) \mapsto \left(\frac{x}{1+z}, \frac{y}{1+z}\right)$$

This is also a homeomorphism from $S^2 \backslash \{(0,0,-1)\}$ to $\mathbb{R}^2$. So S^2 is a topological surface: $\forall p \in S^2$, either p lies in the domain of π_+ or of π_- (or both) so it lies in an open set $S^2 \backslash \{(0,0,1)\}$ or $S^2 \backslash \{(0,0,-1)\}$ homeomorphic to $\mathbb{R}^2$. (Housdorff and second countable from $\mathbb{R}^2$)

Remark. S^2 is **compact** as a topological space, since it is a closed bounded set in $\mathbb{R}^3$

Examples. (v) The real projective plane: the group $\mathbb{Z}/2$ acts on S^2 by homeomorphisms via the **antipodal map** $a : S^2 \to S^2$

$$a(x,y,z) = (-x,-y,-z)$$

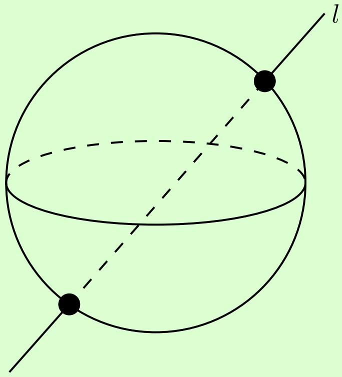
i.e. $\exists$ homeomorphism $\mathbb{Z}/2 \to \text{Homeo}(S^2)$, the groups of all homeomorphisms under composition of maps. Non-trivial element $\mapsto a$

Definition. The **real projective plane** is the quotient space of S^2 given by identifying every point with its antipodal image:

$$\mathbb{RP}^2 = S^2/(\mathbb{Z}/2) = S^2/\sim \quad x\tilde{a}(x)$$

Lemma. As a set, $\mathbb{RP}^2$ is naturally in bijection with the set of straightlines in $\mathbb{R}^2$ through 0

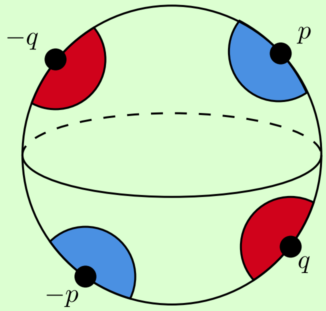
Proof.



Any straight line through $0 \in \mathbb{R}^3$ meets S^2 in exactly a pair of antipodal points and each such pair determines a straight line

Lemma. $\mathbb{RP}^2$ is a topological surface

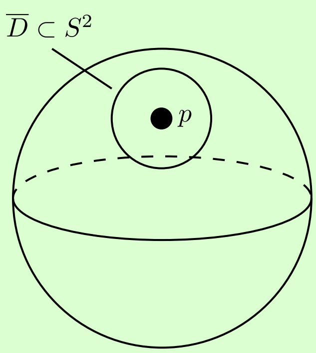
Proof. We check that it is Housdorff: Recall if x is a space and $q : X \to Y$ is a quotient map, $V \subset Y$ is open $\iff$ $q^{-1}V \subset X$ is open



If $[p] \neq [q] \in \mathbb{RP}^2$, then $\pm p$ and $\pm q$ are distinct antipodal pairs. Take small open discs centered on p, q and their antipodal images, as in the diagram. This gives us disjoint open neighbourhoods of $[p], [q]$ in $\mathbb{RP}^2$.
Note we could take small balls $B_{\pm p}(\delta), B_{\pm q}(\delta))$ ($\delta << 1$ small), which meet S^2 in open sets. If $q : S^2 \to \mathbb{RP}^2$ is the quotient map, then $q(B_q(p))$ is open since

$$q^{-1}(qB_\delta(p)) = B_\delta(p) \cup (-B_\delta(p))$$

$\mathbb{RP}^2$ is also countable.
Let $\mathcal{U}$ be a countable base for topology on S^2, and (wlog) $\forall U \in \mathcal{U}$, the antipodal image is in $\mathcal{U}$. Let $\bar{\mathcal{U}}$ be the family of open sets in $\mathbb{RP}^2$ of the form $q(U) \cup q(-U)$, $U \in \mathcal{U}$.
Now if $V \leq \mathbb{RP}^2$ is open, by definition $q^{-1}V$ is open in S^2. So $q^{-1}V$ contains some $U \in \mathcal{U}$ and hence containce $U \cup (-U)$. So $\bar{U}$ is a countable base for the quotient topology on $\mathbb{RP}^2$.
Finally, let $p \in S^2$ and $[p] \in \mathbb{RP}^2$ its image. Let $\bar{D}$ be a small closed disc neighbourhood of $p \in S^2$.



Quotient map $q|_{\bar{D}} : \bar{D} \to q(\bar{D}) \subset \mathbb{RP}^2$ is continuous from a compact space to a Houssdorff space. Also on $\bar{D}$, the map q is injective. Recall "Topological inverse function theorem": a continuous bijection from a compact space to a Housdorff space is a homeomorphism. So $q|_{\bar{D}} : \bar{D} \to q(\bar{D})$ is a homeomorphism inducting a homeomorphism $q|_D : D \to q(D) \in \mathbb{RP}^2$ where D is the open disc interior of $\bar{D}$. So $[p] \in q(D)$ has an open neighbourhood in $\mathbb{RP}^2$ homeomorphic to an open dic and we are done.

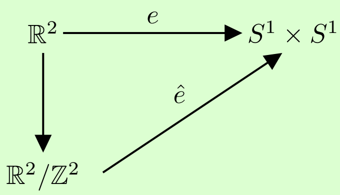
Examples. (vi) Let $S^1 = \{z \in \mathbb{C} : |z| = 1\}$. The torus $S^1 \times S^1$ with the subspace topology from $\mathbb{C}^2$ (which is the product topology)

Lemma. The torus is a topological surface

Proof. We consider the map

$$\mathbb{R}^2 \to S^1 \times S^1 \subset \mathbb{C} \times \mathbb{C} \quad (s,t) \mapsto (e^{2\pi i s}, e^{2\pi i t})$$

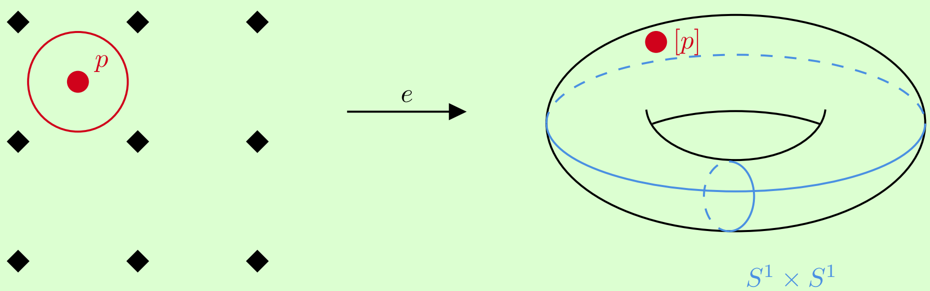
Note: this induces a map:



i.e. on the equivalence relation on $\mathbb{R}^2$ given by translating by $\mathbb{Z}^2$, e is constant on the equivalence classes so induces a map of sets

$$\mathbb{R}^2/\mathbb{Z}^2 \to S^1 \times S^1$$

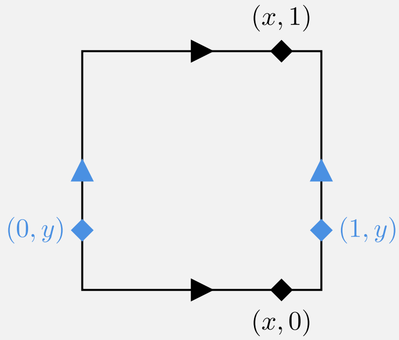
View $\mathbb{R}^2/\mathbb{Z}^2$ as the quotient space for $q : \mathbb{R}^2 \to \mathbb{R}^2/\mathbb{Z}^2$. The map $[0,1]^{\to}\mathbb{R}^2 \to \mathbb{R}^2/\mathbb{Z}^2$ is onto, so $\mathbb{R}^2/\mathbb{Z}^2$ is compact. So $\hat{e}$ is a continuous map from a compact space to a Housdorff space, and a bijection, so a homeomorphism (T.I.F.T)



Note we already know $S^1 \times S^1$ is compact and Housdorff (closed and bounded in $\mathbb{R}^4$). As for $S^2 \to \mathbb{RP}^2$. Pick $[p] = q(p),\ p \in \mathbb{R}^2$ and a small closed disc $\bar{D}(p) \subset \mathbb{R}^2$ s.t. $\forall (n,m) \in \mathbb{R} \times \mathbb{R}$

$$\bar{D}(p) \cap (\bar{D}(p) + (n,m)) = \varnothing$$

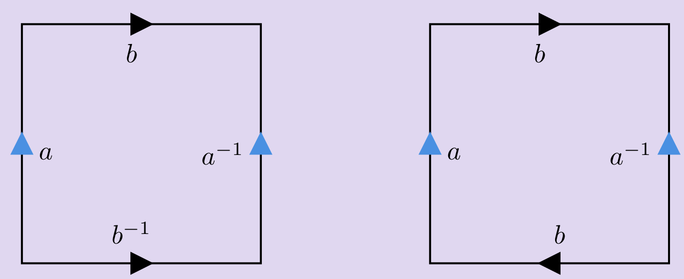
Then $e|_{\bar{D}(p)}$ is injective and $a|_{\bar{D}(p)}$ is injective. Now restricting to the open disc as before, we get an open disc neighbourhood of $[p] \in S^1 \times S^1$. Since $[p]$ arbitrary, $S^1 \times S^1$ is a topological surface

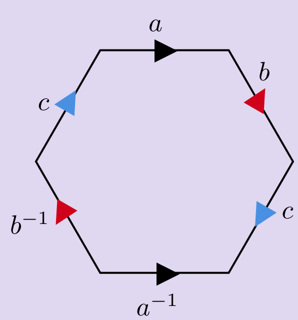


Another viewpoint: $\mathbb{R}^2/\mathbb{Z}^2$ is also given by imposing on $[0,1]^2$ the equivalence relation generated by:

$$\begin{aligned}(x,0) &\sim (x,1) \quad \forall 0 \le x \le 1\\ (0,y) &\sim (1,y) \quad \forall 0 \le y \le 1\end{aligned}$$

Examples. (vii) Let p be a planar Euclidean polygon. Assume the edges are oriented and paired, and (for simplicity) assume the Euclidean lengths of the e and $\hat{e}$ are equal if $\{e,\hat{e}\}$ are paired.





Label by letters and describe orientation by a sign $a^{\pm}$ relative to the clockwise orientation of $\mathbb{R}^2$

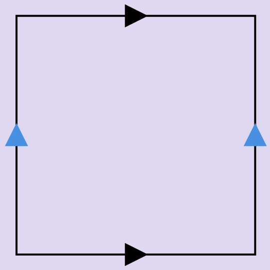
If $\{e,\hat{e}\}$ are paired edges, there is a unique isometry from e to $\hat{e}$ respected their orientations, say

$$f_{e\hat{e}} : e \to \hat{e}$$

These maps generate an equivalence relation on p, where we identify $x \in P$ with $f_{e\hat{e}}(x)$ whenever $x \in e$.

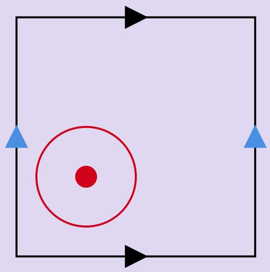
Lemma. $P/\sim$ (with the quotient topology) is a topological surface

Example. The torus on $[0,1]/\sim$

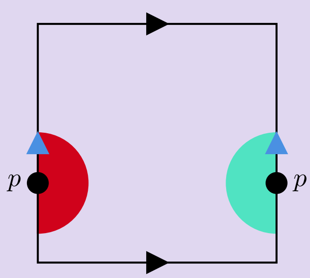


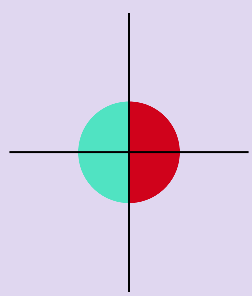
$P = [0,1]^2$

If $p \in \text{interior}(P)$



I pick $\delta > 0$ sufficiently small that $B_\delta(p)$ and $\overline{B_\delta(p)}$ in $\mathbb{R}^2$ lie in interior(p). Now argue as before: the quotient map is injective on $\overline{B_\delta(p)}$ and a homeomorphism on its interior.
If $p \in \text{edge}(p)$,

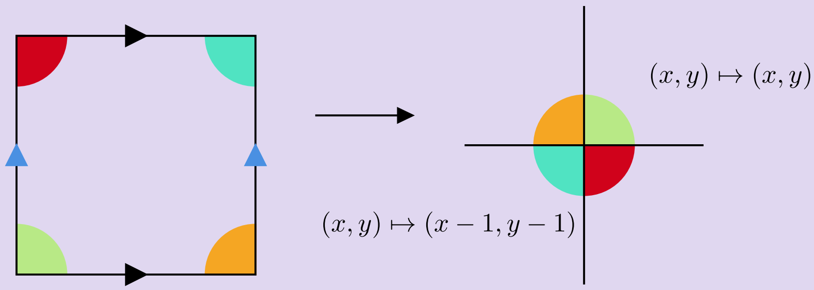




Say $p = (0, y_0) \sim (1, y_0)$ and $\delta > 0$ sufficiently small that half discs of radius δ as shown don't meet vertices(P). Define a map from the union of these hald-discs to $B(0,\delta) \subset \mathbb{R}^2$ via $(x,y) \mapsto (x, y - y_0)$ say f_U on the right half-disc (V) and $(x,y) \mapsto (x-1, y-y_0)$ say f_V on the left half disc (V).
Recall: if $X = A \cup B$ is a union of closed subspaces and $f : A \to Y,\ g : B \to Y$ are continuous and $f|_{A\cap B} = g|_{A \cap B}$ then they define a continuous map on X
Explicitly: f_U, f_V are continuous on $[0,1]^2 \implies$ they induce continuous maps on $qU, qV \subset T^2$

$$q : [0,1]^2 \to [0,1]^2/\sim = T^2$$

Example (continued)**.** Thus in T^2, the half discs qU, qV overlap but our maps agree on the closed intersection locus (as f_U, f_V compatible with equivalence relation). Hence, f_U, f_V give and define a continuous map on an open neighbourhood of $[p] - T^2$ to $B(0,\delta) \subset \mathbb{R}^2$
Now "usual argument" (pass to closed disc, use T.I.F.T, pass back to interior) shows that if $[p] \subset T^2$ lies on the image of edge of $[0,1]^2$, it has an open neighbourhood homeomorphic to a disc.
Analagously, at the vertex of $[0,1]^2$

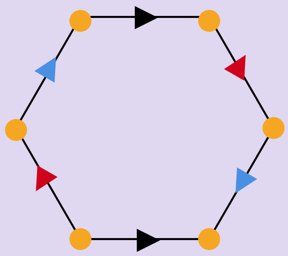


This shows $[0,1]^2/\sim$ is a topological surface.

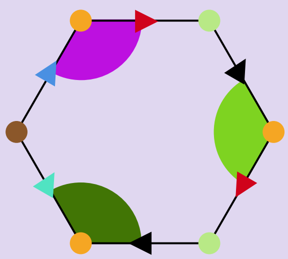
Example. For a general planar polygon $P \subset \mathbb{R}^2$:
Our equivalence relation

$$x \mapsto f_{e\hat{e}}(x) \quad x \in e \subset \mathrm{Edge}(p)$$

$\{e, \hat{e}\}$ pairs, $f : e \to \hat{e}$ compatible walk orientation. This induces an equivalence relation on $\mathrm{Vert}(P)$:



All vertices in one equivelnce class

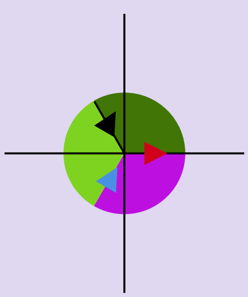


3 equivalence classes of vertices (†)

If $v \in \mathrm{Vert}(P)$ has r vertices in its equivalence class, $\exists r$ sectors in P, of total angle α_V.
Any sector can be identified with our favourite sector

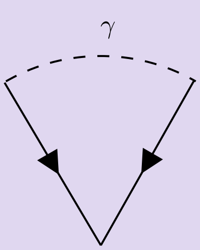
$$(x, y) \in \mathbb{R}^2 \text{ or } (r, \theta) \in \mathbb{R}^2 \quad 0 \leq r < \delta,\ \theta \in [0, 2\pi/r]$$

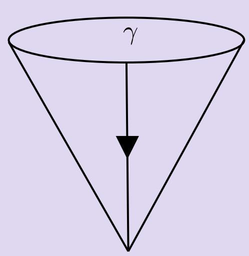
In (†), we get an open disc neighbourhood of v (red dot) via



compare (†)

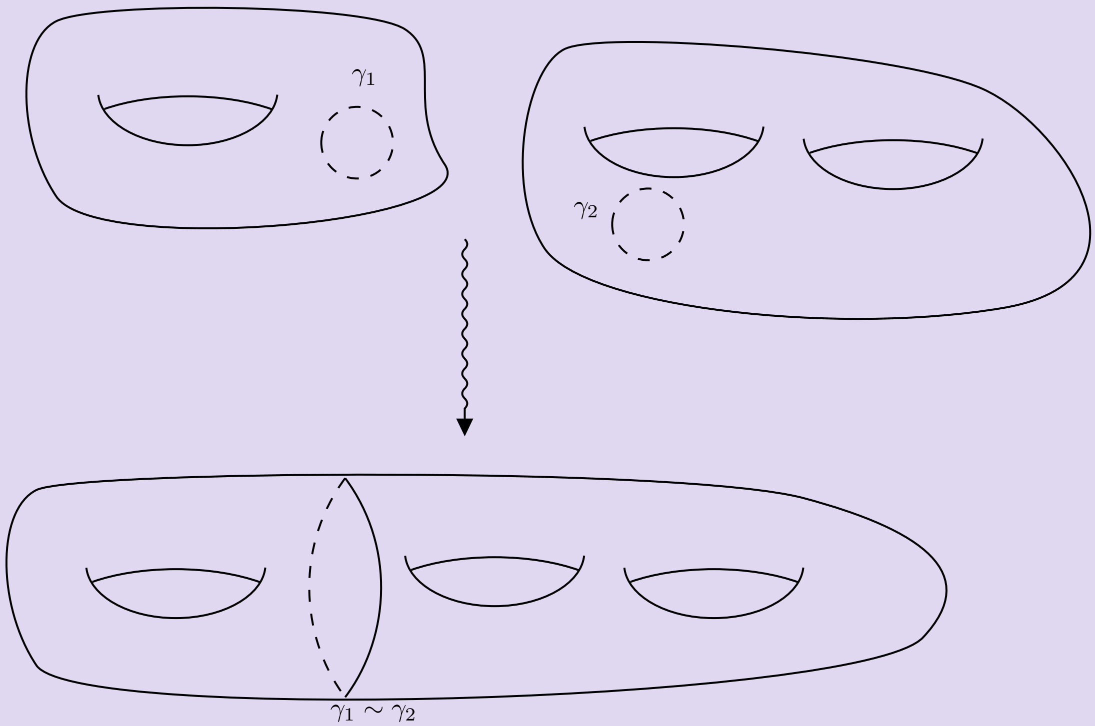
If $r = 1$, we have two arrows pointing outward or two arrows pointing inward. In either case, our quotient space is a cone, homeomorphic to $\mathbb{R}^2$.





These open neighbourhoods of points in $P/\sim$ show $P/\sim$ is locally homeomorphic to a disc. We can also see $P/\sim$ is Housdorff and second countable: By construction, if the δ discs, half discs or sections are sufficiently small and $p, q \in P$ lie in different equivalence classes, these are disjoint. So $P/\sim$ is Housdorff.
For second countable, we can consider discs in interior of P with rational centres and radii, and if $e \in \mathrm{edge}(P)$ and $e \to [0, length(e)]$ an isometry, take only $1/2$ discs on e which are centered at rational values in $[0, length(e)]$ and have rational radius and at vertices allow rational radius sectors. This gives us a countable base

Examples. (vii) Given topological surface, Σ_1, Σ_2, we can remove an open disc from each and give the resulting boundary circles

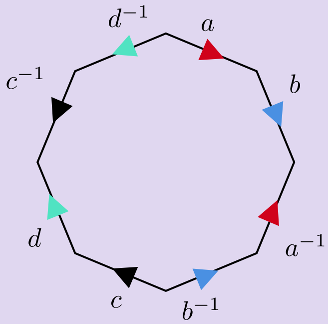


Explicitly, we take $\Sigma_1 \backslash D_1 \cup \sigma_1 \backslash D_2$ and impose a quotient relation $\theta \in \partial D_1 \sim \theta \in \partial D_2$ where ∂D_i is the boundary of D_i and θ parameter.

The result $\Sigma_1 \# \Sigma_2$ is called the connect sum of Σ_1 and Σ_2. (In principle, this depends on many choices, suppressed from the notation)

Lemma. The connect sum $\Sigma_1 \# \Sigma_2$ is a topological surface

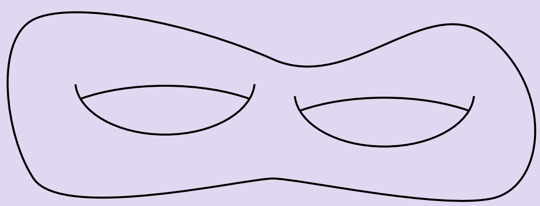
Examples.

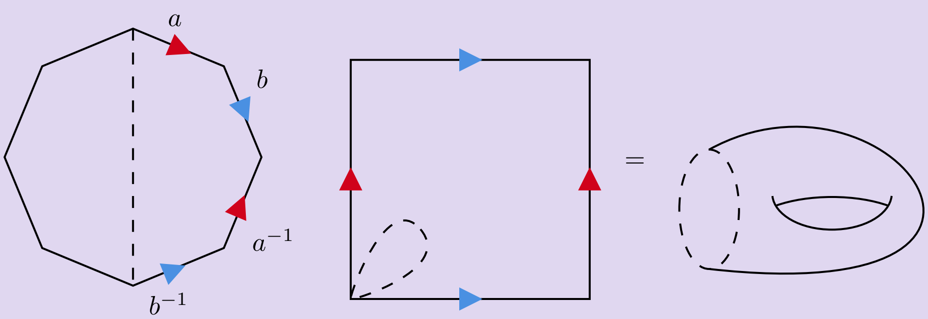


Consider octagon P

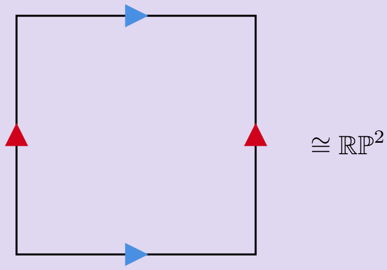
Claim:

$P/\sim\cong$

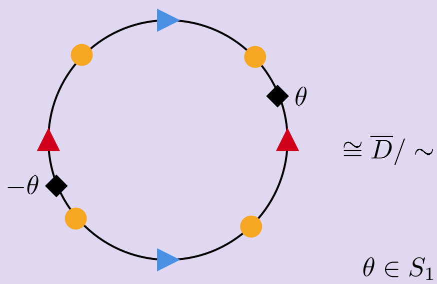




Example.

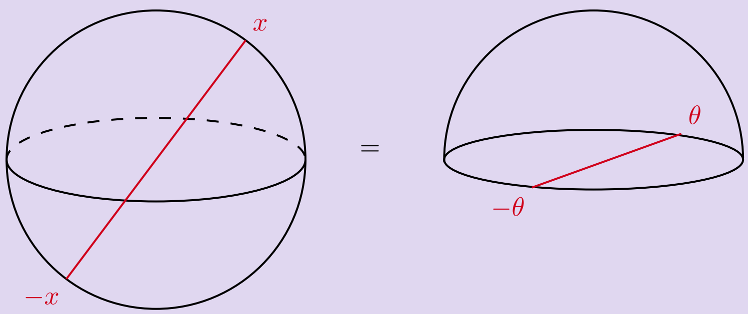


Since: LHS is the same identification space as



$\theta \in S_1 \sim -\theta$

$\mathbb{RP}^2 = S^3/\pm 1 =$ (Closed upper hemisphere)$/\theta \sim -\theta$ for points on equator



Definition. A **subdivision** of a compact topological surface Σ comprises:

(i) A finite set $V \subset \Sigma$ of vertices

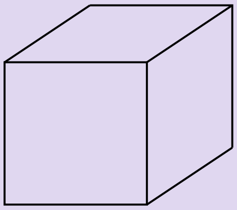
(ii) A finite collection $E = \{e_i : [0,1] \to \Sigma\}_{i \in \Sigma}$ of edges s.t.

- $\forall i : e_i$ is a continuous injection on its interior and $e_i^{-1}V = \{0,1\}$
- e_i and e_j have disjoint images except perhaps their endpoints in V

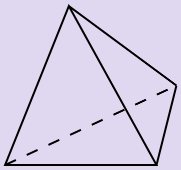
(iii) Such that each connected component of $\Sigma \backslash (\bigcup e_i[0,1] \cup V)$ is homeomorphic to an open disc called a **face**. (So the closure of a face has a boundary $\bar{F} \backslash F$ lying in $E \cup V$)

A subdivision is a **triangulation** if each closed face (closure of a face) contains exactly 3 edges, and two closed faces are disjoint ir meet in exactly one edge (or possible just one vertex)

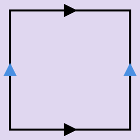
Examples.



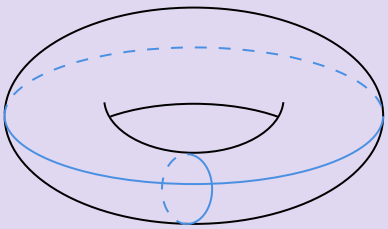
displays a subdivision of S^2



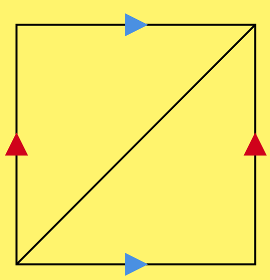
displays a triangulation of S^2



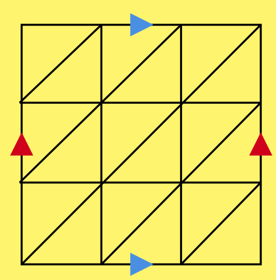
shows a subdivision of T^2 with 1 vertex, 2 edges, 1 face



Note.

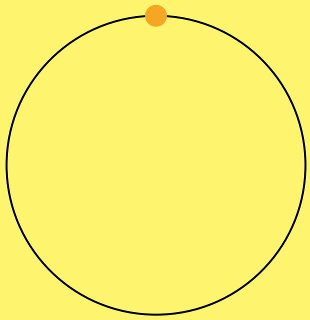


NOT a triangulation
(faces meet in more than 1 edge)



Is a triangulation of T^2

Remark.



Denotes a very degenerate subdivision of S^2 (1 vertex, 0 edges, 1 face)

Definition. The **Euler characteristic** of a subdivision is the number $\#V - \#E + \#F$. (no. of vertices - no. of edges + no. of faces)

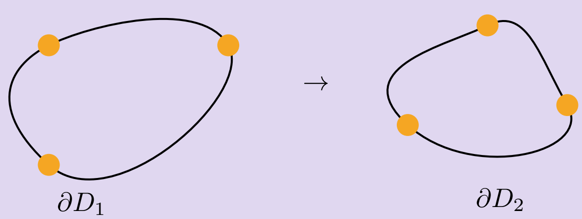
Theorem. (i) Every compact topological surface admits subdivisions (and indeed triangulations)

(ii) The Euler characterstic, denoted $\chi(\Sigma)$ does not depend on the choice of subdivision and describes a topological invariant of the surface (depends only on the homeomorphism type of Σ)

Examples. (i) $\chi(S^2) = 2$

(ii) $\chi(T^2) = 0$

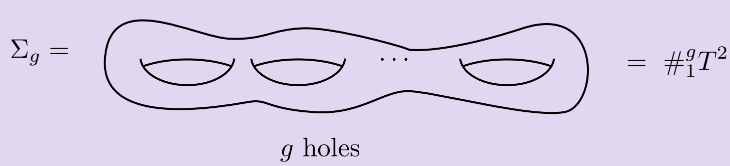
(iii) If Σ_1 and $\Sigma)2$ are compact topological spaces, we can form $\Sigma_1 \# \Sigma_2$ by removing an oen disc $D_i \subset \Sigma_i$ which is a face of a triangulation, and giving the boundary circles ∂D_i by a homeomorphism taking edges to edges



Then $\Sigma_1 \# \Sigma_2$ inherits a subdivision, and

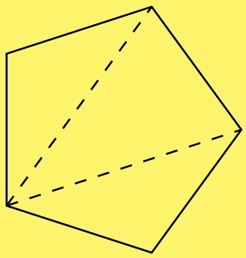
$$\chi(\Sigma_1 \# \Sigma_2) = \chi(\Sigma_1) + \chi(\Sigma_2) - 2$$

In particular, if

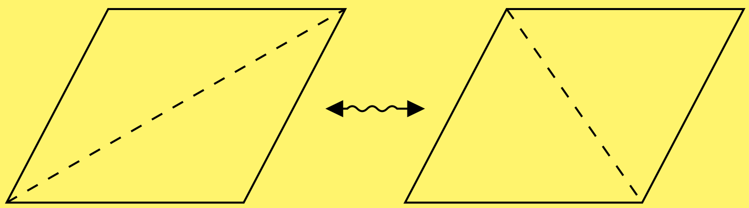


Then $\chi(\Sigma_g) = 2 - 2g$; f is called the **genus** of Σ

Remark. For the theorem earlier, (i) is hard. (ii) We should believe as we can turn a subdivision into a triangulation



and we can relate triangulations by local means



(changing the set of edges), and easy to see these preserve χ. But

- Hard to rigorise this
- You learn essentially nothing

A much cleaner approach is disclosed in Part II Algebraic Topology

Note. Recall, if Σ is a topological surface, each $p \in \Sigma$ lies in an oen neighbourhood $p \in U \subset \Sigma$ with U homeomorphic to an open disc (or $\mathbb{R}^2$)

Definition. A pair (U, φ) where $U \subset \Sigma$ open and $\varphi : U \to V \subset \mathbb{R}^2$ a homeomorphism is called a **chart** for Σ
(If $p \in U$ we might say "a chart for Σ at p")

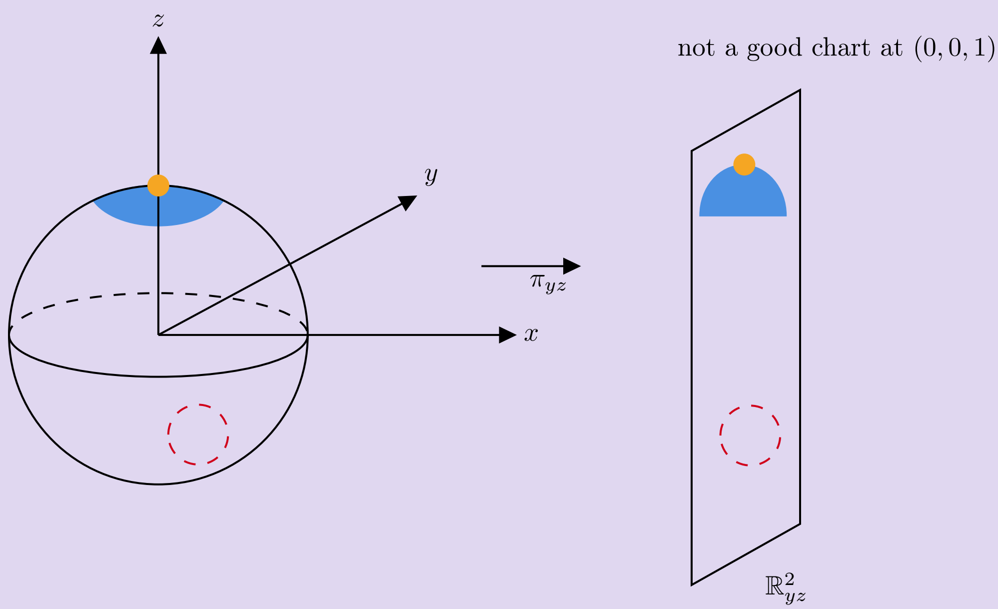
Definition. A collection $\{(U_i, \varphi_i)_{i \in I} : \varphi_i : U_i \to V_i\}$ of charts such that $\bigcup_{i \in \Sigma} U_i = \Sigma$ is called an **atlas** for σ. The inverse

$$\sigma = \varphi^{-1} : V \to U \subset \Sigma$$

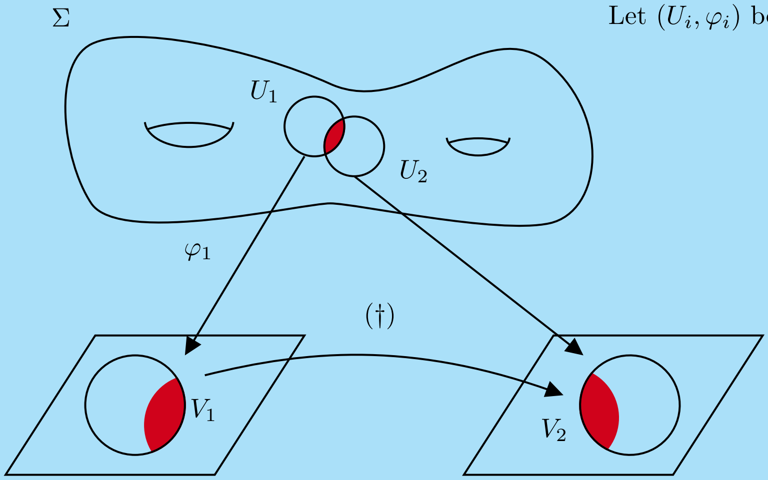
is called a **local parameterisation** for Σ

Examples. (i) If $Z \in \mathbb{R}^2$ is clsoed, $\mathbb{R}^2 \backslash Z$ is a topological surface eith an atlas with one chart $(\mathbb{R}^2 \backslash Z, \varphi = \text{ id.})$

(ii) For S^2, we have an atlas with 2 charts, the 2 stereographic projections



Definition.



Let (U_i, φ_i) be charts containing p

(†) is $\varphi_1(U_1 \cap U_2) \xrightarrow[\varphi_2 \circ \varphi_{1-1}]{} \varphi_2(U_1 \cap U_2)$

The map $\varphi_2 \circ \varphi_1^{-1}|_{\varphi_1(U_1 \cap U_2)}$ is called the **transition map** between the charts. This is a homeomorphism of open sets in $\mathbb{R}^2$

Note. Recall if $V \subset \mathbb{R}^n$ and $V^1 \subset \mathbb{R}^n$ are open, then a map $f : V \to V^1$ is called smooth if it is infinitely differentiable, i.e. it has partial derivatives of all orders

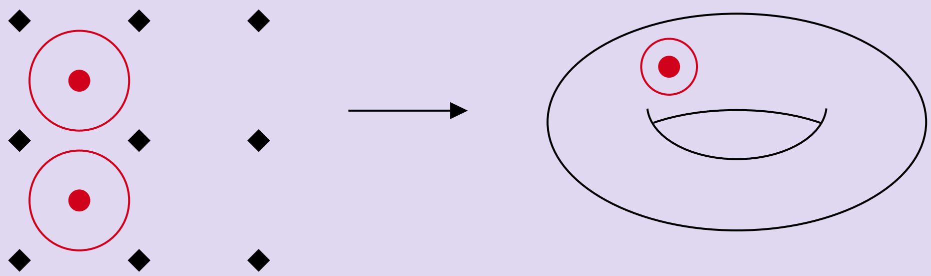
Definition. If $V \subset \mathbb{R}^n$ and $V^1 \subset \mathbb{R}^n$ a homeomorphism, $f : V \to V^1$ is called a **diffeomorphism** if it is smooth and its inverse is smooth

Definition. Ab **abstract smooth surface** Σ is a topological surface with an atlas of charts $\{(U_i, \varphi_i) : \varphi_i : U_i \to V_i \subset \mathbb{R}^2\}_{i\in I}, \bigcup_{i\in I} = \Sigma_i$ s.t. all transition maps $\varphi_i \circ \varphi_j : \varphi_j(U_i \cap U_j) \to \varphi_i(U_i \cap U_j)$ are diffeomorphisms of open sets in $\mathbb{R}^2$

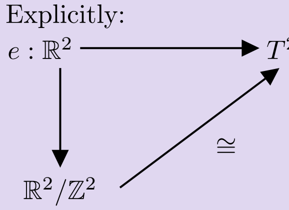
Note. It would NOT make sense to ask for the φ_i themselves to be smooth, as Σ is just a topological space.

Example. The atlas of 2 charts with stereographic projections gives S^2 the structure of an abstract smooth surface

Example. The torus $S^2 = \mathbb{R}^2/\mathbb{Z}^2$



Recall, we obtained charts from (the inverses of) the projection restricted to small discs in $\mathbb{R}^2$ (ones disjoint from translation by $\mathbb{Z} \oplus \mathbb{Z}\backslash\{0, 0\}$). The transition maps are translations fo T^2 inherits the structure of an abstract smooth surface.
Explicitly:



$(t, s) \mapsto \left(e^{2\pi it}, e^{2\pi is}\right)$

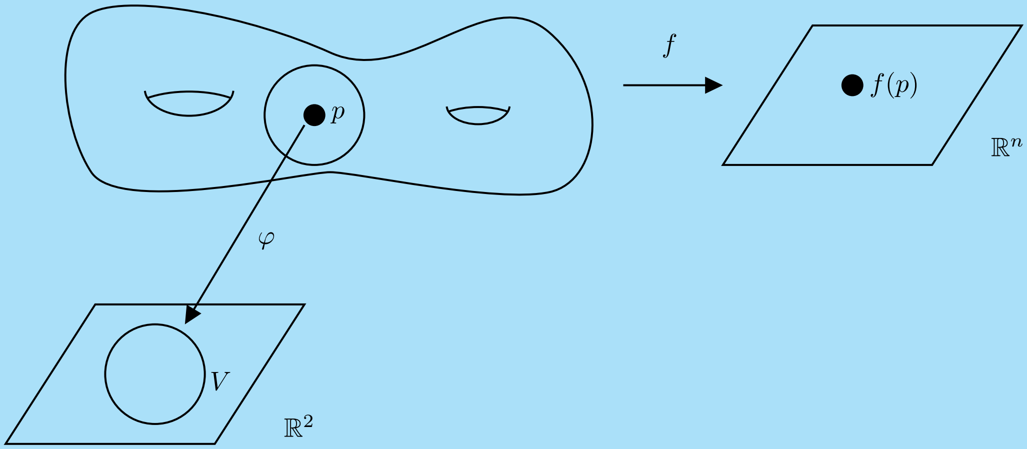
Consider the atlas

$$\{e(D_\varepsilon(x, y)) : e^{-1} \text{ on this image}\} \text{ where } \varepsilon < \frac{1}{2}$$

These are charts on T^2 and the transition maps are (restrictions), and the transition maps are (restrictions to the appropriate domain of) translations in $\mathbb{R}^2$. So T^2 has the structure (via this atlas) of an abstract smooth surface

Remark (Philosophical)**.** Being a topological surface is structure. (One can ask if a topological space X is a topological surface or not).
Being an abstract smooth surface is data. (I have to you an atlas of charts with smooth transition maps with smooth inverses: there could be many choices)

Definition. Let Σ be an abstract smooth surface and $f : \Sigma \to \mathbb{R}^n$ a continuous map. We say f is **smooth** at $p \in \Sigma$ if

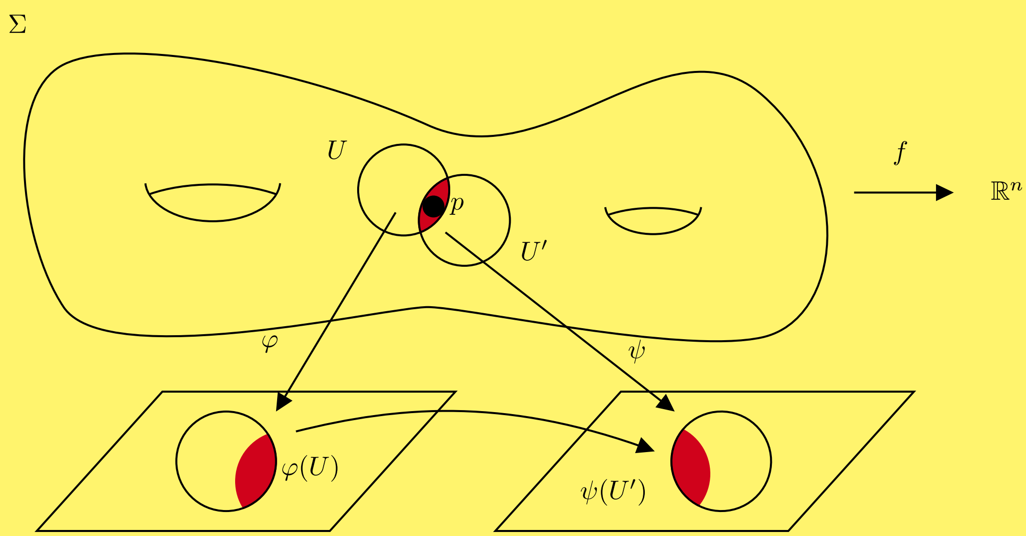


whenever (U, φ) is a chart at p belonging to my smooth atlas for Σ, the map

$$f \circ \varphi^{-1} : \varphi(U) \to \mathbb{R}^n$$

is smooth. ($\varphi(U) \subset \mathbb{R}^2$ an open set)

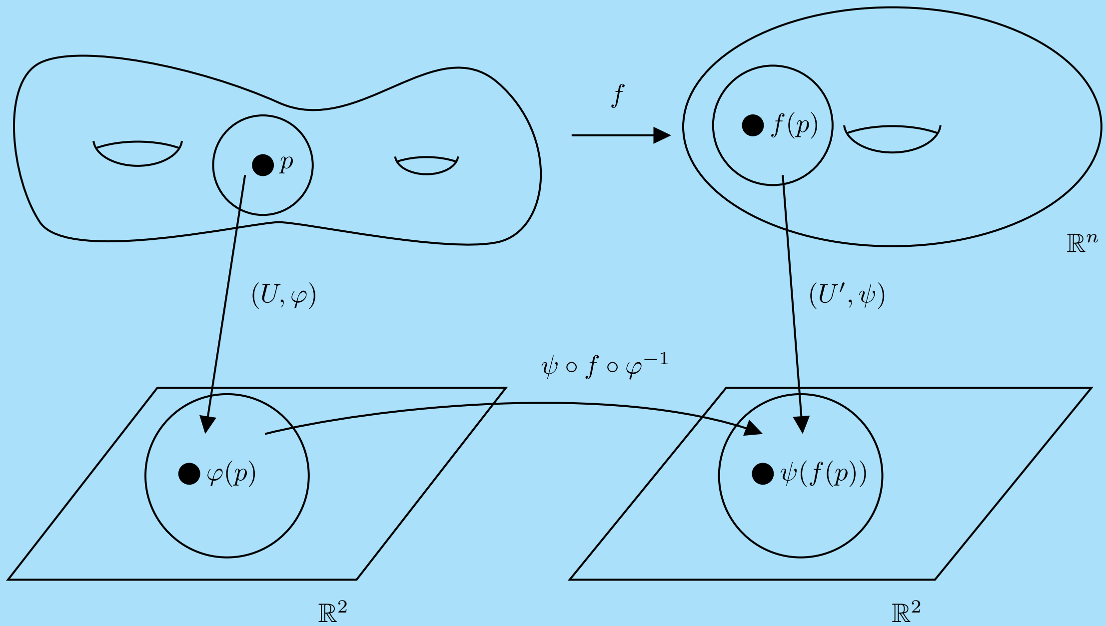
Note. Smoothness of f at p is independent of te choice of chart (U, φ) at p in the smooth atlas, since the transition maps between two such are diffeomorphisms.



$$\psi \circ \varphi^{-1} : \varphi(U \cap U') \xrightarrow{diff^m} \psi(U \cap U')$$

We have a related definition

Definition. If Σ_1 and Σ_2 are abstract smooth surfaces, a map $f : \Sigma_1 \to \Sigma_2$ is **smooth** if "it is smooth in the local charts"



I.e. given charts (U, φ) at p and (U', φ) at $f(p)$ (in our chosen smooth atlases), we want $\psi \circ f \circ \varphi^{-1}$ smooth at $\varphi(p)$

Note. Again: smoothness of f does not depend on the choices of charts at $p.f(p)$ provided we take charts from our smooth atlas

Definition. abstract smooth surfaces Σ_1 and Σ_2 are **diffeomorphic** if $\exists$ a homeomorphism

$$f : \Sigma_1 \to \Sigma_2$$

which is smooth and has smooth inverse

Remark. We often pass from a given smooth atlas for an abstract smooth surface Σ to the maximal "compatible" such atlas: i.e. we add to our atlas $\{(U_1, \varphi_i)_{i\in x}\}$ for Σ all charts (V, ψ) with the property that the transition maps are still all diffeomorphisms. (Technically use Zorn's Lemma)

Recall: if $V \subset \mathbb{R}^n$ and $V' \subset \mathbb{R}^m$ are open, then $f : V \to V'$ is smooth if it is infinitely differentiable.

Definition. If $Z \subset \mathbb{R}^n$ is an arbitrary subset, we say $f : Z \to \mathbb{R}^m$ (continuous) is **smooth** at $p \in Z$ if $\exists$ open $p \in B \subset \mathbb{R}^n$ and a smooth map $F : B \to \mathbb{R}^m$ s.t.

$$F|_{B\cap Z} = f|_{B\cap Z}$$

i.e. f is locally the restriction of a smooth map defined on an open set

Definition. If $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ are subsets, we say X and Y are **diffeomorphic** if $\exists f : X \to Y$ continuous s.t. f is a smooth homeomorphism with smooth inverse

Definition. A **smooth surface in** $\mathbb{R}^3$ is a subspace $\Sigma \subset \mathbb{R}^3$ s.t. $\forall p \in \Sigma, \exists$ an open set $p \in U \subset \Sigma$ s.t. U is diffeomorphic to an open set in $\mathbb{R}^2$

> $\forall p \in \Sigma, \exists$ open ball $p \in B \subset \mathbb{R}^3$ s.t. if $U = B \cap \Sigma$, and a smooth map $f : B \to V \subset \mathbb{R}^2$ oen s.t. $f|_U \to V$ is a homeomorphism and the inverse map $V \to U \subset \Sigma \subset \mathbb{R}^3$ is also smooth

Theorem. For a subset $\Sigma \subset \mathbb{R}^3$, the following are equivalent:

(i) Σ is a smooth surface in $\mathbb{R}^3$

(ii) Σ is locally the graph of a smooth function over one of the coordinate planes, i.e. $\forall p \in \Sigma \exists$ open $p \in B \subset \mathbb{R}^2$ and open $V \subset \mathbb{R}^2$ s.t.

$$\Sigma \cap B = \{(x, y, g(x, y)) : g : V \to \mathbb{R} \text{ smooth}\}$$

(or a graph over the xz or yz plane, locally)

(iii) Σ is locally cut out by a smooth function with nonzero derivative, i.e. $\forall p \in \Sigma, \exists$ open $p \in B \subset \mathbb{R}^3$ and $f : B \to \mathbb{R}$ smooth s.t.

$$\Sigma \cap B = f^{-1}(0) \text{ and } Df_x \neq 0 \forall x \in B$$

(iv) Σ is locally the image of an allowable parametrization, i.e. if $p \in \Sigma, \exists$ open $p \in U \subset \Sigma$ and $\sigma : V \to U$ ($V \subset \mathbb{R}^2, U \subset \Sigma \subset \mathbb{R}^3$ open) s.t. σ^2 is a homeomorphism and $D\sigma|_x$ has rank 2 $\forall x \in V$

Proof. (i) (ii) $\implies$ all others.

- If Σ is locally $\{x, y, g(x, y)\}$, then one gets a chart from projection π_{xy} which is smooth and defined on an open neighbourhood of points of Σ in its domain ((ii) $\implies$ (i))
- If Σ is locally $\{(x, y, g(x, y))\}$, it is locally cut out by $f(x, y, z) = z - g(x, y)$. Clearly $\frac{\partial f}{\partial z} \neq 0$ ((ii) $\implies$ (iii))
- The parametrisation $\sigma(x, y) := (x, y, g(x, y))$ is allowable as smooth and $\sigma_x = (1, 0, g_x)$, $\sigma_y = (0, 1, g_y)$ are linearly independent (and σ is injective) ((ii) $\implies$ (iv))

(ii) (i) $\implies$ (iv) is part of the definition of being a smooth surface in $\mathbb{R}^3$ and hence locally diffeomorphic to $\mathbb{R}^2$. [At $p \in \Sigma, \Sigma$ is locally diffeomorphic to $\mathbb{R}^2$ and the inverse of such a local diffeomorphism gives an allowable parametrisation]

(iii) (iii) $\implies$ (ii) was "illustrative example # 2" for the implicit function theorem

(iv) We'll show (iv) $\implies$ (ii), (i) and then done. Let $p \in \Sigma$ and $V \to \Sigma \subset \mathbb{R}^3$ $\sigma(0) = q \in U \subset \Sigma$. If $\sigma = (\sigma_1(u, v), \sigma_2(u, v), \sigma_3(u, v))$

$$D_\sigma = \begin{bmatrix} \frac{\partial \sigma_1}{\partial u} & \frac{\partial \sigma_1}{\partial v} \\ \frac{\partial \sigma_2}{\partial u} & \frac{\partial \sigma_2}{\partial v} \end{bmatrix}$$

so $\exists 2$ rows defining an invertible matrix our $\Theta \mapsto p$. Suppose the 1st 2 rows and let $pr := \pi_{xy}$ and consider $pr \circ \sigma : V \to \mathbb{R}^2$. Inverse function theorem (since $D(pr \circ \sigma)|_0$ isomorphism) says this is locally invertible. So Σ is locally a graph, i.e. (ii) holds. Moreover, if we let $\phi := pr \circ \sigma$

$$B(p, \delta) \ni (x, y, z) \mapsto \phi^{-1}(x, y)$$

Here $\phi^{-1} : W \to \Sigma$. This is locally defined, smooth and open in $\mathbb{R}^3$

Example. The unit sphere $S^2 \subset \mathbb{R}^3$ is $f^{-1}(0)$ for

$$f : \mathbb{R}^3 \to \mathbb{R} \quad (x, y, z) \mapsto x^2 + y^2 + z^2 - 1$$

If $p \in S^2$, $Df|_p \neq 0$, so f is a smooth surface in $\mathbb{R}^3$

Example. Surfaces of revolution:
Let $\gamma : [a, b] \to \mathbb{R}^2$ be a smooth map with image in the xz-plane

$$\gamma(t) = (f(t), 0, g(t))$$

Assume γ is injective, $\gamma'(t) \neq 0 \ \forall t,\ f > 0$. The associated surface of revolution has (allowable) parametrisation

$$\sigma(u, v) = (f(u) \cos v, f(u) \sin v, g(u))$$

$$(u, v) \in (a, b) \times (\theta, \theta + 2v) \quad v \in (0, 2\pi)$$

Note

$$\sigma_u = (f_u \cos v, f_u \sin v, g_u)$$

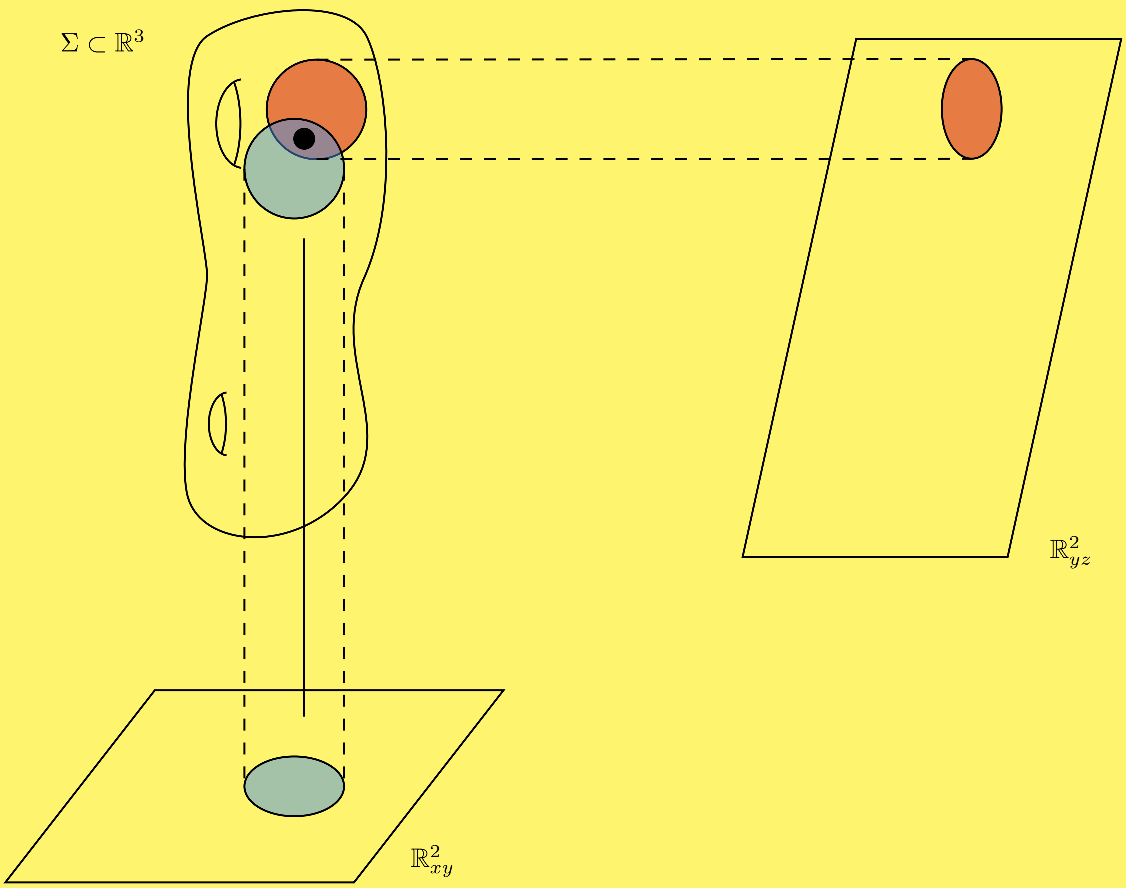
$$(\sigma_v = -f(u) \sin v, f(u) \cos v, 0)$$

and $\|\sigma_u + \sigma_v\| = f^2((f')^2 + (g')^2) \neq 0$
so $D\sigma$ has rank 2 and σ is injective on given domain, so allowable
diagram

Example. The orthogonal group $O(3)$ acts on S^2 by diffeomorphisms

Proof. $A \in O(3)$ defines an invertible linear (smooth) map $\mathbb{R}^3 \to \mathbb{R}^3$ preserving S^2, so inducted map on S^2 is a homeomorphism which is smooth in our definition. (globally so locally restriction of a smooth map).
Compare: action of Möb on $^{@} = \mathbb{C} \cup \{\infty\}$

Remark. (ii) above says that if Σ is a smooth surface in $\mathbb{R}^3$, each $p \in \Sigma$ belongs to a chart (U, φ), where φ is (the restriction of) $\pi_{xy}, \pi_{yz}, \pi_{xz}$ from $\mathbb{R}^3 \to \mathbb{R}^2$ (co-ordinate plane projection)



The transition map

$$(x, y) \mapsto (x, y, g(x, y)) \mapsto (y, g(x, y))$$

has inverse

$$(y, z) = (h(y, z), y, z) \mapsto (h(y, z), y)$$

All the transition maps between such charts involve projection maps and the smooth maps involved in Σ as a graph. This gives Σ the structure of an abstract sooth surface

Our next goal is to prove the Theorem. The non-trivial work comes from the inverse function theorem and its friends

Theorem (Inverse function theorem)**.** Let $U \subset \mathbb{R}^n$ and $f : U \to \mathbb{R}^n$ be continuously differentiable. Let $p \in U, f(p) = q$ and suppose $Df|_p$ is invertible. Then thre is an open neighbourhood V of q and a differentiable map

$$g : V \to \mathbb{R}^n, \quad g(q) = p$$

with image an open neighbourhood $U' \subset U$ of p, s.y. $f \circ g = \mathrm{id}_V$. If f is smooth, so is g

Remark. $Dg|_q = (Df|_p)^{-1}$ by Chain Rule.
Inverse function Theorem concerns $f : \mathbb{R}^n \to \mathbb{R}^n$ with $Df|_p$ -. If we have a map $f : \mathbb{R}^n \to \mathbb{R}^m$ where $n > m$, can ask about what to conclude if $Df|_p$ onto? $Df|_p = (\frac{\partial f_i}{\partial x_j})_{n\times m}$ having full rank means, permuting co-ordinates if necessary, I can assume last m columns linearly independent

Theorem (Implicit function Theorem)**.** Let $p = (x_0, y_0) \in U \subset \mathbb{R}^k \times \mathbb{R}^l$ and a map $f : U \to \mathbb{R}^l$ where $p \mapsto 0$ with $(\frac{\partial f_i}{\partial y_j})_{l\times l}$ is an isomorphism at p. Then there's an open neighbourhood $x_0 \in V \subset \mathbb{R}^k$ and a continuously differentiable map $g : V \to \mathbb{R}^l$, $x_0 \mapsto y_0$ s.t. if $(x, y) \in \cap(V - \mathbb{R}^l)$, then

$$f(x, y) = 0 \iff y = g(x)$$

Addendum: If f is smooth, so is g

Proof. Introduce $F : U \to \mathbb{R}^k \times \mathbb{R}^l$ with $(x, y) \mapsto (x, f(x, y))$ then

$$DF = \begin{bmatrix} I & * \\ 0 & \frac{\partial f_i}{\partial y_j} \end{bmatrix}$$

o $DF|_{(x_0,y_0}$ is isomorphism. So inverse function theorem says F is locally invertible near $F(x_0, y_0) = (x_0, f(x_0, y_0)) = (x_0, 0)$. Take a product open neighbourhood

$$(x_0, 0) \in V \times V' \quad V \subset \mathbb{R}^k, \quad 0 \in V^1 \subset \mathbb{R}^l$$

And the continuously differentiable inverse

$$G : V \times V' \to U' \subset U \subset \mathbb{R}^k \times \mathbb{R}^l$$

s.t. $F \times G = \mathrm{id}_{V\times V'}$. Write $G(x, y) = (\varphi(x, y), \psi(x, y))$ then

$$\begin{aligned} F \times G(x, y) &= (\varphi(x, y), f(\varphi(x, y), \psi(x, y))) \\ &= (x, y) \end{aligned}$$

So $\varphi(x, y) = x$. So G has form

$$(x, y) \mapsto (x, \psi(x, y))$$

And $f(x, \psi(x, y)) = y$ when $(x, y) \in V \times V'$ so $f(x, y) = 0 \iff y = \psi(x, 0)$. Define $g : V \to \mathbb{R}^l$, $x \mapsto \psi(x, 0) = y$ and this does what we want

Example. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be smooth, $f(x_0, y_0) = 0$ and suppose $\frac{\partial f}{\partial y}|_{(x_0,y_0)} \neq 0$. Then $\exists$ smooth $g : (x_0 - \varepsilon, x_0 + \varepsilon) \to \mathbb{R}$, $g(x_0) = y_0$ s.t.

$$f(x, y) = 0 \iff y = g(x)$$

for (x, y) in some open neighbourhood of (x_0, y_0)
Since $f(x, g(x)) = 0$

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \cdot g'(x) = 0$$

$$\implies g'(x) = -\frac{f_x}{f_y} \text{ noting } f_y \neq 0$$

(Idea: set $f(x, y) = 0$ is "implicitly" desribed in g, a function for which we have an integral expression)

Example. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be smooth

$$f(x_0, y_0, z_0) = 0$$

Let $\Sigma = f^{-1}(0)$ and assume $Df|_{(x_0,y_0,z_0)} \neq 0$. Permuting coordinates if necessary, $\frac{\partial f}{\partial z}|_{(x_0,y_0,z_0} \neq 0$. Then $\exists$ an open neighbourhood $(x_0, y_0) \in V \subset \mathbb{R}^2$ and a smooth $g : V \to \mathbb{R}$, $(x_0, y_0) \mapsto z_0$ s.t. in open $(x_0, y_0, z_0) \in U$, $f^{-1}(0) \cap U = \Sigma \cap U = \text{Graph}(g)$ i.e. is $\{(x, y, g(x, y)) : (x, y) \in V\}$

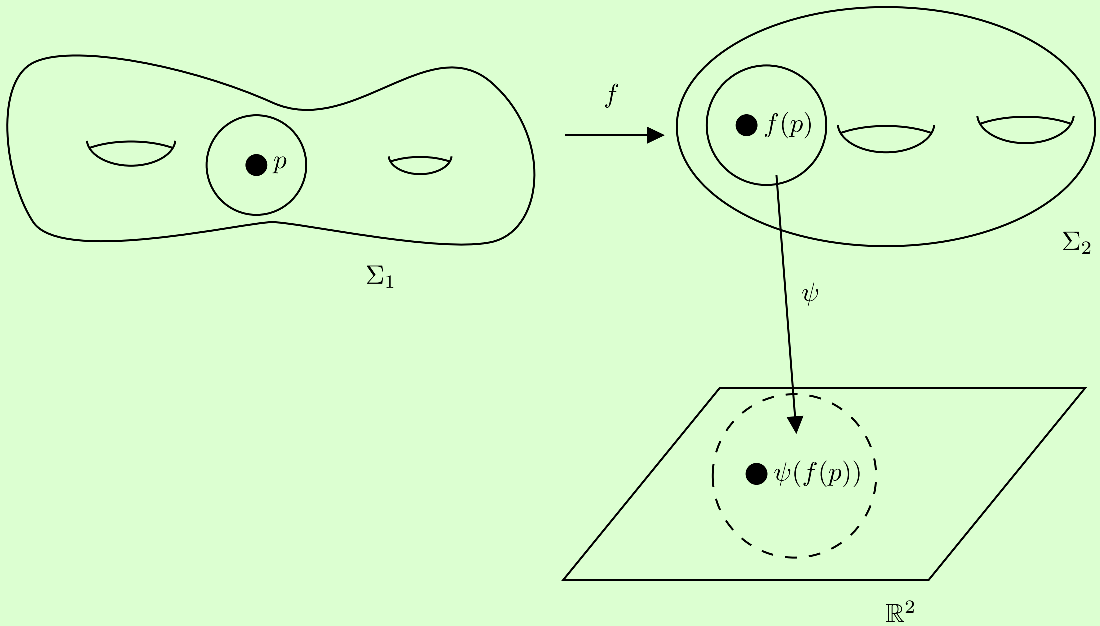
Note. If V, V' are open subsets of $\mathbb{R}^2$, and $f : V \to V'$ a diffeomorphism, then at $x \in V, Df|_x \in GL(2, \mathbb{R})$. Invertible as f is a diffeomorphism.
Let $GL^+(2, \mathbb{R}) \leq GL(2, \mathbb{R})$ be the subgroup of matrices of positive determinant. We say f is orientation-oreserving if $Df|_x \in GL^+(2, \mathbb{R})\ \forall x \in V$.

Definition. An abstract smooth surface Σ is **orientable** if it admits an atlas $\{(U_i, \varphi_i) : \cup U_i = \Sigma\}$ s.t. the transition maps are orientation-preserving diffeomorphisms of open subsets of $\mathbb{R}^2$.
A choice of such an atlas is an **orientation** of Σ and we say Σ is **oriented**

Remark. An oriented atlas (in this sense) belongs to a maximal compatible oriented smooth atlas

Lemma. If Σ_1 and Σ_2 are abstract smooth surfaces and they are diffeomorphic, then Σ, is orientable if and only if Σ_2 is orientable

Proof. Suppose $f : \Sigma_1 \to \Sigma_2$ is a diffomorphism and Σ_2 is orientable and equipped with an oriented smooth atlas.

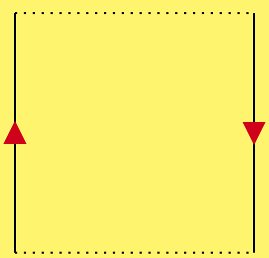


Let's consider the atlas on Σ_1 of charts of form $(f^{-1}U, \psi \circ f|_{f^{-1}U})$ where (U, ψ) is a chart at $f(p)$ in our atlas for Σ_2. A transition map between 2 such is exactly a transition map in the Σ_2 atlas. Put differently, it maximal smooth atlas, we already have for Σ_1 (an abstract smooth surface), w'll allow $(\tilde{U}, \tilde{\psi})$ exactly when for any chart (U, ψ) at $f(p)$ in the Σ_2 atlas, the map $\psi \circ f \circ \tilde{\psi}^{-1}$ preserves orientation.
If the atlas on Σ_2 was maximal as an oriented atlas, this recovers previous set of charts.

Remarks.

(i) There's no really sensible classification of all smooth or topological surfaces, e.g. $\mathbb{R}^2 \backslash Z$ for Z closed in $\mathbb{R}^2$ realises unvountably many homeomorphism types (Hard Exercise).
By contrast, copact smooth surfaces up to diffeomorphism are classified by (Euler characteristic, orientability)

(ii) There is a definition of orientation-preserving homeomorphism, which needs Algebraic Topology
The Möbius band is the surface



It turns out that an abstract smooth surface is orientable $\iff$ it contains no subsurface homeomorphic to the Möbius band. So we say a topological surface is orientable $\iff$ it contains no subsurface (open set) homeomorphic to a Möbius band, as an ad hoc definition

(iii) We can get other structures on an abstract smooth surface by asking for a smooth atlas s.t. if $\varphi_1 \varphi_2^{-1}$ is one of our transition maps, then $D(\varphi_1 \varphi_2^{-1})|_x \in G \le GL(2, \mathbb{R})$ e.g. $G = \{e\}$ leads to "Euclidean surfaces" (or $\{\pm I\}$)

(iv) $G = GL(1, \mathbb{C}) \le GL(2, \mathbb{R})$ is the theory of Riemann surfaces

Examples. (i) For S^2 with the atlas of two stereographic projections, we computed the transition map

$$(u, v) \mapsto \left(\frac{u}{u^2 + v^2}, \frac{v}{u^2 + v^2} \right) \text{ on } \mathbb{R}^2 \backslash \{0\}$$

and (check) this is orientation-preserving

(ii) For T^2, we exhibited an atlas s.t. all the transition maps were translations of $\mathbb{R}^2$ (restricted to small open discs)

We want to investigate orientability for surfaces in $\mathbb{R}^3$. Recall an affine subspace of a vector space is a translate of a linear subspace

Definition. Let Σ be a smooth surface in $\mathbb{R}^3$ and $p \in \Sigma$. Fix an allowable parametrisation

$$\sigma : V \to U \subset \Sigma \quad 0 \mapsto p \in U$$

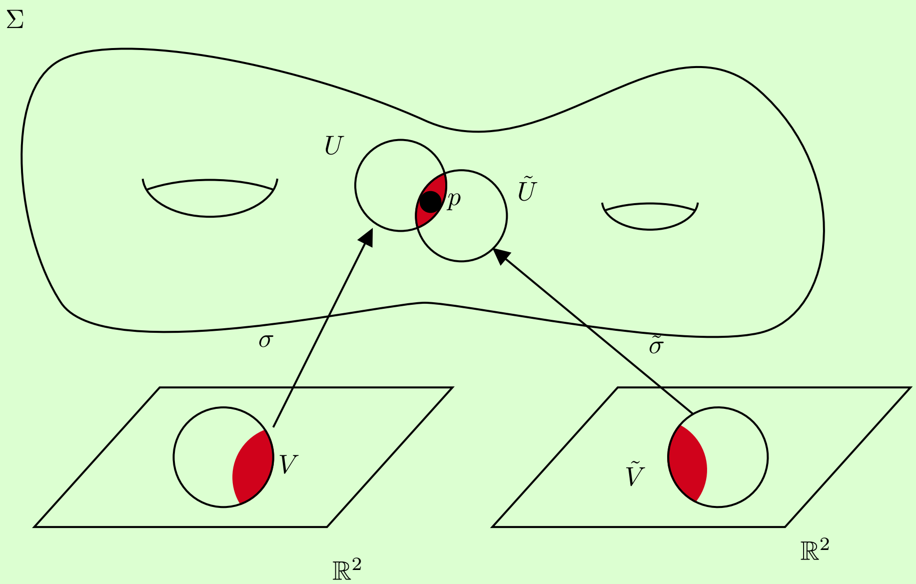
where V an open subset of $\mathbb{R}^2$.
Then the **tangent plane** T_p of Σ at p is $\text{image}(D_\sigma)_0 \subset \mathbb{R}^3$, a 2d vector subspace of $\mathbb{R}^3$. The **affine tangent plane** of Σ at p is $p + T_p\Sigma \subset \mathbb{R}^3$

Lemma. $T_p\Sigma$ is well-defined, i.e. independent of the choice of allowable parametrisation near p

Proof. (i) If

$$\sigma : V \to U \subset \Sigma \quad \sigma(0) = p$$
$$\tilde{\sigma} : \tilde{V} \to \tilde{U} \subset \Sigma \quad \tilde{\sigma}(0) = p$$

are two allowable parametrisations, near p



There's a transition map $\sigma^{-1} \circ \tilde{\sigma}$ is a diffeomorphism of open sets in $\mathbb{R}^2$. This means we can write

$$\tilde{\sigma} = \sigma \circ (\sigma^{-1} \circ \tilde{\sigma})$$

where $\sigma^{-1} \circ \tilde{\sigma}$ a diffeomorphsm so $D(\sigma^{01} \circ \tilde{\sigma})|_0$ is an isomorphism. This means image$(D\tilde{\sigma}|_0)$ and image$(D\sigma|_0)$ agree

(ii) Let $\gamma : (-\varepsilon, \varepsilon) \to \mathbb{R}$ be a smooth map s.t. γ has image inside Σ and $\gamma(0) = p$. Then we claim $\gamma'(0) \in T_p\Sigma$. Well, if $\sigma : V \to U \subset \Sigma$ is our allowable parametrisation near p, and ε is small enough so image$(\gamma) \subset U \subset \Sigma$, then we can write

$$\gamma(t) = \sigma(u(t), v(t))$$

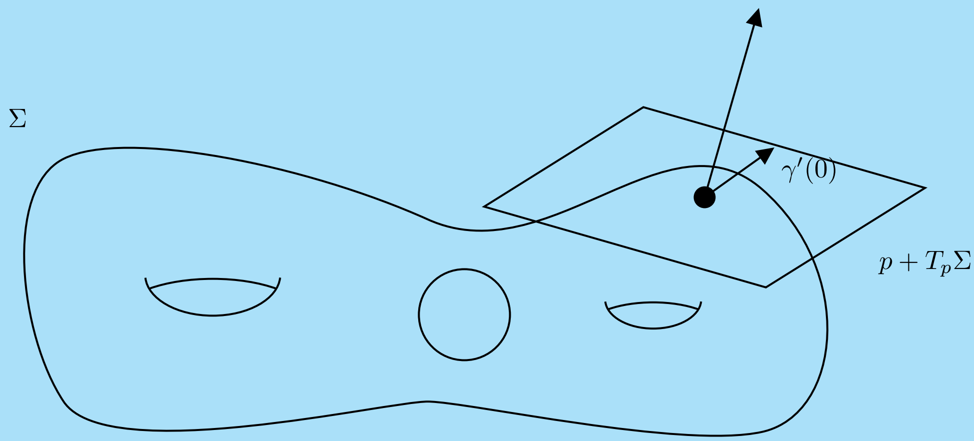
for smooth functions, $u, v : (-\varepsilon, \varepsilon) \to V$. Then

$$\gamma'(t) = \sigma_u \cdot u'(t) + \sigma_v \cdot v'(t) \in \text{image}(D\sigma)$$

This exhibits

$$T_p\Sigma = \mathbb{R}\{\gamma'(0) : \gamma \text{ is a smooth curve as above}\}$$

Definition. If Σ is a smooth surface in $\mathbb{R}^3$, and $p \in \Sigma$, the **normal direction** to Σ at p is just $(T_p\Sigma)^\perp$ (the Euclidean orthogonal complement to $T_p\Sigma$ w.r.t. $\langle \cdot, \cdot \rangle_{eucl}$)
So at each $p \in \Sigma$, there are two unit normal vectors



Affine tangent plane is the "best linear approximation" to Σ at p

Definition. A smooth surface in $\mathbb{R}^3$ is **two-sided** if it admits a continuous global choice of unit normal vector

Lemma. A smooth surface in $\mathbb{R}^3$ is orientable with its abstract smooth surface structure if and only if it is two-sided

Proof. Let $\sigma : V \to U \subset \Sigma$ be an allowable parametrisation for $U \subset \Sigma$ and say $\sigma(0) = p$ (V, U open). Define the positive unit normal w.r.t. σ at p to be the normal $n_\sigma(p)$ s.t.

$$\{\sigma_u, \sigma_v, n_\sigma(p)\} \text{ and } \{e_1, e_2, e_3\}$$

are related by a positive determinant change of basis matrix, where $\{e_1, e_2, e_3\}$ is the standard basis. Explicitly

$$n_\sigma(p) = \frac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|}$$

If $\tilde{\sigma}$ is another allowable parametrisation $\tilde{\sigma} : \tilde{V} \to \tilde{U} \subset \Sigma$, $0 \mapsto p$ and suppose Σ is orientable as an abstract surface and $\tilde{\sigma}$ belongs to the same oriented smooth atlas. So

$$\sigma = \tilde{\sigma} \circ \varphi$$

with $\varphi = \tilde{\sigma}^{-1} \circ \sigma$. Write $D\varphi|_0 = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$. Chain rule says

$$\sigma_u = \alpha\tilde{\sigma}_u + \gamma\tilde{\sigma}_v$$
$$\sigma_v = \beta\tilde{\sigma}_u + \delta\tilde{\sigma}_v$$

and

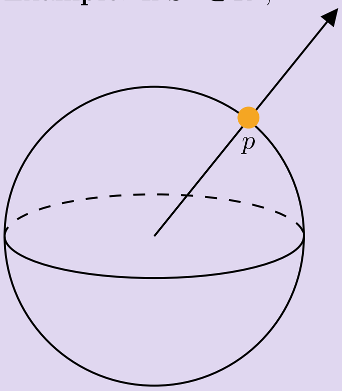
$$\sigma_u \times \sigma_v = \underbrace{\det(D\varphi|_0)}_{>0} \cdot \tilde{\sigma}_u \times \tilde{\sigma}_v \qquad (\dagger)$$

. Determinant > 0 as $\sigma, \tilde{\sigma}$ belong to the same oriented atlas. So the positive unit normal at p was intrinsic; it depends on the orientation fo Σ but not the choice of allowable parametrisation in the oriented atlas. And the expression $\sigma_u \times \sigma_v / \|\sigma_u \times \sigma_v\|$ is continuous, so Σ is 2-sided.
Conversely, if Σ is 2-sided and we have a continuous choice of normal vector, we can consider the subatlas of the natural smooth atlas s.t. allow a chart (U, φ) if the associated parametrisation $\varphi^{-1} = \sigma$ has $\{\sigma_u, \sigma_v, n\}$ is a positive basis for $\mathbb{R}^3$.
Same ($\dagger$) shows transition maps between such charts are orientation-preserving. So Σ is orientable.

Lemma. If Σ is a smooth surface in $\mathbb{R}^3$ and $A : \mathbb{R}^3 \to \mathbb{R}^3$ is a smooth map preserving Σ setwise, then $DA|_p : \mathbb{R}^3 \to \mathbb{R}^3$ sends $T_p\Sigma$ to $T_{A(p)}\Sigma$ whenever $p \in \Sigma$.
Suppose $\gamma : (-\varepsilon, \varepsilon) \to \mathbb{R}^3$ is a smooth map s.t. image$(\gamma) \subset \Sigma$ and $\gamma(0) = p$. (Recall $T_p\Sigma$ is spanned by $\gamma'(0)$ for such γ.) Now $A \circ \gamma : (-\varepsilon, \varepsilon) \to \mathbb{R}^3$ also has image in Σ and

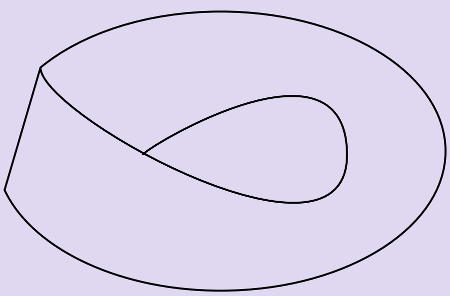
$$DA|_{\gamma(0)} \circ D\gamma|_0 = DA|_p(\underbrace{\gamma'(0)}_{\in T_p\Sigma})$$
$$= \underbrace{D(A \circ \gamma)|_0}_{\in T_{A(p)}\Sigma}$$

Example. If $S^2 \subset \mathbb{R}^3$,

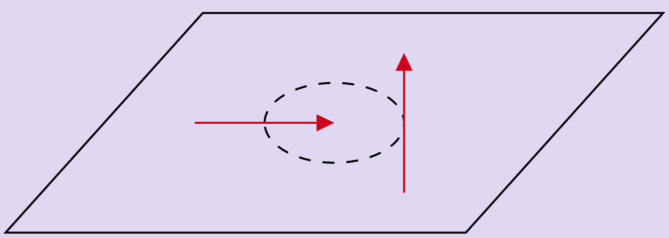


Then the normal line $(T_p\Sigma)^\perp = (T_pS^2)^\perp = \mathbb{R}\langle p \rangle$ is the line through p. (Since SO_3 acts transitively on S^2, check this at the north pole.) So there is at each point an outwards-pointing normal $n(p)$ (s.t. $p \notin \mathbb{R}_{\geq 0}n(0) + p$). So S^2 is 2-sided, and so orientable

Example (A Möbius band)**.** Let $\sigma(t, \theta) = ((1 - t\sin\theta/2)\cos\theta, (1 - t\sin\theta/2)\sin\theta, t\cos\theta/2)$ where $(t, \theta) \in V_1 = \{t \in (-1/2, 1/2), \theta \in (0, 2\pi)\}$ or $V_2 = \{t \in (-1/2, 1/2, \theta \in (-\pi, \pi))$



We start with the unit circle $x^2 + y^2 = 1$ in the xy-plane ($t = 0$), and take an open interval of length 1, and this line rotates as you move around the circle s.t. has rotated by $\theta/2$ at point θ

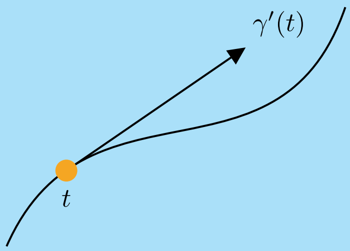


Check: if σ_i is σ on V_i then σ_i is allowable (smooth, injective, $D\sigma_i$ injective)
Also: $\sigma_t \times \sigma_\theta = (-\cos\theta\cos\theta/2, -\sin\theta\cos\theta/2, -\sin\theta/2) =: n_\theta$ (already unit length).
As $\theta \to 0^+$, $n_\theta \to (-1, 0, 0)$, as $\theta \to 2\pi_-, n_\theta \to (+1, 0, 0)$ and so this surface is not 2-sided

1 Geometry of Surfaces in $\mathbb{R}^3$ - Length, Area and Curvature

1.1 Length

Definition. Let $\gamma : (a, b) \to \mathbb{R}^3$ smooth.



The **length** of γ is:

$$L(\gamma) := \int_a^b \|\gamma'(t)\| \,\mathrm{d}t$$

Note. If $s : (A, B) \to (a, b)$ is monotone increasing, and let $\tau(t) = \gamma(s(t))$, then

$$L(\tau) = \int_A^B \|\tau'(t)\| \,\mathrm{d}t = \int_A^B \|\gamma(s(t))\| \underbrace{|s'(t)|}_{\geq 0} \,\mathrm{d}t = L(\gamma)$$

Lemma. If $\gamma : (a, b) \to \mathbb{R}^3$ is continuously differentiable and $\gamma'(t) \neq 0 \; \forall t$, then γ can be parametrised by arc-length (i.e. in a parameter s s.t. $|\gamma'(s)| = 1 \; \forall s$)

Proof. Exercise.

Let Σ be a smooth surface in $\mathbb{R}^3$ and let $\sigma : V \to U \subset \Sigma$ allowable. If $\gamma : (a, b) \to \mathbb{R}^3$ is smooth and has image $\subset U$ then $\exists u(t), v(t) : (a, b) \to V$ s.t. $\gamma(t) = \sigma(u(t), v(t))$

$$\implies \gamma'(t) = \sigma_u u'(t) + \sigma_v v'(t)$$

$$\implies \|\gamma'(t)\|^2 = Eu'(t)^2 + 2Fu'(t)v'(t) + Gv'(t)$$

where

$$\begin{aligned} E &= \langle \sigma_u, \sigma_u \rangle = \|\sigma_u\|^2 \\ F &= \langle \sigma_u, \sigma_v \rangle = \langle \sigma_v, \sigma_u \rangle \\ G &= \langle \sigma_v, \sigma_v \rangle = \|\sigma_v\|^2 \end{aligned}$$

are smooth functions on V, and $\langle \cdot, \cdot \rangle$ is usual Euclidean inner product. Note E, F, G depend only on σ, NOT on γ

Definition. The first fundamental form (FFF) of Σ in the parametrisation of σ is the expression

$$E\,\mathrm{d}u^2 + 2F\,\mathrm{d}u\,\mathrm{d}v + G\,\mathrm{d}v^2$$

The notation is designed to remind you that if $\gamma : (a,b) \to \mathbb{R}^3$ lands in $\sigma(V) = U \subset \Sigma$, then

$$\mathrm{length}(\gamma) = \int_a^b \sqrt{Eu'(t)^2 + 2Fu'(t)v'(t) + Gv'(t)}\,\mathrm{d}t$$

where $\gamma(t) = \sigma(u(t), v(t))$

Remark. Really, the Euclidean inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^3$ gives me an inner product on $T_p\Sigma \subset \mathbb{R}^3$. If we pick a parametrisation σ, $T_p\Sigma = \mathrm{image}(D\sigma|_p) = \langle \sigma_u, \sigma_v \rangle_{\mathbb{R}\text{ span}}$. FFF is a symmetric bilinear form on $T_p\Sigma$ (varying smoothly in p) expressed in a basis coming from the parametrisation σ, so it is often helpful to consider $\begin{bmatrix} E & F \\ F & G \end{bmatrix}$

Example.
(i) The plane $\mathbb{R}^2_{xy} \subset \mathbb{R}^3$ has parametrisation $\sigma(u,v) = (u,v,0)$ so $\sigma_u = (1,0,0); \sigma_v = (0,1,0)$, FFF: $\mathrm{d}u^2 + \mathrm{d}v^2$
(ii) Or in polar coordinates $\sigma(r,\theta) = (r\cos\theta, r\sin\theta, 0)$ for $r \in (0,\infty)$, $\theta \in (0, 2\pi)$. Now $\sigma_r = (\cos\theta, \sin\theta, 0)$ and $\sigma_\theta = (-r\sin\theta, r\cos\theta, 0)$ and FFF $\mathrm{d}r^2 + r^2\mathrm{d}\theta^2$

Definition. Let Σ, Σ' be smooth surfaces in $\mathbb{R}^3$. We say Σ and Σ' are isometric if there is a diffeomorphism

$$f : \Sigma \to \Sigma'$$

s.t. for every smooth curve $\gamma : (a,b) \to \Sigma$

$$\mathrm{length}_\Sigma(\gamma) = \mathrm{length}_{\Sigma'}(f \circ \gamma)$$

Example. If $\Sigma' = f(\Sigma)$ where $f : \mathbb{R}^3 \to \mathbb{R}^3$ is a "rigid motion", i.e.

$$f : v \mapsto Av + b, \quad A \in O(3), \quad b \in \mathbb{R}^3$$

(so f preserves $\langle \cdot, \cdot \rangle_{eucl}$ on $\mathbb{R}^3$), then $f : \Sigma \to \Sigma'$ is an isometry

Note. In the definition, imortantly, f is only a priori defined on Σ, not all of $\mathbb{R}^3$.
Often, we are really intereted in a local statement.

Definition. We say Σ, Σ' are locally isomorphic (near point $p \in \Sigma$ and $q \in \Sigma'$) if $\exists$ open neighbourhoods $p \in U \subset \Sigma$ and $q \in U' \subset \Sigma'$ which are isometric

Lemma. Smooth surfaces Σ, Σ' in $\mathbb{R}^3$ are locally isometric near $p \in \Sigma$ and $q \in \Sigma'$ if and only if there exist allowable parametrisations

$$\begin{aligned} \sigma &: V \to U \subset \Sigma \\ \sigma' &: V \to U' \subset \Sigma' \end{aligned}$$

with $p \in U, q \in U'$, for which the FFF's are equivalent (equal as functions on V)

Proof. We know (by definition) that the FFF of σ determines lengths of all curves on Σ inside $\sigma(V) = U$.
We will show lengths of curves determine the FFF of a parametrisation.
Given $\sigma : V \to U \subset \Sigma$, w.l.o.g. $V = B(0, \delta)$ for some $\delta > 0$, with $\sigma(0) = p$, and consider

$$\gamma_\varepsilon : [0, \varepsilon] \to U \subset \Sigma, \quad \varepsilon < \delta, \quad t \mapsto \sigma(t, 0)$$

Then

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\varepsilon} L(\gamma_\varepsilon) &= \frac{\mathrm{d}}{\mathrm{d}\varepsilon} \int_0^\varepsilon \sqrt{E(t,0)}\,\mathrm{d}t \\ &= \sqrt{E(\varepsilon, 0)} \end{aligned}$$

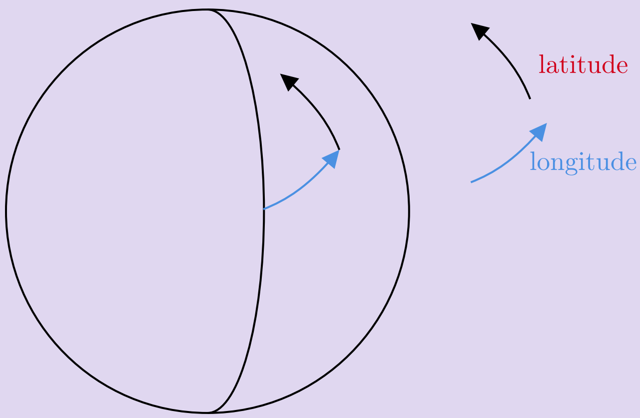
so $\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0} L(\gamma_\varepsilon) = \sqrt{E(0,\varepsilon)}$ so lengths of curves γ_ε determine E at p.
Analagously $\chi_\varepsilon : [0, \varepsilon] \to \Sigma,\ t \mapsto \sigma'(0, t)$ and we find their lengths determine $\sqrt{E(0, \varepsilon)}$ then $\lambda_\varepsilon : [0, \varepsilon] \to \Sigma, t \mapsto \sigma(t, t)$ determines $\sqrt{(E = 2F + g)(0, 0)}$, so (knowing E, G) we get F

Examples. (i) The sphere $\{x^2+y^2+z^2=a^2\} \in \mathbb{R}^3$ has an open set with allowable parametrisation

$$\sigma(u,v) = (a\cos u\cos v, a\cos u \in v, a\sin u)$$

$$u = \text{ latitude } = (-\pi, \pi)$$

$$v = \text{ longitude } = (0, 2\pi)$$



(parametrises the complement of a 1/2 great circle)

$$\sigma_u = (-a\sin u\cos v, -a\sin u\sin v, a\cos u)$$
$$\sigma_v = (-a\cos u\sin v, a\cos u\cos v, 0)$$

$$E = \sigma_u \cdot \sigma_v = a^2, \quad F = 0, \quad G = a^2\cos^2 u$$

FFF: $a^2\,\mathrm{d}u^2 + a^2\cos^2(u)\,\mathrm{d}v^2$

(ii) Surface of revolution: take $q(t) = (f(t), 0, g(t))$ in xz-plane and rotate about z-acis

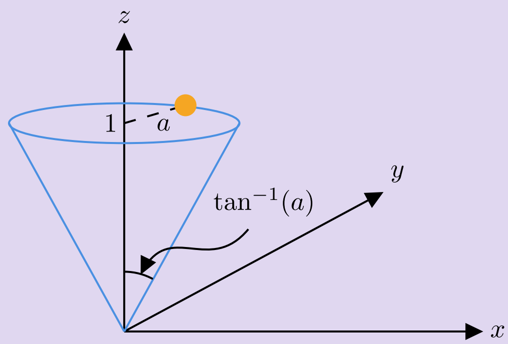
$$\sigma(u,v) = (f(u)\cos v, f(u)\sin v, g(u))$$

$$\sigma_u = (f_u\cos v, f_u\sin v, g_u)$$

$$\sigma_v = (-f\sin v, f\cos v, 0)$$

FFF: $(f_u^2 + g_u^2)\,\mathrm{d}u^2 + f^2\,\mathrm{d}v^2$

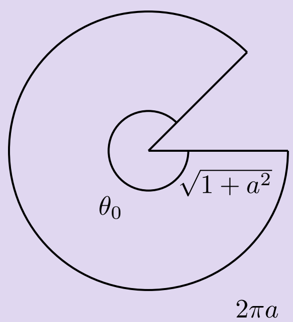
Examples (continued). (iii) Cone:



For $u > 0$ and $v \in (0, 2\pi)$

$$\sigma(u, v) = (au \cos v, au \sin v, u)$$

parametrises complement of one line on the cone
FFF: $(1 + a^2)\,\mathrm{d}u^2 + a^2u^2\,\mathrm{d}v^2$
If we cut open the cone and unfold it, we get a plane sector:



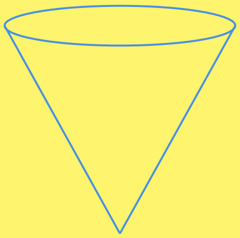
$\theta_0 = \frac{2\pi a}{\sqrt{1+a^2}}$

Parametrise this plane sector by

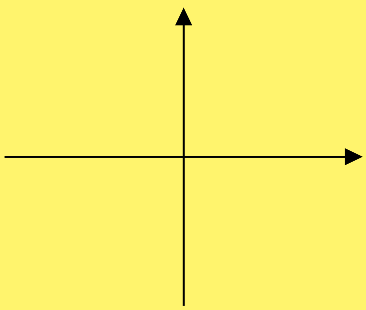
$$\sigma(r, \theta) = (\sqrt{1 + a^2}r \cos\left(\frac{a\theta}{\sqrt{1 + a^2}}\right), \sqrt{1 + a^2}r \sin\left(\frac{a\theta}{\sqrt{1 + a^2}}\right), 0)$$

$r > 0, \theta \in (0, \theta_0)$. We can check that we have:
FFF: $(1 + a^2)\,\mathrm{d}r^2 + r^2a^2\,\mathrm{d}\theta^2$
So the cone is locally isometric to the plane

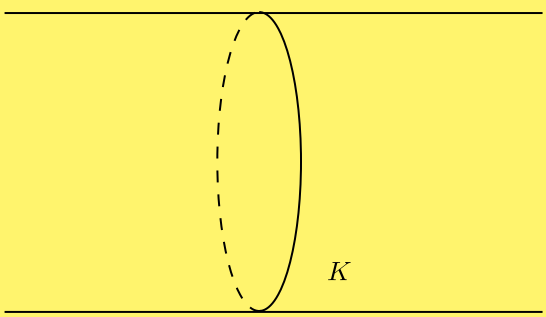
Note. The cone



and the plane



cannot be globally isometric, since these two spaces are not homeomorphic
(The cone $\cong_{C^0}$ $S^1 \times \mathbb{R}$, in the plane $\mathbb{R}^2$, every compact set K lies inside a larger compact set $K = \overline{V(0,N)} N >> 0$, s.t. $\mathbb{R}^2 \backslash K$ is connected. But on $S^1 \times \mathbb{R}$



for any compact $K' \supset K$, $(S^1 \times \mathbb{R}) \backslash K'$ is disconnected)

Let Σ be a smooth surface in $\mathbb{R}^3$, $p \in \Sigma$, and take two allowable parametrisations near p

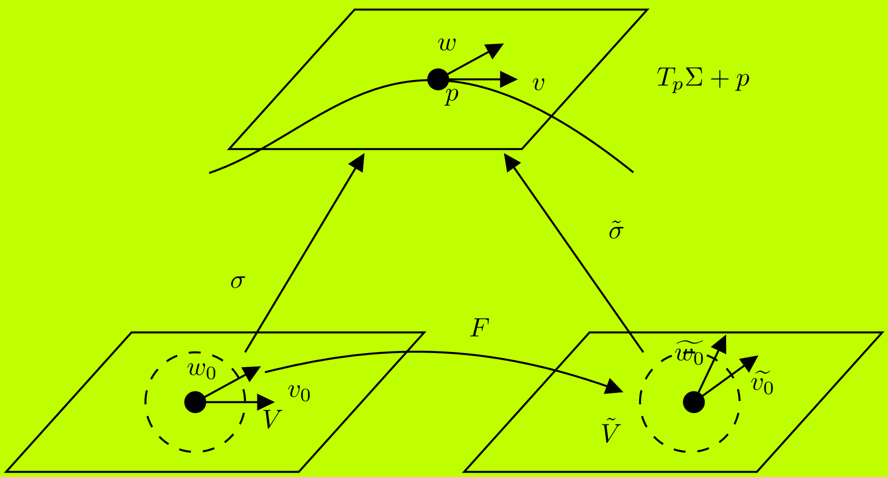
$$\sigma : V \to U \subset \Sigma, \quad \sigma(0) = p$$
$$\tilde{\sigma} : \tilde{V} \to U \subset \Sigma, \quad \tilde{\sigma}(0) = p$$

We have a transition map $F = \tilde{\sigma}^{-1} \circ \sigma : V \to V'$ (diffeomorphism) of open sets of $\mathbb{R}^2$
We have FFFs

$$\begin{bmatrix} E & F \\ F & G \end{bmatrix} \text{ for } \sigma, \quad \begin{bmatrix} \tilde{E} & \tilde{F} \\ \tilde{F} & \tilde{G} \end{bmatrix} \text{ for } tilde\sigma$$

Lemma.

$$\begin{bmatrix} E & F \\ F & G \end{bmatrix} = (DF)^T \begin{bmatrix} \tilde{E} & \tilde{F} \\ \tilde{F} & \tilde{G} \end{bmatrix} DF$$



$D\sigma : v_0 \mapsto v,\ w_0 \mapsto w$

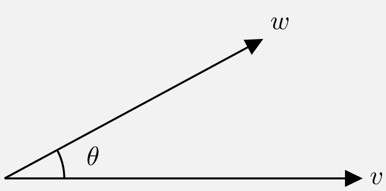
Proof.

$$\begin{bmatrix} E & F \\ F & G \end{bmatrix} = \begin{bmatrix} \sigma_u\sigma_u & \sigma_u\sigma_v \\ \sigma_v\sigma_u & \sigma_v\sigma_v \end{bmatrix}$$
$$= (D\sigma)^T D\sigma$$

Now $\sigma = \tilde{\sigma} \circ F$ amd result follows

Remark. Later, we will define "FFF" (called abstract Riemannian metrics) on abstract smooth surfaces by making local definitions on charts and insisting they transform in this way

If $v, w \in \mathbb{R}^3, v \cdot w = |v||w| \cos\theta$



$\subseteq \mathbb{R}^2 \sigma \langle v, w \rangle_{\mathbb{R}\text{ span}}$

Considering $v, w \in T_p\Sigma, \quad \cos\theta = \frac{v \cdot w}{|v||w|}$. If we have an allowable parametrisation σ for Σ near p and $D\sigma|_0(v_0) = v$, $\sigma(0) = p$, and $D\sigma|_0(w_0) = w$ then

$$\cos\theta = \frac{I(v_0, w_0)}{\sqrt{I(v_0, w_0)}\sqrt{I(w_0, v_0)}}$$

where I denotes the FFF of σ at 0 (so $I(v_0, w_0) = v_0^T \begin{bmatrix} E & F \\ F & G \end{bmatrix} w_0$)

Lemma. If Σ is a smooth surface in $\mathbb{R}^3$ and $\sigma : V \to U \subset \Sigma$ is an allowable parametrisation, then σ is conformal (preserves angles) exactly when $E = G,\ F = 0$

Proof. Consider curves

$$\gamma : t \mapsto (u(t), v(t)) \text{ in } V$$
$$\tilde{\gamma} : t \mapsto (\tilde{u}(t), \tilde{v}(t))$$

with $\gamma(0) = \tilde{\gamma}(0) = 0 \in V$ and

$$\sigma : V \to U \subset \Sigma \text{ s.t. } \sigma(0) = p \in \Sigma$$

Then the curves $\sigma \circ \gamma$ and $\sigma \circ \tilde{\gamma}$ meet at angle θ on Σ, where

$$\cos\theta = \frac{E\dot{u}\dot{\tilde{u}} + F(\dot{u}\dot{\tilde{v}} + \dot{v}\dot{\tilde{u}}) + G(\dot{v}\dot{\tilde{v}})}{(E\dot{u}^2 + F\dot{u}\dot{v} + G\dot{v})^{1/2}(E\dot{\tilde{u}}^2 + 2F\dot{\tilde{u}}\dot{\tilde{v}} + G\dot{\tilde{v}}^2)^{1/2}}$$

If σ is conformal and $\gamma(t) = (t, 0)$, $\tilde{\gamma}(t) = (0, t)$, meeting at $\pi/2$ in V, they meet at $\pi/2$ on Σ, and then $F = 0$.
Similarly, if $\gamma(t) = (t, t)$ and $\tilde{\gamma}(t) = (t, -t)$ these are orthogonal in V, so images are orthogonal on Σ, and then $E = G$.
Conversely, if σ is s.t.

$$E = G \text{ and } F = 0$$

then wrt σ, the FFF of Σ is of the form $\rho(\,\mathrm{d}u^2 + \mathrm{d}v^2)$ for $\rho(= E) : V \to \mathbb{R}$ a smooth function. i.e. the FFF is a pointwise rescaling of the Euclidean fundamental form $\mathrm{d}u^2 + \mathrm{d}v^2$. But rescaling doesn't change angles

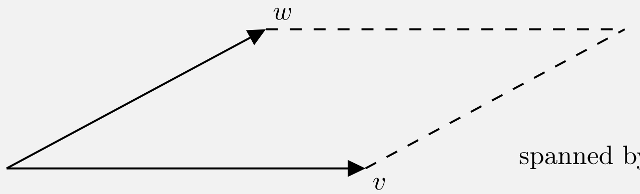
Remarks.

(i) Historically important for maps, cf ES2

(ii) Existence of conformal charts is closely connected to "Riemann surfaces", topological surfaces locally modelled on $\mathbb{C}$

1.2 Area

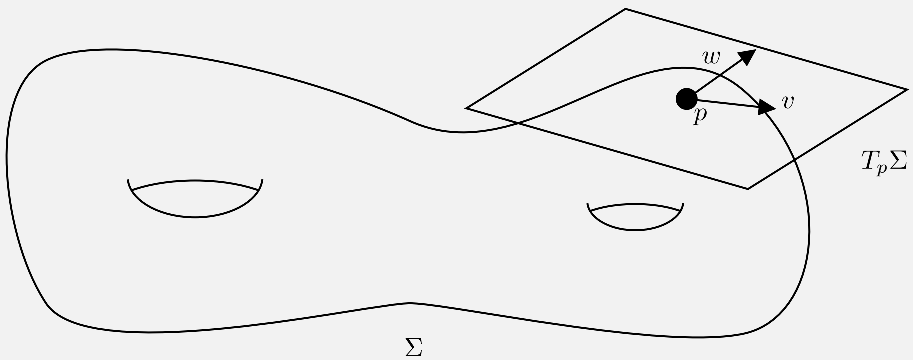
Recall a parallelogram



spanned by vectors v and w

has area

$$(\langle v \times w, v \times w \rangle)^{1/2} = (\langle v, v \rangle \langle w, w \rangle - \langle v, w \rangle^2)^{1/2}$$



Suppose we have an allowable parametrisation

$$\sigma : V \to U \subset \Sigma, \quad \sigma(0) = p$$

and consider $\sigma_u, \sigma_v \in T_p\Sigma$. These span a parallelogram in $T_p\Sigma$ which we think of as an "infinitesimal" parallelogram on Σ of area

$$(\langle \sigma_u, \sigma_u \rangle \langle \sigma_v, \sigma_v \rangle - \langle \sigma_u, \sigma_v \rangle^2)^{1/2} = \sqrt{EG - F^2}$$

Definition. Let Σ be a smooth surface in $\mathbb{R}^3$ and $\sigma : V \to U \subset \Sigma$ allowable. Then $\text{Area}(U) = \int_V \sqrt{EG - F^2}\, \mathrm{d}u\, \mathrm{d}v$

Note. Suppose $\sigma : V \to U$, $\tilde{\sigma} : \tilde{V} \to U$ are both allowable. So $\tilde{\sigma} = \sigma \circ \varphi$, $\varphi = \sigma^{-1} \circ \tilde{\sigma}$ transition map and

$$\begin{bmatrix} \tilde{E} & \tilde{F} \\ \tilde{F} & \tilde{G} \end{bmatrix} = (D\tilde{\sigma})^T D\tilde{\sigma} = (D\varphi)^T \begin{bmatrix} E & F \\ F & G \end{bmatrix} D\varphi$$

So

$$\sqrt{\tilde{E}\tilde{G} - \tilde{F}^2} = |\det(D\varphi)| \cdot \sqrt{EG - F^2}$$

Now the change-of-variables formula for integration, and fact that $\varphi : \tilde{V} \to V$ is a diffeomorphism, shows

$$\int_{\tilde{V}} \sqrt{\tilde{E}\tilde{G} - \tilde{F}^2}\, \mathrm{d}\tilde{u}\, \mathrm{d}\tilde{v} = \int_V \sqrt{EG - F^2}\, \mathrm{d}u\, \mathrm{d}v$$

So $\text{Area}(U)$ is intrinsic and well-defined.
ALSO: we can compute $\text{area}(U)$ for any open $U \subset \Sigma$, not necessarily lying in a single parametrisation, by covering it in pieces which do lie in sets $\sigma(V)$.

Example. Consider a graph

$$\Sigma = \{(u, v, f(u, v)) : (u, v) \in \mathbb{R}^2,\ f : \mathbb{R} \to \mathbb{R} \text{ smooth}\}$$

We take the obvious parametrisation

$$\sigma : (u, v) \mapsto (u, v, f(u, v))$$

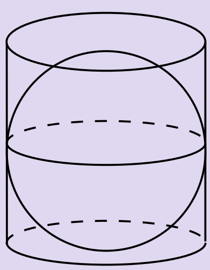
$$\sigma_u = (1, 0, f_u)$$
$$\sigma_v = (0, 1, f_v)$$

and $\sqrt{EG - F^2} = \sqrt{1 + f_u^2 + f_v^2}$.
If $U_R \subset \Sigma$ is part of the graph lying over $B(0, R) \subset \mathbb{R}^2$,

$$\begin{aligned} \text{Area}(U_R) &= \int_{B(0,R)} \sqrt{1 + f_U^2 + f_v^2}\, \mathrm{d}u^2\, \mathrm{d}v^2 \\ &\geq \pi R^2 \end{aligned}$$

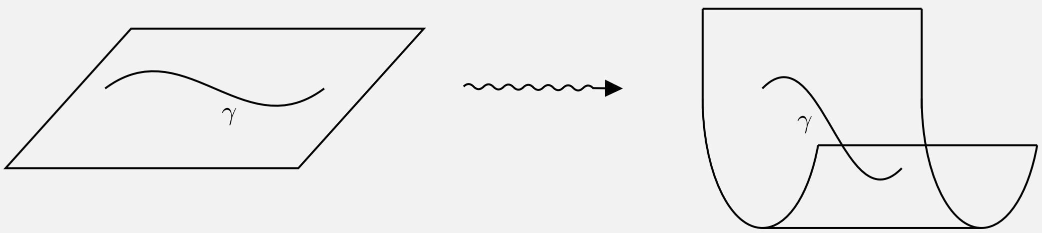
With equality only when $f_u = 0 = f_v$ throughout $B(0, R$ i.e. only when $U_R \leq)$ plane ($z = \text{const.}$)
So projection from Σ to $\mathbb{R}^2_{xy}$ is not area preseving unless Σ is plane parallel to $\mathbb{R}^2_{xy}$

Example. Contrast above with the following (Archimedes)



The radial projection from S^2 to the cylinder is area-preserving. (cf ES2)

Take a plane and bend it:



This won't change lengths and areas, but we clearly change the way the surface sits in $\mathbb{R}^3$. We measure that change by considering how Σ deviated from its own tangent planes.
Let $\sigma : V \to U \subset \Sigma$ be allowable. Use Taylor's theorem

$$\sigma(u+h, v+l) = \sigma(u,v) + h\sigma_u(u,v) + l\sigma_v(u,l) + \frac{1}{2}(h^2\sigma_{uu}(u,v) + 2hl\sigma_{uv}(u,v) + l^2\sigma_{vv}(u,v)) + O(h^3, l^3)$$

where h, l are small so (u, v) and $(u+h, v+l) \in V$.
Recall, if $p = \sigma(u,v)$

$$T_p\Sigma = \langle \sigma_u, \sigma_v \rangle$$

So the distance from $\sigma(u+h, v+l)$ to $T_p\Sigma + p$, measured orthogonally, is given by projection to the normal direction:

$$\langle n, \sigma(u+h, v+l) - \sigma(u,v) \rangle = \frac{1}{2}(h^2\langle n, \sigma_{uu}\rangle + 2hl\langle n, \sigma_{uv}\rangle + l^2\langle n, \sigma_{vv}\rangle) + O(h^3, l^3)$$

Definition. The **second fundamental form** of the smooth surface Σ in $\mathbb{R}^3$ in the allowable parametrisation σ is the quadratic form

$$L\,\mathrm{d}u^2 + 2M\,\mathrm{d}u\,\mathrm{d}v + N\,\mathrm{d}v^2$$

where

$$L = \langle n, \sigma_{uu}$$
$$M = \langle n, \sigma_{uv}\rangle$$
$$N = \langle n, \sigma_{vv}\rangle$$

where (as usual) $n = \frac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|}$ the positive unit normal

Note. Again, $\begin{bmatrix} L & M \\ M & N \end{bmatrix}$ defines a quadratic form on $T_p\Sigma$, varying smoothly in p

Lemma. Let V be connected and $\sigma : V \to U \subset \Sigma$ an allowable parametrisation s.t. 2nd FF vanishes identically w.r.t. σ. Then U lies in an affine plane $\mathbb{R}^2 \subset \mathbb{R}^3$

Proof. Recall
$$\langle n, \sigma_u \rangle = 0 = \langle n, \sigma_v \rangle$$

$$\begin{aligned} \implies \langle n_u, \sigma_u \rangle + \langle n, \sigma_{uu} = 0 \\ \langle n_v, \sigma_v \rangle + \langle n, \sigma_{vv} = 0 \\ \langle n_v, \sigma_u \rangle + \langle n, \sigma_{uv} = 0 \end{aligned}$$

So in the second fundamental form

$$\begin{aligned} L &= \langle n, \sigma_{uu} \rangle = -\langle n_u, \sigma_v \rangle \\ M &= \langle n, \sigma_{uv} \rangle = -\langle n_v, \sigma_u = -\langle n_u, \sigma_v \rangle \\ N &= \langle n, \sigma_{vv} \rangle = -\langle n_v, \sigma_v \rangle \end{aligned}$$

So if 2nd FF vanishes then n_u is orthogonal to σ_u, σ_v and $\langle n, n \rangle = 1 \implies \langle n, n_u \rangle = 0 \implies n_u$ orthogonal to n.
So n_u orthogonal to $\{\sigma_u, \sigma_v, n\} \implies n_u \equiv 0$ and similarly $n_v \equiv 0$, so n is constant by MVT.
So U is constrained in the affine hyperplane $\{\mathbf{x} \cdot \mathbf{n} = \text{constant}\}$

Remark. Recall the FFF is a non-degenerate symmetric bilinear form on $T_p\Sigma$.
Contrast with 2nd FF

Remark. The FFF in parametrisation σ was

$$(D\sigma)^T D\sigma \cdot \begin{bmatrix} E & G \\ F & G \end{bmatrix} \cdot \begin{bmatrix} \sigma_u \cdot \sigma_v & \sigma_u \sigma_v \\ \sigma_v \sigma_u & \sigma_v \sigma_v \end{bmatrix}$$

Analagously the 2nd FF

$$(Dn)^T D\sigma = \begin{bmatrix} L & M \\ M & N \end{bmatrix} = -\begin{bmatrix} n_u \cdot \sigma_u & n_u \cdot \sigma_v \\ n_v \cdot \sigma_u & n_v \cdot \sigma_v \end{bmatrix}$$

(using the alternative expressions for L, M, N derived in the previous proof).
So if $\sigma : V \to \Sigma$ and $\tilde{\sigma} : \tilde{V} \to \Sigma$ are 2 allowable parametrisations for $U \subset \Sigma$ with transition map

$$\varphi : \tilde{V} \to_{\cong_{C^\infty}} V \quad \varphi = \sigma^{-1} \circ \sigma$$

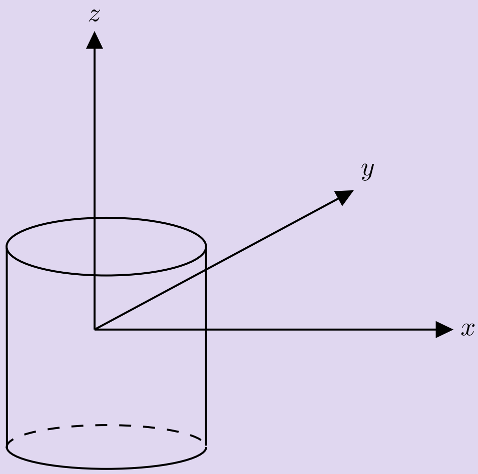
then
$$\begin{bmatrix} \tilde{L} & \tilde{M} \\ \tilde{M} & \tilde{N} \end{bmatrix} = \pm (D\varphi)^T \begin{bmatrix} L & M \\ M & N \end{bmatrix} D\varphi$$

where we get a minus sign if φ is orientation reversing. Here note that

$$n_{\sigma \circ \varphi}|_{(\tilde{u}, \tilde{v})} = \pm\, n_\sigma|_{\varphi(\tilde{u}, \tilde{v})}$$

for $(\tilde{u}, \tilde{v}) \in \tilde{V}$. With sign depending on $\det(D\varphi)$ (assume $V, \tilde{V}$ connected)

Example.



The cylinder has allowable parametrisation

$$\sigma(u,v) = (a\cos u, a\sin u, v), \quad u \in (0,2\pi), \quad v \in \mathbb{R}$$

Note $\sigma_{uv} = 0 = \sigma_{vv}$

$$\implies M = N = 0$$

Check 2nd FF is $\begin{bmatrix} -a & 0 \\ 0 & 0 \end{bmatrix}$ i.e. $-a\,\mathrm{d}u^2$

Next goal: "intrinsic" description of 2nd FF for an oriented smooth surface in $\mathbb{R}^3$

Definition. Let $\Sigma \subset \mathbb{R}^3$ be a smooth oriented surface. The **Gauss map**

$$n : \Sigma \to S^2 = \{x^2 + y^2 + z^2 = 1\} \subset \mathbb{R}^3$$

is the map $p \mapsto n(p)$ normal unit vector at p, well-defined as σ oriented

Lemma. The Gauss map $n : \Sigma \to S^2$ is smooth

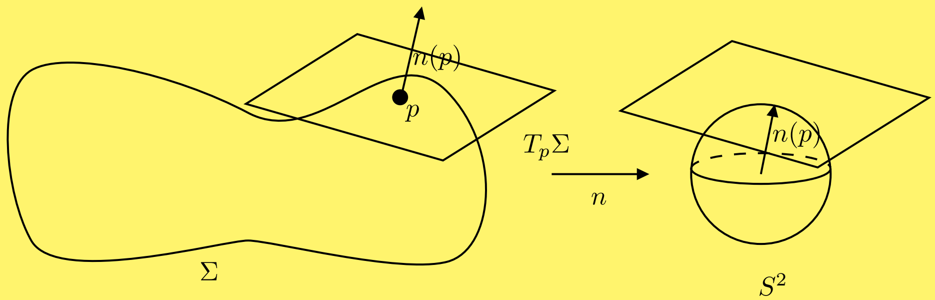
Proof. Smoothness can be checked locally. We know if $\sigma : V \to U \subset \Sigma$ is allowable and compatible with orientation, then at $\sigma(u,v) = p \in \Sigma$

$$n(p) = \frac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|}$$

and this is smooth since σ is smooth (and denominator never vanishes)

Remark. If $\Sigma = F^{-1}(0),\ F:\mathbb{R}^3 \to \mathbb{R}$ where F is smooth and $DF|_{\mathbf{x}} \neq 0\ \forall \mathbf{x} \in \Sigma$ (so Σ is a smooth surface in $\mathbb{R}^3$), recall

$$n(p) = \frac{\nabla F}{\|\nabla F\|}$$



Note. By definition, if $n:\Sigma \to S^2$ is the Gauss map

$$T_p\Sigma = T_{n(p)}S^2 \quad (= n(p)^\perp)$$

Concretely: if $v \in T_p\Sigma$ is $\gamma'(0)$ where $\gamma:(-\varepsilon,\varepsilon) \to \Sigma,\ \gamma(0)=p,\ \gamma$ smooth, then $n\circ\gamma:(-\varepsilon,\varepsilon)\to S^2$ has

$$n\circ\gamma(0) = n(p)$$

and

$$Dn|_p(v) = (n\circ\gamma)'(0) = T_{n(p)}S^2 = T_p\Sigma$$

So $Dn|_p : T_p\Sigma \to T_p\Sigma$ can be viewed as an endomorphism of a fixed 2d subspace of $\mathbb{R}^3$.
Recap of the funamental forms:
Take Σ an oriented (so two-sided) smooth surface in $\mathbb{R}^3$

(i) The FFF is a symmetric bilinear form $\langle\cdot,\cdot\rangle : T_p\Sigma \times T_p\Sigma \to \mathbb{R}$ which is restriction of $\langle\cdot,\cdot\rangle_{eucl}$ to $T_p\Sigma \subset \mathbb{R}^3$. We'll write

$$I_p(v,w) \quad v,w \in T_p\Sigma$$

(ii) The 2nd FF is the symmetric bilinear form $T_p\Sigma\times T_p\Sigma \to \mathbb{R},\ (v,w)\mapsto \mathbb{I}_p(v,w)$ defined by

$$\mathbb{I}_p(v,w) = I_p(-Dn|_p(v), w)$$

where $n:\Sigma\to S^2$ is the Gauss map

If we choose an allowable (oriented) parametrisation $\sigma : V \to U \subset \Sigma$ near $p\in\Sigma$ and if

$$D\sigma|_0(\hat{v}) = v \quad \sigma(0) = p$$
$$D\sigma|_0(\hat{w}) = w \quad v,w \in T_p\Sigma$$

then

$$I_p(v,w) = \hat{v}^T \begin{bmatrix} E & F \\ F & G \end{bmatrix} \hat{w}$$
$$\mathbb{I}_p(v,w) = \hat{v}^T \begin{bmatrix} L & M \\ M & N \end{bmatrix} \hat{w}$$

Lemma. The Gauss map $n : \Sigma \to S^2$ satisfies

$$Dn|_p : T_p\Sigma \to T_p\Sigma$$

is a self-adjoint linear map w.r.t. the non-degenerate inner product I_p on $T_p\Sigma$

Proof. The lemma means

$$I_p(Sn|_p(v), w) = I_p(v, Dn|_p(w)) \quad v, w \in T_p\Sigma$$

We know (e.g. from our local expression) that I_p and $\mathbb{I}$ are symmetric so

$$\begin{aligned} I_p(Dn|_p(v), w) &= -\mathbb{I}_p(v, w) \\ &= -\mathbb{I}_p(w, v) \\ &= I_p(Dn|_p(w), v) \\ &= I_p(v, Dn|_p(w)) \end{aligned}$$

Remark. The "fundamental theorem of surfaces in $\mathbb{R}^3$" says that a smooth oriented connected surface in $\mathbb{R}^3$ is determined up to rigid motion (global isometry of $\mathbb{R}^3$) by its first and 2nd FF

Definition. Let Σ be a smooth surface in $\mathbb{R}^3$. The **Gauss curvature** $\kappa : \Sigma \to \mathbb{R}$ of Σ is the function

$$p \mapsto \det(Dn|_p : T_p\Sigma \to T_p\Sigma)$$

Remark. This is always well-defined, even if Σ is not oriented. (since Σ is always locally orientable, e.g. in the open subset of an allowable parametrisation, and the unit normal to Σ is at most ambiguous up to sign. But for a 2×2 matrix, det is unchanged on reversing the sign)

Method. Computing κ:
Take Σ smooth in $\mathbb{R}^3$ and σ an allowable parametrisation for an open subset.
Recall:

$$I_p : T_p\Sigma \times T_p\Sigma \to \mathbb{R} \quad (v, w) \mapsto \langle v, w\rangle_{eucl}$$

$$\mathbb{I}_p : T_p\Sigma \times T_p\Sigma \to \mathbb{R} \quad (v, w) \mapsto I_p(-Dn|_p(v), w)$$

and have $Dn|_p : T_p\Sigma \to T_p\Sigma$.
The choice of parametrisation σ for $p \in U$ gives me a preferred basis $\{\sigma_u, \sigma_v\}$ for $T_p\Sigma$ and in this basis

$$I_p = \underbrace{\begin{bmatrix} E & F \\ F & G \end{bmatrix}}_{A}, \quad \mathbb{I}_p = \underbrace{\begin{bmatrix} L & M \\ M & N \end{bmatrix}}_{B}$$

and we write $\mathbb{S}$ for $Dn|_p$ in this same basis $\{\sigma_u, \sigma_v$
The identity $\mathbb{I}_p(v, w) = I_p(-Dn|_p(v), w)$ says $B = -\mathbb{S}^T A$

$$\implies \mathbb{S} = -(BA^{-1})^T = -A^{-1}B$$

$$\implies \kappa(p) = \det(\mathbb{S}|_p) = \frac{LN - M^2}{EG - F^2}$$

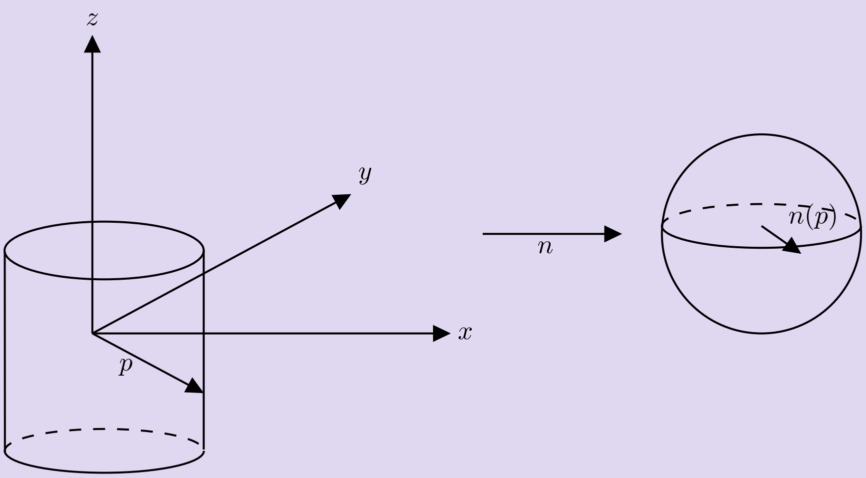
Note. Recall: we already saw that if $\sigma, \tilde{\sigma}$ are 2 allowable parametrisations for U, and $\varphi = \sigma^{-1} \circ \tilde{\sigma}$, we saw

$$\begin{bmatrix} \tilde{E} & \tilde{F} \\ \tilde{F} & \tilde{G} \end{bmatrix} = (D\varphi)^T \begin{bmatrix} E & F \\ F & G \end{bmatrix} D\varphi$$

$$\begin{bmatrix} \tilde{L} & \tilde{M} \\ \tilde{M} & \tilde{N} \end{bmatrix} = \pm(D\varphi)^T \begin{bmatrix} L & M \\ M & N \end{bmatrix} D\varphi$$

from which we directly see that the expression $\frac{LN-M^2}{EG-F^2}$ is intrinsic

Example.



The image of $n : \Sigma \to S^2$ is contained in the equator $S^1 \subset S^2$.
So $\forall p \in \Sigma$, $Dn|_p : T_p\Sigma \to T_p\Sigma$ has 1 dimensional image since $\forall \gamma : (-\varepsilon, \varepsilon) \to \Sigma$, $n \circ \gamma \subset S^1$. So $\det(Dn|_p) = 0$ and $\kappa(p) = 0\ \forall p$

Definition. If Σ is a smooth surface in $\mathbb{R}^3$ and if $\kappa \equiv 0$ on Σ, we say Σ is **flat**

Remark. We saw before that if σ is an allowable parametrisation $\sigma : V \to U \subset \Sigma$ and if we write n_σ for $n \circ \sigma$ so $n_\sigma : V \to S^2$ then

$$Dn_\sigma|_0 : \sigma_u \mapsto (n_\sigma)_u \text{ and } \sigma_v \mapsto (n_\sigma)_v$$

so $\kappa(p) = 0 \iff (n_\sigma)_u \times (n_\sigma)_v = 0$
Usually, we just write n for n_σ and the above as $n_u \times n_v = 0$

Example. If Σ is the graph of a smooth function of f then (ES2)

$$\kappa = \frac{f_{uu}f_{vv} - f_{uv}^2}{(1 + f_u^2 + f_v^2)^2}$$

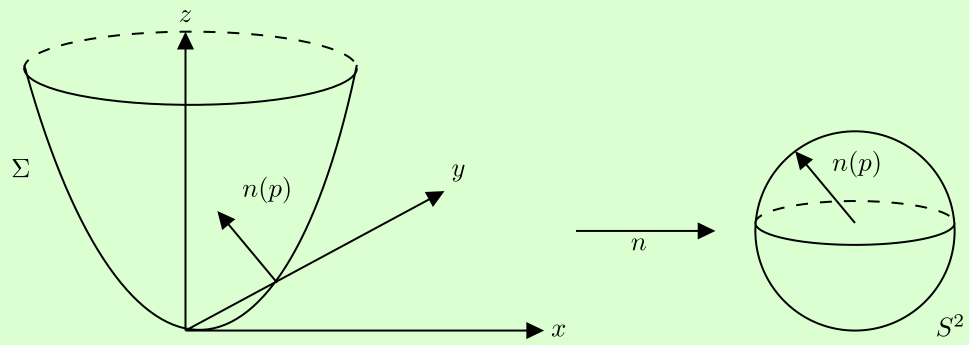
so depends on the Hessian of f.
If $f(u, v) = \sqrt{r^2 - u^2 - v^2}$, then

$$f_{uu}|_0 = f_{vv}|_0 = -\frac{1}{r}, \quad f_{uv}|_0 = 0$$

and $\kappa(0, 0, r) = 1/r^2$.
Since $O(3)$ acts transitively on the sphere, we see $\kappa(p) = 1/r^2 \ \forall p \in S_r^2$

Example. Let Σ be the smooth surface in $\mathbb{R}^3$ $\{z = x^2 + y^2\}$. We claim for a suitable choice of orientation, the image of the Gauss map is the open northern hemisphere

Proof.



Note Σ is invariant under $R_\theta : \mathbb{R}^3 \to \mathbb{R}^3$.
ES2: $n(R_\theta(p)) = R_\theta(n(p))$
So it suffices to compute $n(p)$ for

$$p = (x, 0, t) = (x, 0, x^2) \in \Sigma$$

$\Sigma = F^{-1}(0)$, for $F : \mathbb{R}^2 \to \mathbb{R}$

$$(x, y, z) \mapsto z - x^2 - y^2$$

(which has $DF|_p \neq 0$) so

$$n(p) = \frac{\nabla F}{\|\nabla F\|} = \frac{(-2x, 0, 1)}{\sqrt{1 + 4x^2}}$$

at $(x, 0, x^2)$. We can check

$$x \mapsto \frac{(-2x, 0, 1)}{\sqrt{1 + 4x^2}}$$

sends $\mathbb{R} \mapsto \{y = 0, z > 0\} \subset S^2$

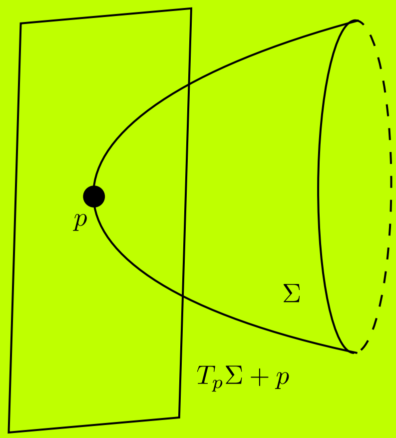
Note. If we choose the other orientation on Σ, then we get the open lower hemisphere as image of Gauss

Definition. If Σ is a smooth surface in $\mathbb{R}^3$ and $p \in \Sigma$, we say p is:

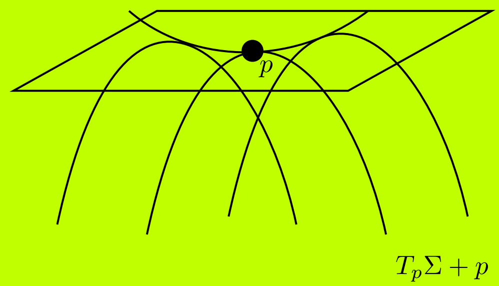
- **elliptic** if $\kappa(p) > 0$
- **hyperbolic** if $\kappa(p) < 0$
- **parabolic** if $\kappa(p) = 0$

Lemma. (i) In a sufficiently small neighbourhood of an elliptic point p, Σ lies entirely on one sde of the affine tangent plane $p + T_p\Sigma$

(ii) In a sufficiently small neighbourhood of a hyperbolic point, Σ, meets both sides of its affine tangent plane



Elliptic p, $\kappa(p) > 0$



Hyperbolic p, $\kappa(p) < 0$

Proof. Take a local parametrisation σ near p. Recall $\kappa = (LN - M^2)/(EG - F^2)$ and $EG - F^2 > 0$ since I_p is positive definite. Recall that if

$$w = h\sigma_u + l\sigma_v \in T_p\Sigma$$

then $\frac{1}{2}\mathbb{I}_p(w, w)$ measured the signed distance from $\sigma(h, l)$ to $T_p\Sigma$ (here $\sigma(0, 0) = p$), measured via inner product with the positive normal distance

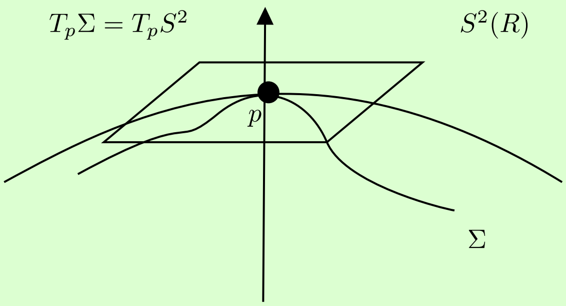
$$\frac{1}{2}(Lh^2 + 2Mhl + Nl^2) + O(h^3, l^3)$$

If p elliptic, $\begin{bmatrix} L & M \\ M & N \end{bmatrix}$ has eigenvalues of same sign so is positive or negative definite at p, so in a neighbourhood of p, this signed distance only has one sign locally. But if p is hyperbolic, then $\mathbb{I}_p(w, w)$ takes both signs in a neighbourhood of p, so Σ meets both sides of $p + T_p\Sigma$

Remark. If p is parabolic, cannot conclude either (Monkey Saddle)

Prop. Let Σ be a compact smooth surface in $\mathbb{R}^3$. Then Σ has an elliptic point

Proof. As Σ is compact, we know it is closed and bounded as a cubset of $\mathbb{R}^3$. So for $R >> 0$, $\Sigma \subset \overline{B(0,R)}$. Decrease R to the minimal such value: up to applying a global isometry (rotation of $\mathbb{R}^3$)



i.e. we've used rotation to put a point of contact of Σ and $S^2(R)$ on the positive z-axis.
Locally near p, we can view Σ as the graph of a smooth function f st.

$$f - \sqrt{R^2 - u^2 - v^2} \leq 0$$

$f : V \to \mathbb{R}^3$, V open in $\mathbb{R}^2$.
Consider the Taylor serries o f and note $f_u = 0 = f_v$ at $(u,v) = (0,0)$ as $f(0,0) = p$ is a maximum

$$\implies \frac{1}{2}(f_{uu}u^2 + 2f_{uv}uv + f_{vv}v^2) + \frac{1}{2R}(u^2 + v^2) \leq 0$$

for sufficiently small u, v so $\begin{bmatrix} L & M \\ M & N \end{bmatrix}$ is locally negative definite near $(0,0)$

$$\implies \kappa|_{\sigma(0,0)} = \kappa(p) > 0$$

(In fact at p, $\kappa(p) \geq 1/R^2$, which is $\kappa(p)$ of $S^2(R)$)

Moral. There is a nice reformulation of Gauss curvature using area

Theorem. Let Σ be a smooth surface in $\mathbb{R}^3$ and $p \in \Sigma$ where $\kappa(p) \neq 0$. Pick a small open neighbourhood $p \in U \subset \Sigma$ and a decreasing sequence

$$p \in A_i \subset U \subset \Sigma$$

where A_i open neighbourhoods which "shrink to" p in the sense that

$$\forall \varepsilon > 0 \; A_i \subset B(p, \varepsilon) \subset \mathbb{R}^3 \quad \forall i >> 0$$

Then

$$|\kappa(p)| = \lim_{i \to \infty} \frac{\text{Area}_{S^2}(n(A_i))}{\text{Area}_{\Sigma}(A_i)}$$

i.e. Gauss curvature is an infinitesimal measure of how much Gauss map n distorts area

Proof. Fix an allowable parametrisation $\sigma : V \to U \subset \Sigma$ near p, s.t. $\sigma(0,0) = p$. Using σ, we get $\sigma^{-1} A_i = V_i \subset V$ open. since A_i shrink to p

$$\bigcap_{i \in I} V_i = \{(0,0)\}$$

$$\begin{aligned} \text{Area}_{\Sigma})A_i &= \int_{V_i} \sqrt{EG - F^2} \, \mathrm{d}u \, \mathrm{d}v \\ &= \int_{V_i} \|\sigma_u \times \sigma_v\| \, \mathrm{d}u \, \mathrm{d}v \end{aligned} \qquad (\dagger)$$

Recall: (chain rule applied to $n \circ \gamma$ for γ a curve in Σ) that

$$Dn|_{(u,v)} \sigma_u \mapsto n_u, \quad \sigma_v \mapsto n_v$$

Since $\kappa(p) = \kappa(\sigma(0,0)) \neq 0$, the map $n \circ \sigma : V \to S^2 \subset \mathbb{R}^2$ has rank 2 derivative near $(0,0)$, so it defines an allowable parametrisation for an open neighbourhood of $n(p) \in S^2$, by inverse function theorem. Therefore

$$\text{Area}_{S^2}(n(A_i)) = \int_{V_i} \|n_u \times n_v\| \, \mathrm{d}u \, \mathrm{d}v$$

(i.e. some formula as ($\dagger$) but on S^2) provided $i >> 0$ so $\sigma^{-1} A_i = V_i$ lies in the open neighbourhood of $(0,0)$ where $n \circ \sigma$ is a diffeomorphism

$$\begin{aligned} \int_{V_i} \|n_u \times n_v\| \, \mathrm{d}u \, \mathrm{d}v &= \int_{V_i} \|D_n(\sigma_u) \times D_n(\sigma_v)\| \, \mathrm{d}u \, \mathrm{d}v \\ &= \int_{V_i} |\det(Dn))| \cdot \|\sigma_u \times \sigma_v\| \, \mathrm{d}u \, \mathrm{d}v \\ &= \int_{V_i} |\kappa(u,v)| \cdot \|\sigma_u \times \sigma_v\| \, \mathrm{d}u \, \mathrm{d}v \end{aligned}$$

Since κ is continuous, given $\varepsilon > 0$, $\exists \delta > 0$ s.t. $|\kappa(u,v) - \kappa(0,0)| < \varepsilon$ if $(u,v) \in B((0,0), \delta) \subset V$ so if $i >> 0$

$$|\kappa(u,v)| \in (|\kappa(p)| - \varepsilon, |\kappa(p)| + \varepsilon)$$

throughout V_i.

Proof (continued)**.** So

$$(|\kappa(p)| - \varepsilon) \int_{V_i} \|\sigma_u \times \sigma_v\| \, \mathrm{d}u \, \mathrm{d}v' \leq \int_{V_i} |\kappa(u, v)| \cdot \|\sigma_u \times \sigma_v\| \, \mathrm{d}u \, \mathrm{d}v$$

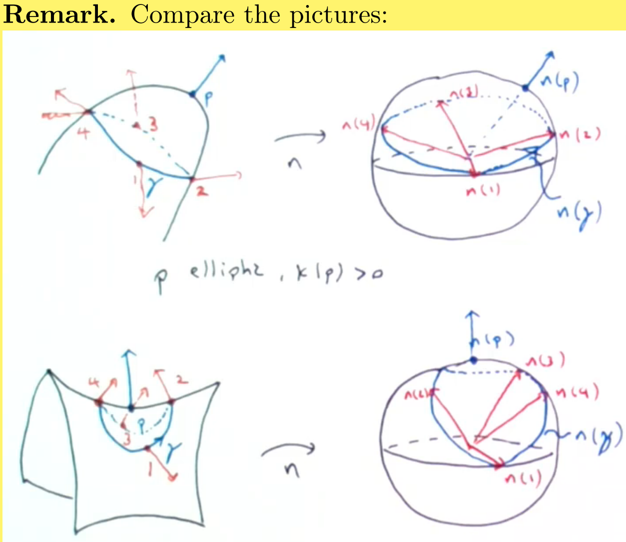
$$\leq (|\kappa(p)| + \varepsilon) \underbrace{\int_{V_i} \|\sigma_u \times \sigma_v\| \, \mathrm{d}u \, \mathrm{d}v}_{\mathrm{Area}_\Sigma(A_i)}$$

i.e.

$$|\kappa(p)| - \varepsilon \leq \frac{\mathrm{Area}_{S^2}(n(A_i))}{\mathrm{Area}_\Sigma(A_i)} \leq |\kappa(p)| + \varepsilon$$

(this holds $\forall i >> 0$) so done

Remark. Compare the pictures:



In the top picture, n locally preserves orientation: γ oriented anticlockwise looking down at Σ along $n(p)$, and $n \circ \gamma$ similarly oriented anticlockwise if we look down at S^2 at $n(p)$ along normal.
But in the lower picture, n locally reverses orientation: the curve $n \circ \gamma$ has opposite sense to curve γ.
Gauss defined this **signed area** of $n(A_i)$ to be area$(n(A_i))$ if $\kappa > 0$, $-$area$(n(A_i))$ if $\kappa < 0$ and then stated

$$\kappa(p) = \lim_{A_i \to p} \frac{\text{signed area}_{S^2}(n(A_i))}{\text{area}_\Sigma(A_i)}$$

Note this result also holds when $\kappa = 0$, with a bit more care

Gauss curvature is constrained by two amazing theorems:

- local result, called the "theorema egregium" (remarkable theorem)
- global rigidity

Theorem. The Gauss curvature of a smooth surface in $\mathbb{R}^3$ is an isometry invariant i.e. if $f : \Sigma_1 \to \Sigma_2$ is a diffeomorphism of surfaces in $\mathbb{R}^3$, which is an isometry then

$$\kappa(p) = \kappa(f(p)) \quad \forall p \in \Sigma$$

(The Gauss curvature can be extracted from I_p even though its definition uses I_p and $\mathbb{I}_p$)

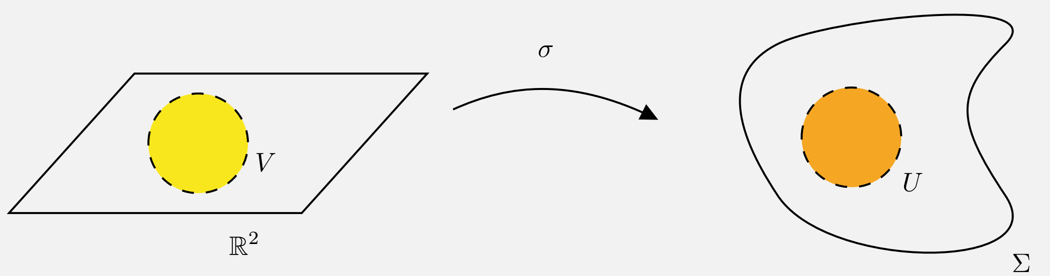
Theorem (Gauss-Bonnet theorem)**.** If Σ is a compact smooth surface in $\mathbb{R}^3$ then

$$\int_\Sigma \kappa \, \mathrm{d}A_\Sigma = 2\pi\chi(\Sigma)$$

($\mathrm{d}A_\Sigma = \sqrt{EG - F^2}$ locally)

How might one prove “theorema egregium”?

- direct proof in part II
- ask a different question: are some allowable parametrisations of a smooth surface in $\mathbb{R}^3$ “better” than others?
  A paramterisation: $\sigma : V \to U \subset \Sigma$ defines distinguished curves, the images of coordinate lines



  So looking for a “best” local parametrisation is related to looking for distinguished local curves in Σ

Later we’ll see every smooth surface in $\mathbb{R}^3$ admits a local parametrisation such that FFF has form $\mathrm{d}u^2 + G(u, v)\,\mathrm{d}v^2$ (i.e. $E = 1, F = 0$), and (ES3) if you have a local parametrisation then

$$\kappa = (\text{expression in } G)$$

This is a (more conceptual?) route to theorema egregium

2 Geodesics

If $\gamma : [a, b] \to \mathbb{R}^3$ is smooth, recall

$$\text{length}(\gamma) := \int_a^b \|\gamma'(t)\| \,\mathrm{d}t$$

Definition. The **energy** of γ

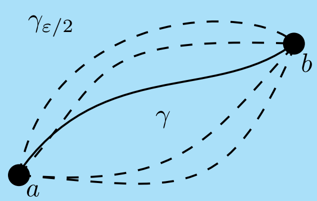
$$E(\gamma) := \int_a^b \|\gamma'(t)\|^2 \, \mathrm{d}t$$

Given $\gamma : [a,b] \to \Sigma$ smooth, for a smooth surface Σ in $\mathbb{R}^3$, then:

Definition. A **one-parameter variation** (with fixed end-points) of γ is a smooth map

$$\Gamma : (-\varepsilon, \varepsilon) \times [a,b] \to \Sigma$$

s.t. if $\gamma_s = \Gamma(s, \cdot)$ then
(i) $\gamma_0(t) = \gamma(t) \ \forall t$
(ii) $\gamma_s(a), \gamma_s(b)$ are independent of s



Definition. A smooth curve $\gamma : [a,b] \to \Sigma$ is a **geodesic** if for every variation (γ_s) of γ with fixed end points, we have

$$\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} E(\gamma_s) = 0$$

i.e. γ is a "critical point" of the energy functional on curves from $\gamma(a)$ to $\gamma(b)$.

Equation. Suppose γ has image in the image of an allowable parametrisation σ, and write

$$\gamma_s(t) = \sigma(u(s,t), v(s,t))$$

Suppose FFF wrt σ is

$$E\,\mathrm{d}u^2 + 2F\,\mathrm{d}u\,\mathrm{d}v + G\,\mathrm{d}v^2$$

and set

$$R := E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2$$

where $\dot{u} = \frac{\partial u}{\partial t}$ and $\dot{v} = \frac{\partial v}{\partial t}$ so

$$E(\gamma_s) = \int_a^b R\,\mathrm{d}t$$

noting R depends on s. so

$$\begin{aligned}\frac{\partial R}{\partial s} &= (E_u\dot{u}^2 + 2F_u\dot{u}\dot{v} + G_u\dot{v}^2)\frac{\partial u}{\partial s} \\ &+ (E_v\dot{u}^2 + 2F_v\dot{u}\dot{v} + G_v\dot{v}^2)\frac{\partial v}{\partial s} \\ &+ 2(E\dot{U} + F\dot{v})\frac{\partial \dot{u}}{\partial s} \\ &+ 2(F\dot{u} + G\dot{v})\frac{\partial \dot{v}}{\partial s}\end{aligned}$$

so

$$\frac{\mathrm{d}}{\mathrm{d}s}E(\gamma_s) = \int_a^b \frac{\partial R}{\partial s}\,\mathrm{d}t$$

We can integrate by parts and note that $\frac{\partial u}{\partial s}$ and $\frac{\partial v}{\partial s}$ vanish at the endpoints a, b (variation has fixed endpoints). So

$$\left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_{s=0} E(\gamma_s) = \int_a^b (A\frac{\partial u}{\partial s} + B\frac{\partial v}{\partial s})\,\mathrm{d}t$$

where

$$A = E_u\dot{u}^2 + 2F_u\dot{u}\dot{v} + G_u\dot{v}^2 - 2\frac{\mathrm{d}}{\mathrm{d}t}(E\dot{u} + F\dot{v})$$
$$B = E_v\dot{u}^2 + 2F_v\dot{u}\dot{v} + G_v\dot{v}^2 - 2\frac{\mathrm{d}}{\mathrm{d}t}(F\dot{u} + G\dot{v})$$

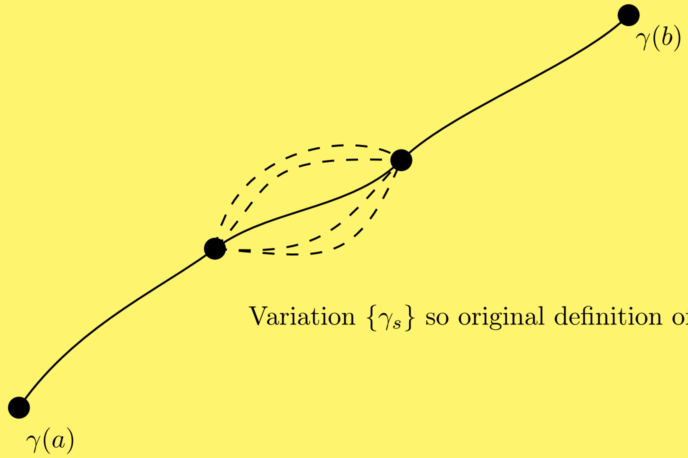
Corollary. A smooth curve $\gamma : [a, b] \to \Sigma$ (with image in the image of σ) is a geodesic if and only if it satisfies the geodesic equations:

$$\frac{\mathrm{d}}{\mathrm{d}t}(E\dot{u} + F\dot{v}) = \frac{1}{2}(E_u\dot{u}^2 + 2F_u\dot{u}\dot{v} + G_u\dot{v}^2)$$
$$\frac{\mathrm{d}}{\mathrm{d}t}(F\dot{u} + G\dot{v}) = \frac{1}{2}(E_v\dot{u}^2 + 2F_v\dot{u}\dot{v} + G_v\dot{v}^2)$$

Note: these only depend on γ not $\{\gamma_s\}$

Remarks.

(i) Solving a differential equation is a local property on γ:



Variation $\{\gamma_s\}$ so original definition of geodesic also has a local character

(ii) Unlike length $L(\gamma)$, the energy $E(\gamma)$ does depend on the paramterisation of γ. If $f, g : [a, b] \to \mathbb{R}$ are smooth, Cauchy-Schwartz says

$$\left(\int_a^b fg \, \mathrm{d}t\right)^2 \leq \int_a^b f^2 \, \mathrm{d}t \int_a^b g^2 \, \mathrm{d}t$$

Apply this with

$$f = \sqrt{R} = (E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v})^{1/2}, \quad g = 1$$

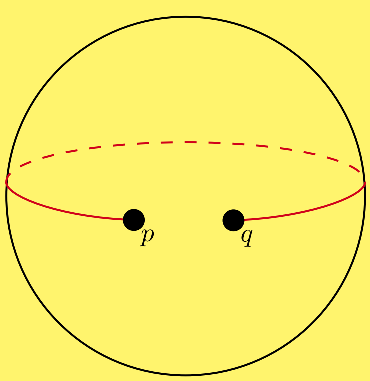
to see $\text{Length}(\gamma)^2 \leq E(\gamma)(b-a)$.

Since we get equality in Cauchy-Schwartz only when $f = c\dot{g}$ ffor a constant c, which here would say $\|\dot{\gamma}(t)\| = \text{constant}$ so γ parametrised proportional to arc length

Corollary. (i) If γ has constant speed and locally minimises length, then γ is a geodesic

(ii) If γ globally minimised energy (amongst paths with the same end-points) then it globally minimises length, and is paramtetrised with constant speed
(so geodesics are naturally constant-speed parametrised)

Remark.



red arc is a geodesic from p to q

Want geodesics to be local but not necessarily global-length minimisers.

Example. The plane $\mathbb{R}^2$ has parametrisation

$$\sigma(u, v) = (u, v, 0) \text{ and FFF: } \mathrm{d}u^2 + \mathrm{d}v^2$$

Geodesic equations:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\dot{u}) = \frac{\mathrm{d}}{\mathrm{d}t}(\dot{v})$$

for a acurve $\gamma(t) = (u(t), v(t), 0) = \sigma(u(t), v(t))$ i.e. $\ddot{u} = 0 = \ddot{(v)}$. So

$$u(t) = \alpha t + \beta$$
$$v(t) = \gamma t + \delta$$

which is a straight line parametrised at constant speed

Example. Take the unit sphere with parametrisation

$$\sigma(u,v) = (\cos u \cos v, \cos u \sin v, \sin u)$$

$u \in (-\pi/2, \pi/2)$ latitude; $v \in (0, 2\pi)$ longitude.
FFF: $\mathrm{d}u^2 + \cos^2(u)\,\mathrm{d}v^2$ $(E = 1,\ F = 0,\ G = \cos^2(u))$
Geodesic equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\dot{u}) = -\cos(u)\sin(u)v^2$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\cos^2(u)\dot{v}\right) = 0$$

$$\implies \ddot{u} + \sin(u)\cos(u)\dot{v}^2 = 0$$
$$\ddot{v} - 2\tan(u)\dot{u}\dot{v} = 0$$

Let's assume our geodesic is parametrised at unit speed. Then

$$u^2 + \cos^2(u)\dot{v}^2 = 1$$

so $\ddot{v}/\dot{v} = 2\tan(u)\dot{u}$

$$\implies \ln(\dot{v}) = -2\ln(\cos u) + \text{ constant}$$
$$\implies \dot{u} = \frac{C}{\cos^2(u)}$$

So $\dot{u}^2 = 1 - C/\cos^2(u)$.
So

$$\dot{u} = \sqrt{\left(\frac{\cos^2(u) - C^2}{\cos^2(u)}\right)}$$

Then

$$\frac{\dot{v}}{\dot{u}} = \frac{\mathrm{d}v}{\mathrm{d}u} = \frac{C}{\cos(u)\sqrt{\cos^2(u) - C^2}}$$

and so

$$V = \int \frac{\partial v}{\partial u}\,\mathrm{d}u = \int \frac{C\sec^2(u)}{\sqrt{1 - C^2\sec^2 u}}\,\mathrm{d}u$$

and if we set $w = \frac{C\tan u}{\sqrt{1-C^2}}$ then

$$r = \int \frac{\mathrm{d}w}{\sqrt{1 - w^2}} = \sin^{-1}(w) + \text{ const} = \sin^{-1}(\lambda \tan u) + \delta$$

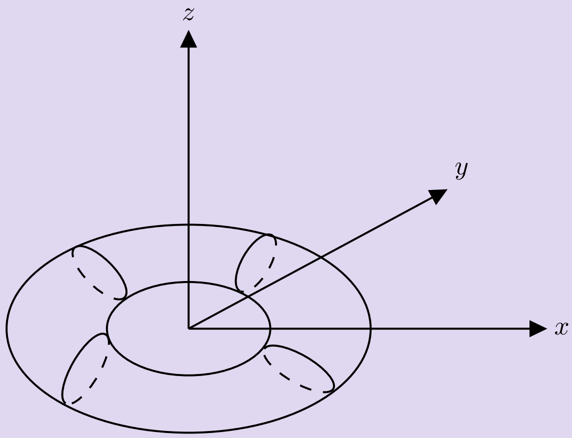
for constants λ, δ. We saw: $\sin(v - \delta) = \lambda \tan u$

$$\sin r \cos\delta - \cos v \sin\delta - \lambda \tan u = -$$

$$\implies \underbrace{(\sin v \cos u)}_{x}\cos\delta - \underbrace{(\cos v \cos u)}_{y}\sin\delta - \lambda\underbrace{\sin u}_{z} = 0$$

So our geodesic γ lies on a plane through $0 \in \mathbb{R}^3$, i.e. γ is an arc of a great circle on S^2

Example. A Torus in $\mathbb{R}^3$



Rotate $(x-a)^2+z^2=1$ about xz-axis. An allowable parametrisation is

$$\sigma(u,v)=((a+\cos u)\cos v,(a+\cos u)\sin v,\sin u)$$

FFF: $\mathrm{d}u^2+(a+\cos u)^2\,\mathrm{d}v^2$

$$E=1,\ F=0,\ G=(a+\cos u)^2$$

Note: if formally we set $a=0$, this recovers the unit sphere and its FFF.
Follow same procedure as for S^2, or formally substitute $\cos u\mapsto a+\cos u$ and we'll get

$$\frac{\mathrm{d}v}{\mathrm{d}u}=\frac{C}{(a+\cos u)\sqrt{(a+\cos u)^2-C^2}}$$

which can't be integrated using classical functions (c.f. "elliptic" functions)

Next goal: give a different characterisation of geodesics on a smooth surface in $\mathbb{R}^3$

Moral. Straight lines in $\mathbb{R}^2$ are not just locally shortest but locally straightest.
Idea: characterise via saying the change in the tangent vector of the curve is as small as it could be subject to the fact that the curve γ stays on the surface

Prop. Let Σ be a smooth surface in $\mathbb{R}^3$. A smooth curve $\gamma : [a,b] \to \Sigma$ is a geodesic if and only if the vector $\ddot{\gamma}(t)$ is everywhere normal to Σ

Proof. Being a geodesic as we defined or having $\ddot{\gamma}(t)$ normal to Σ, are both local conditions on γ, so we can work in one allowable parametrisation. As usual:

$$\sigma : V \to U \subset \Sigma$$

and suppose $\gamma(t) = \sigma(u(t), v(t))$ so

$$\dot{\gamma}(t) = \sigma_u \dot{u} + \sigma_v \dot{v}$$

so $\ddot{\gamma}$ is normal to Σ exactly when it's orthogonal to $T_p\Sigma = \langle \sigma_u, \sigma_v \rangle$ if and only if

$$\langle \frac{\mathrm{d}}{\mathrm{d}t} \left(\sigma_u \dot{u} + \sigma_v \dot{v} \right), \sigma_u \rangle = 0 \tag{†}$$

$$\langle \frac{\mathrm{d}}{\mathrm{d}t} \left(\sigma_u \dot{u} + \sigma_v \dot{v} \right), \sigma_v \rangle = 0 \tag{††}$$

We'll consider (†), which is equivalent to

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle \sigma_u \dot{u} + \sigma_v \dot{v}, \sigma_u \rangle - \langle \sigma_u \dot{u} + \sigma_v \dot{v}, \frac{\mathrm{d}}{\mathrm{d}t} \left(\sigma_u \right) \rangle = 0$$

Noting

$$\langle \sigma_u, \sigma_u \rangle = E, \quad \langle \sigma_u, \sigma_v \rangle = F$$

this is:

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(E\dot{u} + F\dot{v} \right) - \langle \sigma_u \dot{u} + \sigma_v \dot{v}, \sigma_{uu} \dot{u} + \sigma_{uv} \dot{v} \rangle = 0$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(E\dot{u} + F\dot{v} \right) - \left\{ \dot{u}^2 \langle \sigma_u, \sigma_{uu} \rangle + \dot{u}\dot{v} \left(\langle \sigma_u, \sigma_{uv} \rangle + \langle \sigma_v, \sigma_{uu} \rangle \right) + \dot{v}^2 \langle \sigma_v \sigma_{uv} \right\} = 0$$

But

$$E = \langle \sigma_u, \sigma_u \rangle \implies E_u = 2 \langle \sigma_u, \sigma_{uu} \rangle$$

and

$$G = \langle \sigma_v, \sigma_v \rangle \implies G_u = 2 \langle \sigma_v, \sigma_{uv}$$

and

$$F = \langle \sigma_u, \sigma_v \rangle = \langle \sigma_v, \sigma_u \rangle \implies F_u = \langle \sigma_u, \sigma_{uv} \rangle + \langle \sigma_{uu}, \sigma_v \rangle$$

and (†) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(E\dot{u} + F\dot{v} \right) = \frac{1}{2} \left(E_u \dot{u}^2 + 2F_u \dot{u}\dot{v} + G_u \dot{v}^2 \right)$$

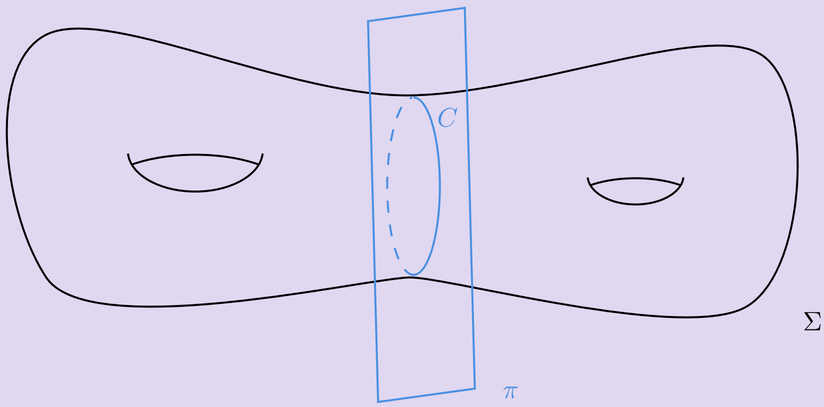
the first of the geodesic equations.
Similarly (††) is equivalent to the second geodesic equation

Remark. Note

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle = 2 \langle \underbrace{\dot{\gamma}(t)}_{\text{tang. to } \Sigma}, \underbrace{\ddot{\gamma}(t)}_{\text{norm. to } \Sigma} \rangle$$

So $\langle \dot{\gamma}(t), \dot{\gamma}(t) \rangle$ is constant i.e. geodesics are indeed parametrised with constant speed, so proportional to arc-length

Example. A surface with symmetry,



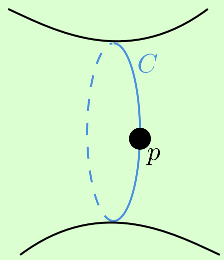
Let Σ be a smooth surface in $\mathbb{R}^3$ and assume there is a plane $\Pi \subset \mathbb{R}^3$ s.t.

(i) $\Pi \cap \Sigma$ is a smooth embedded curve $C \subset \Sigma$

(ii) Σ is setwise preserved by reflection in Π

Claim. C is a geodesic when parametrized at constant speed

Proof.



We can write $\mathbb{R}^3 = \Pi \oplus \Pi^\perp$ (e.g.) suppose p is the origin of our co-ordinates, by translation, and also $\mathbb{R}^3 = T_p\Sigma \oplus \mathbb{R}n_p$
Clearly Refl_Π acts on Π by identity Π^+ by -1.
Since Refl_Π fixes Σ setwise and fixes p, it also preserves $T_p\Sigma$, so it also preserves $\mathbb{R}n_p$

$$\implies \mathbb{R}n_p \subset \Pi$$

as Π not identity on $T_p\Sigma$.
Let's parametrise C locally near p via

$$t \mapsto \gamma(t) \in C \subset \Sigma \subset \mathbb{R}^3$$

at constant speed. Well:

$$\gamma(t) \subset \Pi \implies \dot{\gamma}(t), \ddot{\gamma}(t) \in \Pi$$

But

$$\langle \dot{\gamma}(t), \ddot{\gamma}(t) \rangle = 0$$

so

$$\dot{\gamma}(t) \in \Pi \cap T_p\Sigma$$

and $\ddot{\gamma}(t)$ is orthogonal to this and in Π

$$\implies \ddot{\gamma}(t) \in \mathbb{R}n_p \subset \Pi$$

so γ is a geodesic

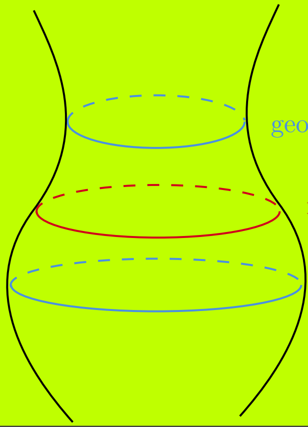
Remark. As given, C is not parametrized at all

Example. Surfaces of revolution revisited
We take $\eta(u) = (f(u), 0, g(u))$ in xz-plane and rotate it about xz-axis.
(η smooth, injective, $f(u) > 0$)

Definition. A circle obtained by rotating a point of η is called a **parallel**.
A curve obtained by rotating η itself by a fixed angle is called a **meridian**.
A plane in $\mathbb{R}^3$ containing the z-axis is a **plane of symmetry**.

Corollary. All meridians are geodesics

Lemma. A parallel $u = U_0$ (constant) is a geodesic (parametrised at constant speed) $\iff f'(u_0) = 0$



Proof. We take our usual allowable parametrisation

$$\sigma(u,v) = (f(u)\cos v, f(u)\sin v, g(u) \quad a < u < b, v \in (0, 2\pi)$$

Then the FFF is

$$((f')^2 + (g')^2)\,\mathrm{d}u^2 + f^2\,\mathrm{d}v^2$$

We can parametrise η by arc-length and then FFF becomes

$$\mathrm{d}u^2 + f^2\,\mathrm{d}v^2$$

i.e. $E = 1,\ F = 0,\ G = f(u)^2$. Geodesic equations are then:

$$\frac{\mathrm{d}}{\mathrm{d}y}(\dot{u}) = \ddot{u} = f f_u \dot{v}^2$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(f^2\dot{v}\right) = 0$$

and we know $\gamma(t) = (u(t), v(t))$ is constant speed so

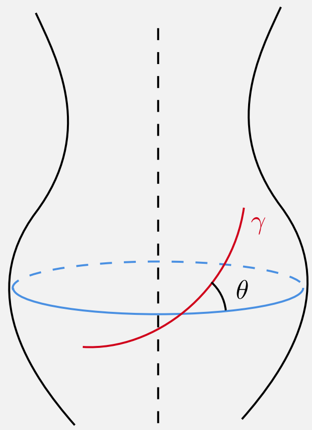
$$\dot{u}^2 + f^2\dot{v}^2 = \text{constant (non-zero)}$$

Up to now this was for any geodesic on surface of revolution.
Parallels: $u = u_0 = \text{constant}$

$$\implies \dot{u} = 0 \quad \dot{v} = \frac{\text{const.}}{f(u_0)}$$

2nd geodesic equation automatically holds.
1st geodesic equation holds exactly if

$$f_u|_{u_0} = 0$$



Consider a curve $\gamma(t)$ on our surface of revolution making angle θ with the parallel of radius $\rho(= f)$

Claim (Clairout's Relation)**.** If γ is a geodesic, then $\rho \cos\theta$ is constant along γ

Proof. As usual, if $\gamma(t) = \sigma(u(t), v(t))$ and $\dot{\gamma}(t) = \sigma_u \dot{u} + \sigma_v \dot{v}$ and note that the tangent vector to the parallel is σ_v, then we know (cf discussion of angles wrt FFF)

$$\cos\theta = \frac{\langle \sigma_v, \sigma_u \dot{u} + \sigma_v \dot{v}}{\|\sigma_v\| \cdot \|\sigma_u \dot{u} + \sigma_v \dot{v}\|}$$

and if γ is parametrised by arc-length then

$$\|\sigma_u \dot{u} + \sigma_v \dot{v}\| = 1$$

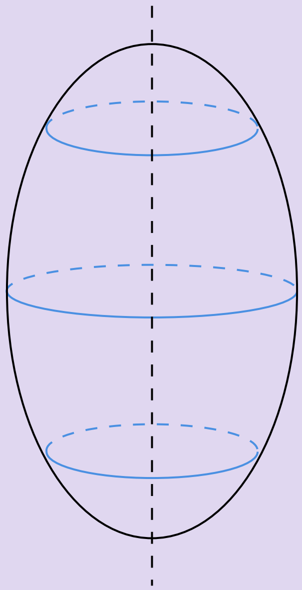
so (using $F = 0$ and $G = f^2$ in our case)

$$\cos\theta = |f(u)\dot{v}| = \rho\dot{v}$$

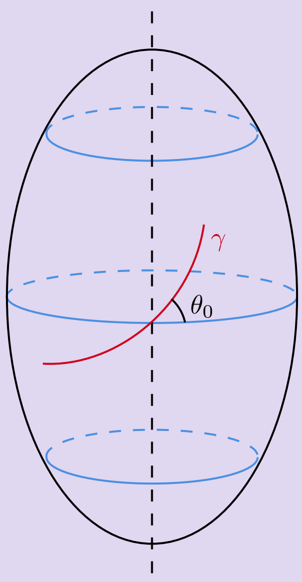
2nd geodesic equation ($\frac{\mathrm{d}}{\mathrm{d}t} f^2 \dot{v} = 0$)

$$\implies \rho\cos\theta \text{ is constant}$$

Example. Ellipsoid of revolution:



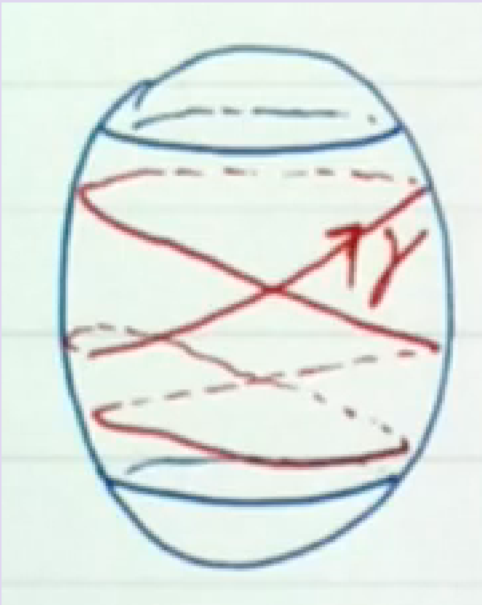
Observe: usually for a surface of revolution we take η in the xz-plane away from the z-axis ($f > 0$) But in fact we can allow η to meet the z axis orthogonally as in the ellipsoid (or for a sphere) [Or, remove the 2 poles]



$\rho \cos\theta$ is constant along geodesic γ

Example (continued)**.** If we meet ρ_0-parallel at angle θ_0, and suppose γ is not a meridian, so $\theta_0 \in [0, \pi/2)$, so

$$\rho \cos\theta \text{ constant} \implies \rho \text{ bounded below}$$



Geodesic that isn't meridian is trapped between parallels coming from bound on ρ.
When we cansay global things about geodesics, because we can't solve the equations, there's an important local existence theory.
Recall Picard's theorem

$$I = [t_0 - a, t_0 + a] \subset \mathbb{R}$$

$$B = \{x : \|x - x_0\| \le b\} \subset \mathbb{R}^n$$

and $f : I \times B \to \mathbb{R}^n$ continuous and Lipchitz in 2nd variable

$$\|f(t, x_1) - f(t, x_2)\| \le N\|x_1 - x_2\|$$

Then

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f)t, x), \quad x(t_0) = x_0$$

has a unique solution for some interval $|t - t_0| < h$ (e.g. for $h = \min\{a, b/s\},\ s = \sup\|f\|$)

Note. If f is smooth, then the solution depends smoothly on the initial consition (and is smooth)

Our setting: recall

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(E\dot{u}+F\dot{v}\right)=\frac{1}{2}\left(E_u\dot{u}^2+2F_u\dot{u}\dot{v}+G_u\dot{v}^2\right)$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(F\dot{u}+G\dot{v}\right)=\frac{1}{2}\left(E_v\dot{u}^2+2F_v\dot{u}\dot{v}+G_v\dot{v}^2\right)$$

i.e.

$$\begin{bmatrix}E & F\\ F & G\end{bmatrix}\begin{bmatrix}\ddot{u}\\ \ddot{v}\end{bmatrix}=[\dots]$$

$M\to M^{-1}$ is smooth in $GL(2,\mathbb{R})\to GL(2,\mathbb{R})$ so the geodesic equations are

$$\ddot{u}=A(u,v,\dot{u},\dot{v})$$
$$\ddot{u}=B(u,v,\dot{u},\dot{v})$$

for smooth A,B. We introduce $p=\dot{u},\ q=\dot{v}$ and rewrite as

$$\dot{u}=p$$
$$\dot{v}=q$$
$$\dot{p}=A(u,v,p,q)$$
$$\dot{q}=B(u,v,p,q)$$

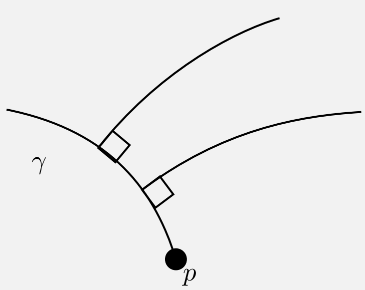
so Picard's theorem applies, noting since A,B smooth, a local bound on $\|DA$ and $\|DB\|$ will give us the required Lipschitz condition

Corollary. Let Σ be a smooth surface in $\mathbb{R}^3$. For $p\in\Sigma$ and $0\neq v\in T_p\Sigma$, there is some $\varepsilon>0$ and a geodesic

$$\gamma:[0,\varepsilon)\to\Sigma$$

s.t. $\gamma(0)=p,\ \dot{\gamma}(0)=v$. Moreover, γ depends smoothly on the initial condition (p,v).

The local existence of geodesics gives rise to parametrisations of Σ with nice properties.
Fix $p\in\Sigma$ and consider a geodesic arc γ starting at p and parametrised by arc-length



For $t>0$, small, let γ_i be the geodesic s.t.

$$\gamma_t(0)=\gamma(t)$$

$\gamma_t'(0)$ is orthogonal to $\gamma'(t)$, γ_t parametrised by arc-length.
Define $\sigma(u,v)=\gamma_v(u)$ defined for $u\in[0,\varepsilon),\ v\in[0,\delta)$

Lemma. For ε, δ sufficiently small, σ defines an allowable parametrisation of an open set in Σ (taking interior of domain)

Proof. Smoothness of σ is immediate from our note/ last line in Picard's theorem.
At $(0,0)$, σ_u, σ_v are orthogonal by construction so they are linearly independent for ε, δ sufficiently small.
So $D\sigma$ has rank 2 and (shrinking set if necessary) σ is injective and parametrisation is allowable

Corollary. Any smooth surface Σ in $\mathbb{R}^3$ admits local parametrisations for which the FFF is of shape

$$\mathrm{d}u^2 + G(u,v)\,\mathrm{d}v^2$$

i.e. $E = 1, \ F = 0$

Proof. We'll consider this parametrisation $\sigma(u,v) = \gamma_v(u)$. If we fix v_0, the curve $u \mapsto \gamma_{v_0}(u)$ is a geodesic parametrised by arc-length so

$$E = \langle \sigma_u, \sigma_u \rangle = 1$$

Also, one of the geodesic equations

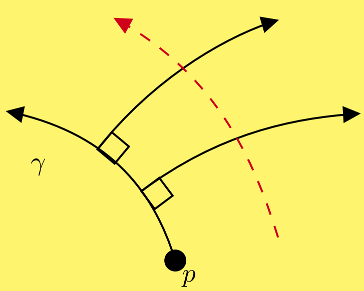
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(F\dot{u} + G\dot{v}\right) = \frac{1}{2}(E_v\dot{u}^2 + 2F_v\dot{u}\dot{v} + G_v\dot{v}^2)$$

and again consider $v = v_0, \ u(t) = t$.
We get $\frac{\mathrm{d}}{\mathrm{d}y}\left(F\right) = 0$ or equivalently

$$F_u\dot{u} = 0 \implies F_u = 0$$

so F is independent of u. But when $u = 0$, then (by construction of γ_v as orthogonal to γ at $\gamma(v)$), we see $F = 0$. so $F = 0$ everywhere

Remark. Co-ordinates built this way are sometimes called "geodesic normal co-ordinates"



Fix v, vary u: geodesics

u fixed, varying v: Typically NOT a geodesic
(except $u = 0$ i.e. γ itself)

Remark. In these co-ordinates
(i) $G(0, v) = 1$
(ii) $G_u(0, v) = 0$
(i) holds because σ_v has length 1 at $u = 0$
(ii) holds because $u = 0$ is a geodesic with arc-length parametrisation and

$$\frac{\mathrm{d}}{\mathrm{d}t}(E\dot{u} + F\dot{v}) = \frac{1}{2}(E_u\dot{u}^2 + 2F_u\dot{u}\dot{v} + G_u\dot{v}^2)$$

which becomes

$$0 = \frac{1}{2}G_u(0, v)$$

Remark. In ES3, we show that for a smooth surface in $\mathbb{R}^3$ with allowable parametrisation s.t. $E = 1,\ F = 0$, have

$$\kappa = -\frac{\sqrt{G}_{uu}}{\sqrt{G}}$$

If Σ is in $\mathbb{R}^3$ and $a : \mathbb{R}^3 \to \mathbb{R}^3$ is just a dilation $(x, y, z) \mapsto (ax, ay, az)$, then

$$\kappa_{a(\Sigma)} = \frac{1}{a^2}\kappa_\Sigma$$

(the coeffs E, F, G rescale by a^2, and L, N, M by a, c.f. our computation for spheres of radii R for different R)

Question: what do constant curvature surfaces look like?
By dilating, it suffices to understand surfaces of constant curvature $1, -1, 0$

Prop. Let Σ be a smooth surface in $\mathbb{R}^3$

(i) If $\kappa_\Sigma \equiv 0$, then Σ is locally isometric to $(\mathbb{R}^2, \, \mathrm{d}u^2 + \mathrm{d}v^2)$

(ii) If $\kappa_\Sigma \equiv 1$, then Σ is locally isometric to $(S^2, \, \mathrm{d}u^2 + \cos^2(u)\,\mathrm{d}v^2)$

Proof. We know Σ admits an allowable parametrisation with $E = 1$, $F =$ (so $\kappa = -\sqrt{G}_{uu}/\sqrt{G}$) and s.t.

$$G(0, v) = 1$$
$$G_u(0, v) = 0$$

If $\kappa = 0$, we get

$$\sqrt{G}_{uu} = 0$$

so $\sqrt{G}_{uu} = A(v)u + B(v)$ and our "boundary conditions" on G show

$$B(v) \equiv 1, \; A(v) \equiv 0$$

Then the FFF is $\mathrm{d}u^2 + \mathrm{d}v^2$.

$$\kappa = +1, \; \sqrt{G}_{uu} + \sqrt{G} = 0$$

so

$$\sqrt{G} = A(v)\sin u + B(v)\cos u$$

Now boundary conditions show

$$A(v) \equiv 0, \quad B(v) \equiv 1$$

so FFF $\mathrm{d}u^2 + \cos^2(u)\,\mathrm{d}v^2$. In parametrisation

$$\sigma(u, v) = (\cos u \cos v, \cos u \sin v, \sin u)$$

this was the FFF of the round unit sphere

Remark. If $\kappa = -1$, and we do the same thing, we'll get FFF: $\mathrm{d}u^2 + \cosh^2(u)\,\mathrm{d}v^2$, which we might not recognise from any smooth surface in $\mathbb{R}^3$

- we can look for one ("tractoid")
- we can widen our imagination and let go of $\mathbb{R}^3$

Remark. In fact, the change of variables

$$V = e^v \tanh u$$
$$W = e^v \operatorname{sech} u$$

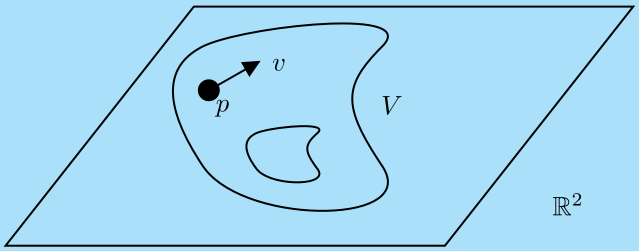
turns $\mathrm{d}u^2 + \cosh^2(u)\,\mathrm{d}v^2$ into $\frac{\mathrm{d}V^2 + \mathrm{d}W^2}{W^2}$ which is a "standard" presentation of the FFF of the "hyperbolic plane"

Definition. Let $V \subset \mathbb{R}^3$ be an open set. An **(abstract) Riemannian metric** on V is a smooth map

$$V \to \{\text{positive-definitite symmetric bilinear forms}\} \subset \mathbb{R}^4$$

$$v \mapsto \begin{bmatrix} E(v) & F(v) \\ F(v) & G(v) \end{bmatrix}$$

s.t. $E > 0,\ G > 0,\ EG - F^2 > 0$



If v is a vector at $p \in V$

$$\|v\|^2_{\begin{bmatrix} E(p) & F(p) \\ F(p) & G(p) \end{bmatrix}} = v^t \begin{bmatrix} E(p) & F(p) \\ F(p) & G(p) \end{bmatrix} v$$

and if $\gamma : [a, b] \to V$ is smooth

$$\text{Length}(\gamma) = \int_a^b (E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2)^{1/2}\,\mathrm{d}t$$

where $\gamma(t) = (u(t), v(t)) : [a, b] \to V$

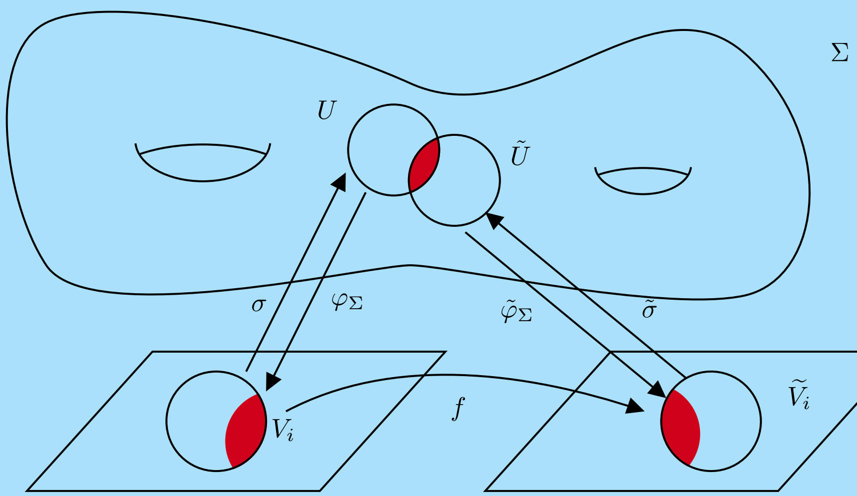
Definition. Let Σ be an abstract smooth surface, so $\Sigma = \bigcup_{i\in I} U_i,\ U_i \subset \Sigma$ and

$$\varphi_i : U_i \to V_i \subset \mathbb{R}^2$$

a homeomorphism and s.t. the transition maps

$$\varphi_i \varphi_j^{-1} : \varphi_j(U_i \cap U_j) \to \varphi_i(U_i \cap U_j)$$

is smooth $\forall i, j$
A Riemannian metric on Σ, usually called g or $\mathrm{d}s^2$, is a choice of Riemannian metric on each V_i which are compatible in the following sense:



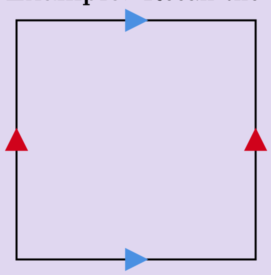
$f : \tilde{\sigma}^{-1} \circ \sigma$ transition map

Require:

$$(\mathrm{d}f)^t \begin{bmatrix} \tilde{E} & \tilde{F} \\ \tilde{F} & \tilde{G} \end{bmatrix} \mathrm{d}f = \begin{bmatrix} E & F \\ F & G \end{bmatrix} \qquad (\dagger)$$

so $\mathrm{d}f$ defines an isometry from an open set in the chart $(U, \varphi(U) = V)$ to one in $(\tilde{U}, \varphi(\tilde{U}) = \tilde{V})$
c.f. $(\dagger)$ was the transformation law for FFF

Example. Recall the torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2 =$



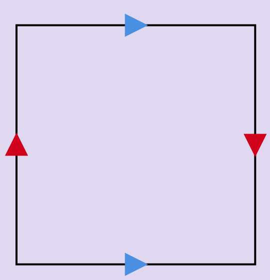
We exhibited an atlas of charts for whcih the transition maps were restrictions of translations of open subsets of $\mathbb{R}^2$.
Equip each $V_i \subset \mathbb{R}^2$ (image of such a chart) with the Euclidean metric $\mathrm{d}u^2 + \mathrm{d}v^2$ i.e. the map $V_i \to \{+\text{ve def. s.b.f}\}$ is constant at I.
If f is a translation, $Df =$ Identity and

$$(Df)^t I(Df) = I \tag{†}$$

holds trivially.
So T^2 inherits a global Riemannian metric which is flat (everywhere locally isometric to $\mathbb{R}^2$).
Contrast:
For a torus in $\mathbb{R}^3$ e.g. our torus of revolution, we know there is an elliptic point (it's compact and smooth in $\mathbb{R}^3$).
So the (abstract) flat Riemannian metric is not the induced metric from any embedding of T^2 as a smooth surface in $\mathbb{R}^3$

Example. The real projective plane $\mathbb{RP}^2$ admits a Riemannian metric with constant curvature $+1$.
Indeed, we built a smooth atlas for $\mathbb{RP}^2$ with charts of the form (U, φ) where $U = q\hat{U}$, $q : S^2 \to \mathbb{RP}^2$ quotient map, $\hat{U} \subset S^2$ small enough such that $\hat{U} \subset$ (open hemisphere), and $\varphi : U \to V \subset \mathbb{R}^2$ was $\hat{\varphi} \circ q^{-1}|_U$ $\hat{\varphi} : \hat{U} \to V$ chart on S^2.
Transition maps for this atlas were all the identity or induced from the antipodal map of S^2.
But $a : S^2 \to S^2$ (antipodal) is an isometry so both transition maps preserve usual round metric on S^2

Example. In ES1, we considered the Klein bottle



Whcih has a smooth atlas s.t. all transition maps are translations or reflections.
These preserve the usual flat metric on $\mathbb{R}^2$, so Klein bottle inherits a flat Riemannian metric.
(Note: $\mathbb{RP}^2$, Klein do NOTE embed in $\mathbb{R}^3$ so we had no "non-abstract" construction of Riemannian metrics on these)

Definition. If (Σ_1, g_1) and (Σ_2, g_2) are abstract smooth surfaces with Riemannian metrics g_i on Σ_i, then a diffeomorphism

$$f : \Sigma_1 \to \Sigma_2$$

is an **isometry** if it preserved the lengths of all cruves

Example. If (Σ_2, g_2) is given and $f : \Sigma_1 \to \Sigma_2$ is a diffeomorphism, we can equiv Σ_1 with a metric (called the pullbac metric $f^* g_2 = g_1$) s.t. f becomes an isometry.

Prop. Given a Riemannian metric g on a connected abstract smooth surface Σ, define the **length** metric

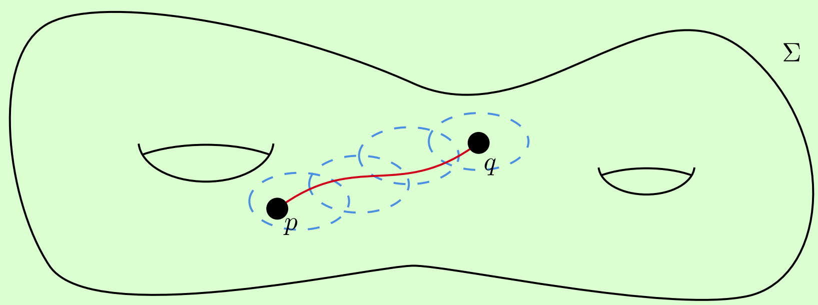
$$d_g(p,q) = \inf_{\gamma} L(\gamma)$$

where γ varies over piecewise smooth paths in Σ from p to q, and $L(\gamma)$ is computed using g. Then

(i) d_g is a metric (in the sense of metric spaces) on Σ, and

(ii) d_g induces the given topology on Σ

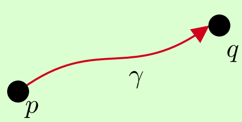
Proof. Σ is path-connected so $\exists$ some piecewise smooth path from p to q so $d_g(p,q) < \infty \; \forall p, q$



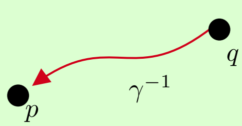
Take some continuous path from p to q and a finite set of charts (U_i, φ_i) with associated parametrisations $\sigma_i = \varphi_i^{-1} : V_i \to U_i \subset \Sigma$ s.t. path $\subset \bigcup_{i=1}^{N} 0i$.
Now pick points

$$\begin{aligned} x_0 = p &\in U_1 \\ x_1 &\in U_1 \cap U_2 \\ x_2 &\in U_2 \cap U_3 \\ &\vdots \\ x_{N-1} = q &\in U_{N-1} \cap U_N \end{aligned}$$

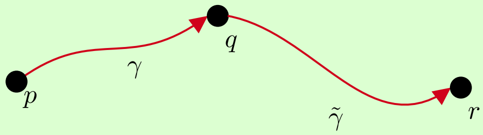
and smooth paths in V_i from $\varphi_i(x_i)$ to $\varphi_{i+1}(x_{i+1})$.
Since our atlas is smooth, being a smooth path in some U_i is the same as being smooth in U_{i+1} whenever $U_i \cap U_{i+1} \neq \varnothing$, as the transition maps are smooth.
So indeed $p, q \in \Sigma$ are joined by some piecewise smooth path. We can reverse paths:



We also have:



Proof (continued)**.** and we can concatenate:



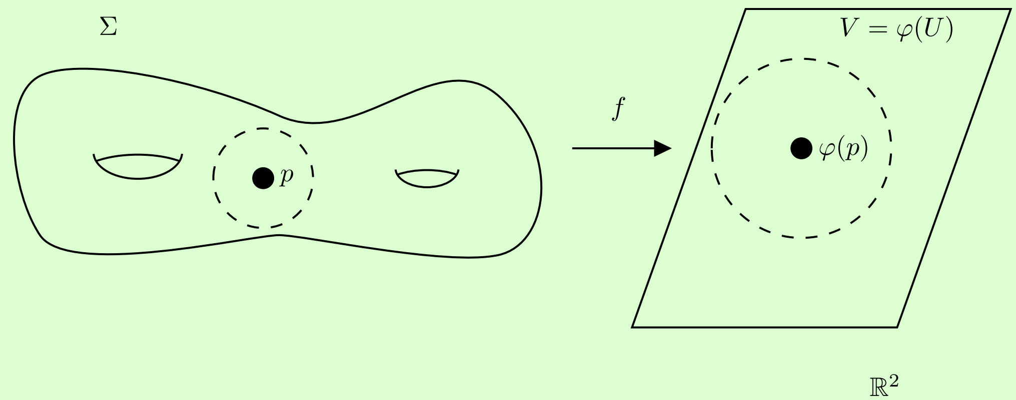
then we have $\tilde{\gamma} \circ \gamma : [0,1] \to \Sigma$ from p to r.
Reversl and concatenation n class of piecewise smooth paths implies

$$d_g(p,q) = d_g(q,p)$$

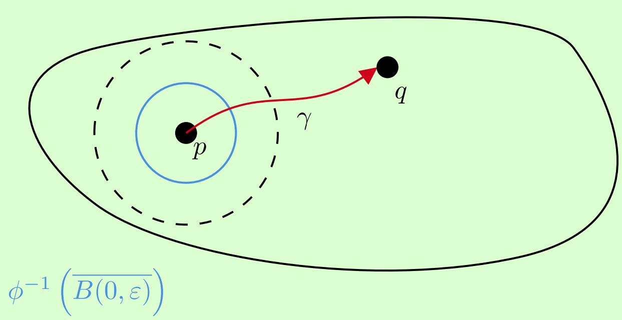
and

$$d_g(p,r) \leq d_g(p,q) + d_g(q,r)$$

remaining $d_g(p,q) = 0 \implies p = 1$ (converse is obvious)



Take $p \in \Sigma$ and fix chart (U, φ) at p. W.l.o.g. suppose $V = B(0,1) \subset \mathbb{R}^2$, with $\varphi(p) = 0$. If $q \neq p \in \Sigma$, $\exists \varepsilon > 0$ s.t. $q \notin \varphi^{-1}(\overline{B(0,\varepsilon)})$.
Suppose $\gamma : [0,1] \to \Sigma$ is a piecewise smooth path from p to q.



Certainly γ must escape $\varphi^{-1}(\overline{B(0,\varepsilon)}) \ni p$. By Δ-inequality, suffices to show $\exists \delta > 0$ s.t. $d_g(p,q) \geq \delta$ whenever $r \in \text{Boundary}(\varphi^{-1}(\overline{B(0,\varepsilon)})) = \varphi^{-1}(\{\text{circle radius } \varepsilon \text{ in } \mathbb{R}^2\})$.
Data of Riemannian metric g on Σ includes $\begin{bmatrix} E_z & F_z \\ F_z & G_z \end{bmatrix}$ for $z \in \overline{B(0,\varepsilon)} \subset V$.
Also have usual Euclidean inner product

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \forall z \in \overline{B(0,\varepsilon)} \subset V$$

Proof (continued)**.** So $\forall z \in \overline{B(0,\varepsilon)}$ we have 2 positive definite inner products, and $\overline{B(0,\varepsilon)}$ is compact so $\exists \delta > 0$ s.t.

$$\begin{bmatrix} E_z - \delta & F_z \\ F_z & G_z - \delta \end{bmatrix} \text{ still +ve definite } \forall z \in \overline{B(0,\varepsilon)}$$

(i.e. $EF - G^2 > 0\ \forall z \in \overline{B(0,\varepsilon)}$ so bounded below by something positive).
So $\text{Length}_g(\hat{\gamma}) \geq \text{Length}_{\delta\cdot\text{euclidean}}(\hat{\gamma})$ (†)
for any $\hat{\gamma}$ contained in $\overline{B(0,\varepsilon)}$. So taking $\hat{\gamma} = \varphi[\gamma \cap \varphi^{-1}(\overline{B(0,\varepsilon)})]$ (part of γ in $\overline{B(0,\varepsilon)}$ w.r.t. our chart).
(†) has RHS $\geq \delta\varepsilon$ so $d_g(p,q) \geq \delta\varepsilon$

Remark. We've proved (i), we should think why the last step of the argument, comparing the inner products $\begin{bmatrix} E & F \\ F & G \end{bmatrix}$ associated to g with Euclidean inner products, also gives (ii) i.e. d_g-metric topology is the one we have from Σ being locally homeomorphic to $\mathbb{R}^2$

Definition. We define an abstract Riemanian metric on the disc

$$D = B(0,1) = \{z \in \mathbb{C} : |z| < 1\}$$

by

$$\begin{aligned} g_{hyp} &= \frac{4(\mathrm{d}u^2 + \mathrm{d}v^2)}{(1-u^2-v^2)^2} \\ &= \frac{4|\mathrm{d}z|^2}{(1-|z|^2)^2} \end{aligned}$$

I.e. if $\gamma : [0,1] \to D$ is smooth,

$$L_{g_{hyp}}(\gamma) = 2\int_0^1 \frac{|\dot{\gamma}(t)|}{1-|\gamma(t)|^2}\,\mathrm{d}t$$

and if $\gamma(t) = (u(t), v(t))$

$$L(\gamma) = 2\int_0^1 \frac{(\dot{u}(t)^2 + \dot{v}(t)^2)^{1/2}}{q - u(t)^2 - v(t)^2}\,\mathrm{d}t$$

(c.f. a FFF with

$$E = G = \frac{4}{(1-u^2-v^2)^2}, \quad F = 0$$

but there is no embedding in $\mathbb{R}^3$ in the background).
The flat metric on $\mathbb{R}^2$ and the round metric on S^2 both have large (transitive) isometry groups.

Recall the Möbius group

$$\text{Möb } = \{z \mapsto \frac{az+b}{cz+d} : \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL(2,\mathbb{C})\}$$

acts on $\mathbb{C} \cup \{\infty\}$

Lemma.

$$\begin{aligned}
\text{Möb} &= \{T \in \text{Möb} : T(D) = D\} \\
&= \{z \mapsto e^{i\theta}\frac{z-a}{1-\bar{a}z} : |a| < 1\} \\
&= \{\begin{bmatrix} a & b \\ \bar{b} & \bar{a} \end{bmatrix} \in \text{ Möb} : |a|^2 - |b|^2 = 1\}
\end{aligned}$$

Proof.

$$\begin{aligned}
\left|\frac{z-a}{1-\bar{a}z}\right| = 1 &\iff (z-a)(\bar{z}-\bar{a}) = (1-\bar{a}z)(1-a\bar{z}) \\
&\iff z\bar{z} - a\bar{z} - \bar{a}z + a\bar{a} = 1 - a\bar{z} - \bar{a}z + a\bar{a}z\bar{z} \\
&\iff |z|^2(1-|a|^2) = 1 - |a|^2 \\
&\iff |z| = 1
\end{aligned}$$

So $z \mapsto e^{i\theta}\frac{z-a}{1-\bar{a}z}$ does preserve $|z| = 1$ and sends $0 \in D$ to $a \in D$. So preserves disc

Lemma. The Riemannian metric $g_{hyp} = \frac{4|\mathrm{d}z|^2}{(1-|z|^2)^2}$ is invariant under $\text{Möb}(D)$, i.e. it acts by hyperbolic isometries.

Proof. $\text{Möb}(D)$ is generated by $e^{i\theta}z$ and $z \mapsto \frac{z-a}{1-\bar{a}z}$, $|a| < 1$. The first (rotations) clearly preserve g_{hyp}.
For second type, let $w = \frac{z-a}{1-\bar{a}z}$ so

$$\begin{aligned}
\mathrm{d}w &= \frac{\mathrm{d}z}{1-\bar{z}z} + \frac{z-a}{(1-\bar{a}z)^2}\bar{a}\,\mathrm{d}z \\
&= \frac{\mathrm{d}z}{(1-\bar{a}z)^2}(1-|a|^2)
\end{aligned}$$

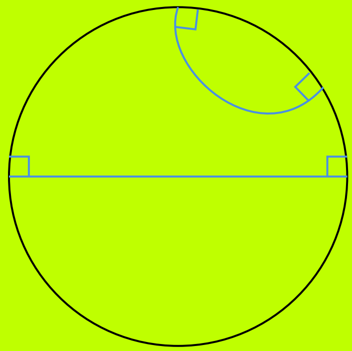
Then

$$\begin{aligned}
\frac{|\mathrm{d}w}{1-|w|^2} &= \frac{|\mathrm{d}z|}{|1-\bar{a}z|^2}\frac{(1-|a|^2)}{\left(1-\left|\frac{z-a}{1-\bar{a}z}\right|^2\right)} \\
&= \frac{|\mathrm{d}z|(1-|a|^2)}{|1-\bar{a}z|^2 - |z-a|^2} \\
&= \frac{|\mathrm{d}z|}{1-|z|^2}
\end{aligned}$$

so done

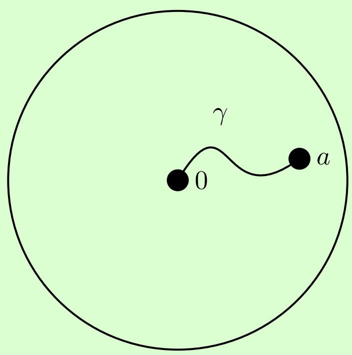
Lemma. (i) Every pair of points in (D, g_{hyp}) is joined by a unique geodesic (up to reparametrisation)

(ii) The geodesics are diameters of the disc and circular arcs orthogonal to ∂D



The whole geodesics (i.e. the ones that are defined on $\mathbb{R}$) are called **hyperbolic lines**

Proof.



Let $a \in \mathbb{R}_+ \cap D$ and γ a smooth path from $0 \in D$ to a. Say $\gamma(t) = (u(t), v(t))$ and note that $\mathrm{Re}(\gamma)(t) = (u(t), 0)$ is also a smooth path from 0 to a

$$\begin{aligned} L(\gamma) &= \int_0^1 \frac{2|\dot{\gamma}(t)|}{1 - |\gamma(t)|^2} \, \mathrm{d}t \\ &= \int_0^1 \frac{2\sqrt{\dot{u}^2 + \dot{v}^2}}{1 - u^2 - v^2} \, \mathrm{d}t \\ &\geq \int_0^1 \frac{2|\dot{u}(t)|}{1 - u^2} \, \mathrm{d}t \end{aligned}$$

$$\geq \int_0^1 \frac{2\dot{u}(t)}{1 - u(t)^2} \, \mathrm{d}t$$

With equalities $\iff \dot{v} \equiv 0 \iff v \equiv 0$ and equality $\iff u$ is monotonic.
So the arc of the diameter (parametrised monotonically) is globally length-minimised among all paths from 0 to a, and hence a geodesic.
Indeed $L(\text{diameter arc}) = 2\tanh^{-1}(a)$.
Now 0 and $a \in \mathbb{R}_+ \cap D$ are joined by a unique geodesic and $\text{Möb}(D)$ acts transitively and can be used to send any $p, q \in D$ to $0, a \in \mathbb{R}_+ \cap D$.
Since isometries send geodesics to geodesics, every $p, q \in D$ is joined by one geodesic.
And Möbius maps send circles to circles, and preserve angles and hence orthogonality to ∂D.
This impies our description of geodesics

Corollary. If $p, q \in D$, then

$$d_{hyp}(p,q) = 2\tanh^{-1}\left|\frac{p-q}{1-\bar{p}q}\right|$$

Definition. The **hyperbolic upper half-plane** (h, g_{hyp}) is the set

$$h = \{z \in \mathbb{C} : \ \mathrm{Im}(z) > 0\}$$

with the abstrat Riemannian metric $\frac{\mathrm{d}x^2 + \mathrm{d}y^2}{y^2}$ (or $\frac{|\mathrm{d}z|^2}{\mathrm{Im}(z)^2}$)

Lemma. The (D^2, g_{hyp}) and (h, g_{hyp}) are isometric

Proof. We have maps

$$h \to^T D \quad D \to h$$
$$w \mapsto \frac{w-i}{w+i} \quad z \mapsto i\left(\frac{1-z}{1+z}\right)$$

whcih are inverse diffeomorphisms (compare to ES4).
If $w \in h$, let $T(w) = \frac{w-i}{w+i} \in D$. Then

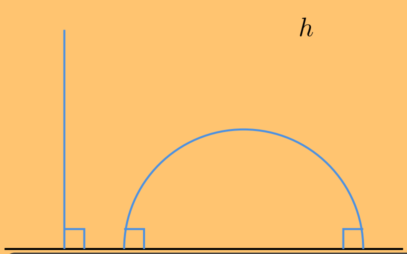
$$T'(w) = \frac{1}{w+i} - \frac{w-i}{(w+i)^2} = \frac{2i}{(1+i)^2}$$

Considering $T(w) = z \in D$

$$\begin{aligned}\frac{|\mathrm{d}z|}{1-|z|^2} &= \frac{|\,\mathrm{d}(Tw)}{1-|T(w)|^2} = \frac{|T'(w)| \cdot |\mathrm{d}w|}{1-|Tw|^2} \\ &= \frac{2|\mathrm{d}w|}{|w+i|^2\left(1-\left|\frac{w-i}{w+i}\right|^2\right)} = \frac{|\mathrm{d}w|}{2\mathrm{Im}(w)}\end{aligned}$$

i.e. $\frac{4|\mathrm{d}z|^2}{(1-|z|^2)^2}$ is the metric obtained under pullback by T from $\frac{|\mathrm{d}w|}{\mathrm{Im}(w)}$

Corollary. In (h, g_{hyp}), every pair of points is joined by a unique geodesic, and the geodesics are vertical straight lines and semi-circles centered on $\mathbb{R}$



Proof. Our isometry $h \to D$ is given my a Möbius map sending $\mathbb{R} \cup \{\infty\} \to \partial D$, and Möbius maps preserve circles and orthogonality

Remarks.

(i) Wehn we discussed surfaces in $\mathbb{R}^3$ with constant Gauss curvature, we saw that if something had $\kappa = -1$, its FFF in geodesic normal co-ordinates was $\mathrm{d}u^2 + \cosh^2(u)\,\mathrm{d}v^2$ and there is a change of variables taking that to $\frac{\mathrm{d}V^2 + \mathrm{d}W^2}{W^2}$ ($= g_{hyp}$ on h)
Gauss' theorema egregium implies Gauss curvature makes sense for an abstract Riemannian metric (lengths, areas angles do; so do geodesics, and hence so do co-ordinate systems letting us express/ define κ)
So h has constant curvature -1

(ii) Suppose we looked for a metric

$$d : D \times D \to \mathbb{R}_{\geq 0}$$

on D^2 with the properties

- Möb(D)-invariant:

$$d(Tx, Ty) = d(x, y) \quad \forall T \in \text{Möb}(D)$$

- $\mathbb{R} \cap D$ to be length-minimising

Möb(D)-invariance means that d is completely determined by $d(0, a)$for $a \in \mathbb{R}_+ \cap F$. Call this $p(a)$.
If $\mathbb{R}_+ \cap D$ is "length-minimising", distance along it should be additive, so if $0 < a < b < 1$,

$$d(0, a) + d(a, b) = d(0, b)$$

i.e.

$$p(a) + p(\frac{b-a}{1-ab}) = p(b)$$

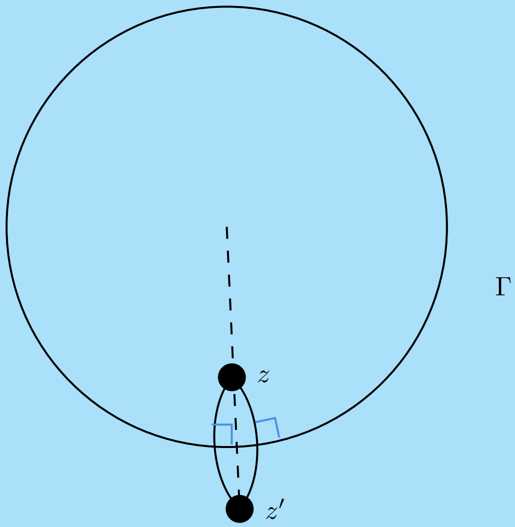
If we furthermore suppose p is differentiable and differentiate w.r.t. b and set $b = a$,

$$p'(A) = \frac{p'(a)}{1-a^2}$$

i.e. $p(a) = \text{const}.\tanh^{-1}(a)$.
So up to scale, length metric associated to g_{hyp} on D is the only metric with these nice properties.
The scale is chosen to make $\kappa \equiv -1$ (and not $-c$ for some other $c > 0$)

We would like to understand the full isometry group of (D, g_{hyp}) or (h, g_{hyp}) The result is we need to add "reflections" in hyperbolic lines, called **inversions**

Definition. Let $\Gamma \subset \hat{\mathbb{C}}$ be a circle or line. We say points $z, z' \in \hat{\mathbb{C}}$ are **inverse** for Γ if every circle through z and orthogonal to Γ also passes through z'



Lemma. For every circle $\Gamma \subset \mathbb{C}$ and $z \in \mathbb{C}$, there is a unique inverse point w.r.t. Γ for z

Proof. Recall Möbius maps send circles (in $\hat{\mathbb{C}}$) to circles and preserve angles. So if z, z' are inverse for Γ, and $\Gamma \in$ Möb, then Tz and Tz' are inverse for $T(\Gamma)$.
If $\Gamma = \mathbb{R} \cup \{\infty\}$, then $Jz = \bar{z}$ gives inverse points (i.e. this map satisfies the requirements and is unique such).
Now if $\Gamma \subseteq \hat{\mathbb{C}}$ is any circle, $\exists T \in$ Möb s.t.

$$T(\mathbb{R} \cup \{\infty\}) = \Gamma$$

Define inversion in Γ by

$$J_\Gamma : z \mapsto T(\text{conj.})T^{-1}(z)$$

This works!

Definition. The map $z \mapsto J_\Gamma(z)$ sending z to the unique inverse point z' for z w.r.t. Γ is called **inversion** in Γ.
(This fixes all points of Γ and exchanges the two complementary regions)

Examples. (i) If Γ is a straight line (circle in $\hat{\mathbb{C}}$ through $\infty \in \hat{\mathbb{C}}$), J_Γ is reflection in Γ

(ii) If $S^1 = \{|z| = 1\}$

$$J_{S^1} : z \mapsto \frac{1}{\bar{z}} \quad (0 \mapsto \infty)$$

(cf. ES4)

Remark. A composition of two inversions is a Möbius map. Let $C : z \mapsto \bar{z}$ be inversion in $\mathbb{R} \cup \{\infty\}$ so if Γ is any circle,

$$J_\Gamma = T \circ C \circ T^{-1} \tag{*}$$

where $T(\mathbb{R} \cup \{\infty\}) = \Gamma$. Now given Γ_1 and Γ_2 circles, and T_i takes $\mathbb{R} \cup \{\infty\}$ to Γ_i, then

$$\begin{aligned} J_{\Gamma_1} \circ J_{\Gamma_2} &= (J_{\Gamma_1} \circ C) \circ (C \circ J_{\Gamma_2}) \\ &= (C \circ J_{\Gamma_1})^{-1} \circ (C \circ J_{\Gamma_2}) \end{aligned}$$

and

$$C \circ J_\Gamma = C \circ T \circ C \circ T^{-1}$$

by (*). So STP $C \circ T \circ C \in$ Möb. But if $T(z) = \frac{az+b}{cz+d}$

$$C \circ T \circ C : z \mapsto \frac{\bar{a}z + \bar{b}}{\bar{c}z + \bar{d}} \in \text{ Möb}$$

Lemma. An orientation preserving isometry of $(\mathbb{H}^2, g_{hyp})$ is an element of Möb($\mathbb{H}$) where

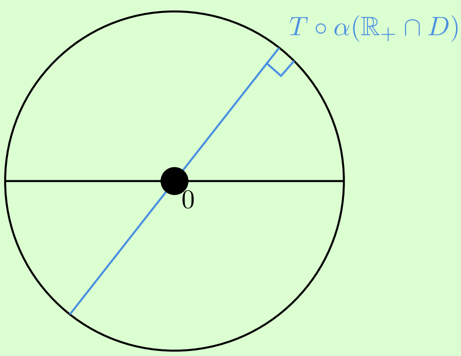
$$\mathbb{H} = D \text{ or } h$$

The full isometry group is generated by inversions in hyperbolic lines (circles $\perp$ to $\partial\mathbb{H}$)

Proof. Suffices to prove this in either model.
In D, inversion in $\mathbb{R} \cap D$, i.e. conjugation, preserves

$$g_{hyp} = \frac{4|\,\mathrm{d}z|^2}{(1-|z|^2)^2}$$

Now Möb($\mathbb{H}$) acts transitively on geodesics and its acting by isometries, so all inversions in hyperbolic lines are isometries.
Now suppose $\alpha \in \text{Isom}(D, g_{hyp})$ is some isometry of the hyperbolic disc. $a := \alpha(0) \in D$ and using $z \mapsto \frac{z-a}{1-\bar{a}z}$, $\exists T \in$ Möb(D) s.t. $R \circ T \circ \alpha$ sends $D \cap \mathbb{R}_+$ to itself.



Composing with C and if necessry, $\exists A \in \text{Isom}(D)$ of the form (inversion)$\circ$(Möbius) s.t. $A \circ \alpha$ fixes $\mathbb{R} \cap D$ pointwise & and fixes $i\mathbb{R} \cap D$ pointwise (unique geodesic through 0 $\perp$ to $\mathbb{R} \cap D$).
Now $A \circ \alpha =$id, so $\alpha = A^{-1}$. If α preserved orientation and fixed $\mathbb{R} \cap D$, it necessarily fixed $i\mathbb{R} \cap D$ pointwise and so in fact $\alpha = (R \circ T)^{-1} \in$ Möb.
In general, $\alpha \in \langle$ Möb($\mathbb{H}$), inversions in hyperbolic geodesics$\rangle$ and we saw compositions of 2 inversions are Möbius maps, and in fact every Möbius map is a product of inversions (cf ES4)

Remark. In upper half-plane model

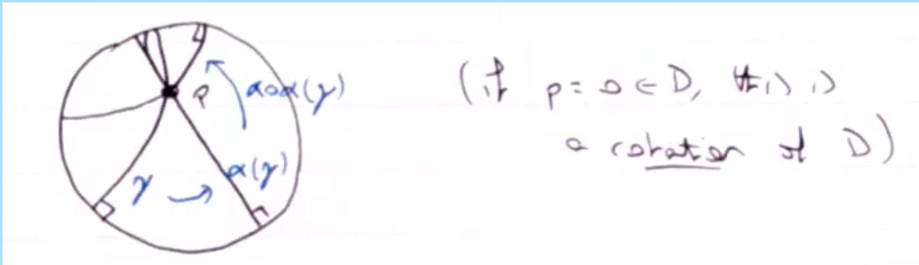
$$\begin{aligned}\text{Möb}(h) &= \mathbb{P}SL(2,\mathbb{R}) \\ &= \{z \mapsto \frac{az+b}{cz+d} : \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL(2,\mathbb{R})\}\end{aligned}$$

and

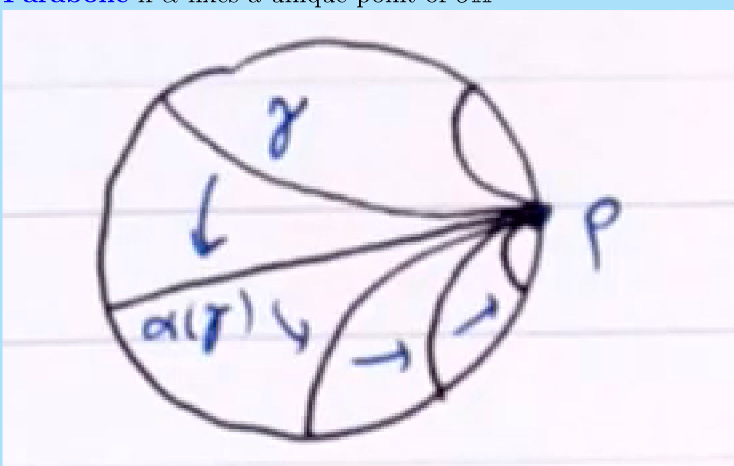
$$d_{hyp}(a,b) = 2\tanh^{-1}\left|\frac{b-a}{b-\bar{a}}\right|$$

Definition. Let $\alpha \in \text{Isom}^+(\mathbb{H}) = \text{Möb}(\mathbb{H})$ orientation preserving isometries. Suppose $\alpha \neq \text{id}$. We say α is

- **Elliptic** if α fixes $p \in \mathbb{H}$

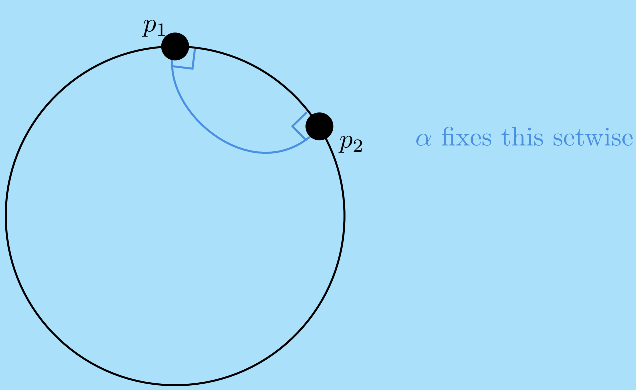


- **Parabolic** if α fixes a unique point of $\partial\mathbb{H}$



(If $p = \infty \in h$, then $\alpha : z \mapsto z + t$)

- **Hyperbolic** if α fixes 2 points on $\partial\mathbb{H}$



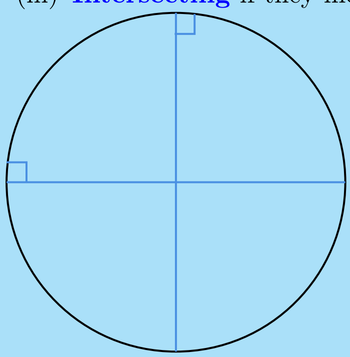
Exercise: All elements of Möb($\mathbb{H}$) falls into one of these 3 cases

Definition. Let l, l' be hyperbolic lines. We say l, l' are

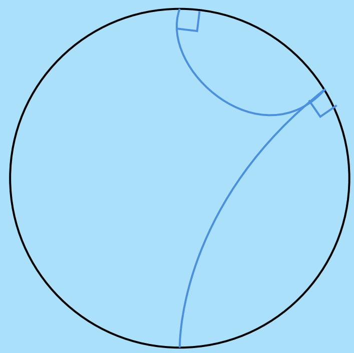
(i) **Parallel** if they meet in $\partial\mathbb{H}$ but not in $\mathbb{H}$

(ii) **Ultraparallel** if they do not meet in $\mathbb{H} \cup \partial\mathbb{H}$

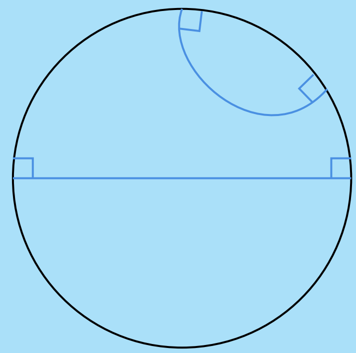
(iii) **Intersecting** if they meet in $\mathbb{H}$



intersecting



parallel

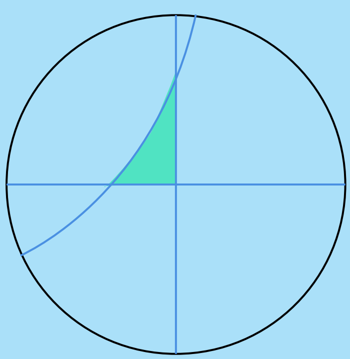


ultra-parallel

Exercise (cf ES4): What is $J_{\Gamma_1} \circ J_{\Gamma_2}$ where $\{\Gamma_1, \Gamma_2\}$ are in the 3 cases?

Remark. The parallel postulate fails!

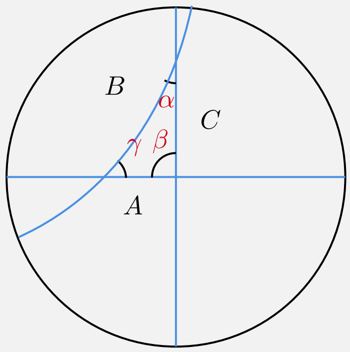
Definition. A **hyperbolic triangle** is the region bound by 3 hyperbolic lines, no two of which are ultraparallel



Vertices lying at infinity (on $\partial\mathbb{H}$) are called **ideal** vertices

Note. Remember points of $\partial\mathbb{H}$ are NOT in the hyperbolic plane

Consider a hyperbolic triangle



We take a triangle in $\mathbb{H}^2$ with hyperbolic side lengths A, B, C and opposite angles α, β, γ

Note. The hyperbolic metric g_{hyp} was

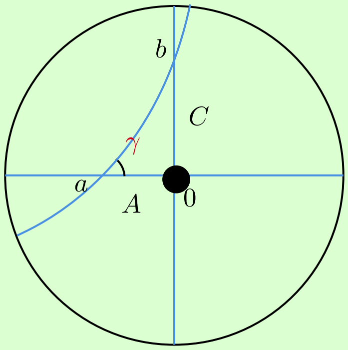
$$\frac{\mathrm{d}u^2 + \mathrm{d}v^2}{(1-u^2-v^2)^2} \text{ with } E = G,\ F = 0$$

So this is conformal: angles computed w.r.t. g_{hyp} agree with Euclidean angles

Equation (Hyperbolic cosine formula)**.**

$$\cosh C = \cosh A \cosh B - \sinh A \sinh B \cos \gamma$$

Proof. To singulify, by an isometry, put vertex of angle γ at $0 \in D$ and put the vertex of angle β on $\mathbb{R}_+ \cap D$



$$d_{hyp}(0, a) = 2 \tanh^{-1}(a)$$

i.e. $a = \tanh \frac{A}{2}$ and $b = e^{i\gamma} \tanh\left(\frac{B}{2}\right)$ and

$$\left| \frac{b - a}{1 - \bar{a}b} \right| = \tanh\left(\frac{C}{2}\right)$$

If $t = \tanh(\lambda/2)$, "recall"

$$\cosh(\lambda) = \frac{1 + t^2}{1 - t^2}$$
$$\sinh(\lambda) = \frac{2t}{1 - t^2}$$

So

$$\cosh(A) = \frac{1 + |a|^2}{1 - |a|^2}$$
$$\cosh(B) = \frac{1 + |b|^2}{1 - |b|^2}$$

and

$$\begin{aligned} \cosh C &= \frac{|1 - \bar{a}b|^2 + |b - a|^2}{|1 - \bar{a}b|^2 - |b - a|^2} \\ &= \frac{(1 + |a|^2)(1 + |b|^2) - 2(\bar{a}b + a\bar{b})}{(1 - |a|^2)(1 - |b|^2)} \end{aligned}$$

but $a \in \mathbb{R}$ and $b + \bar{b} = 2\mathrm{Re}(b) = 2b \cos \gamma$. Using that

$$\cosh C = \cosh A \cosh B - \sinh A \sinh B \cos \gamma$$

as required

Remarks.

(i) If A, B, C small, and

$$\sinh A \approx A$$
$$\cosh A \approx 1 + \frac{A^2}{2}$$

then formula reuces to

$$C^2 = A^2 + B^2 - 2AB\cos\gamma$$

(up to higher order terms), Euclidean cosine formula.

Recall dilating a surface in $\mathbb{R}^3$ rescaled its curvature. Zooming in to any point on an abstract smooth surface with a Riemannian metric, the surface looks closer and closer to being flat

(ii) $\cos\gamma \geq -1$ so formula says

$$\begin{aligned}\cosh C &\leq \cosh A \cosh B + \sinh A \sinh B \\ &= \cosh(A+B)\end{aligned}$$

and cosh increasing so $C \leq A + B$ which is the triangle inequality for g_{hyp}. (We already know the triangle inequality holds for any length metric, but our formula refines it)

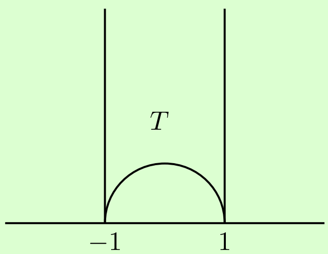
2.1 Area of Triangles

Claim. Let $T \subset \mathbb{H}^2$ be a hyperbolic triangle with internal angles α, β, γ

$$\text{Area}_{hyp}(T) = \pi - \alpha - \beta - \gamma$$

(this is a version of Gauss-Bonnet)

Proof. Möb($\mathbb{H}^2$) acts transitively on triples of points in the boundary with the correct cyclic order. In particular, $\exists$ a unique ideal triangle (all 3 vertices at infinity) up to isometry. Consider:

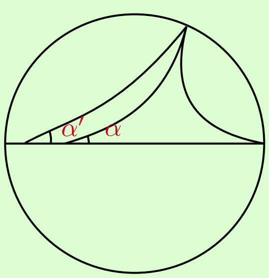


$$\text{Area}_{hyp}(T) = \int_{-1}^{1} \int_{\sqrt{1-x^2}}^{\infty} \frac{1}{y^2} \, \mathrm{d}y \, \mathrm{d}x$$

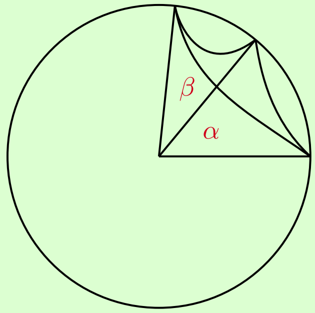
noting $\sqrt{EG - F^2} = \frac{1}{y^2}$. So

$$\text{Area}(T) = \int_{-1}^{1} \frac{\mathrm{d}x}{\sqrt{1-x^2}} = \pi$$

Let $A(\alpha)$ be the area of a triangle with angles $0, 0, \alpha$



$A(\alpha)$ is decreasing in α, and clearly continuous in α.



Proof (continued)**.**

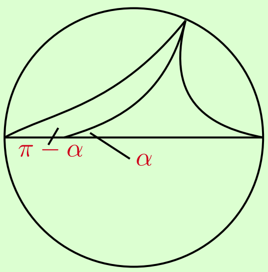
$$A(\alpha) + A(\beta) = A(\alpha + \beta) + \pi$$

Set $F(\alpha) = \pi - A)\alpha)$, this says

$$F(\alpha) + F(\beta) = F(\alpha + \beta)$$

and F is continuous and increasing so $F(\alpha) = \lambda\alpha$ for some $\lambda \in \mathbb{R}_+$

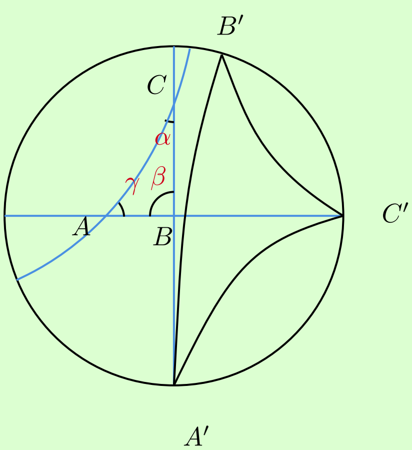
$$\implies A(\alpha) = \pi - \lambda\alpha$$



This picture shows

$$A(\alpha) + A(\pi - \alpha) = \pi \implies \lambda = 1$$

General case:



Now

$$ABC + A'CB' + A'B'C' = AB'C' + A'BC'$$

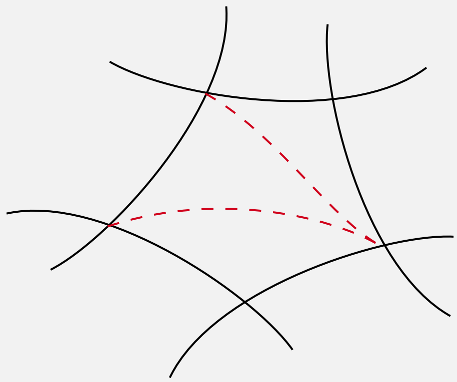
where ABC stands for Area_{hyp}(triangle with vertices A, B, C)

$$\implies ABC + \pi - (\pi - \gamma) + \pi = (\pi - \alpha) + (\pi - \beta)$$

which rearranges to what we want

Note. We allow R to have ideal vertices, i.e. ones at infinity (on $\partial\mathbb{H}$) then the internal angle is zero

If G is a hyperbolic n-gon, i.e. region bound by n hyperbolic geodesics:



then Area(G) is

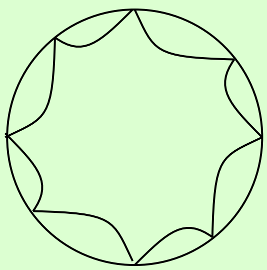
$$(n-2)\pi - \sum_{i=1}^{n} \alpha_i$$

where $\{\alpha_i\}$ are the internal angles. (See this by cutting G into hyperbolic triangles. Recall any two points in $\mathbb{H}^2$ are joined by a unique geodesic)

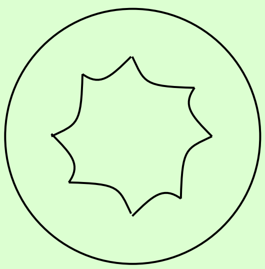
Lemma. If $g \geq 2$, there is a regular $4g$-gon in $\mathbb{H}^2$ with internal angle

$$\frac{2\pi}{4g} = \frac{\pi}{2g}$$

Proof.



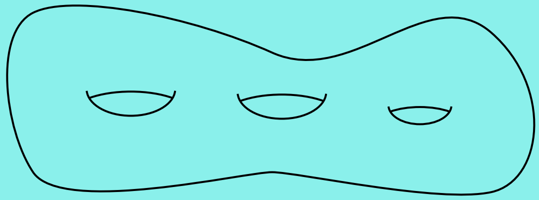
Take an ideal $4g$-gon in D^2, with all vertices at ∂D, being the $4g$-th roots of unity. Slide all-vertices radially inwards



This gives a continuous family of regular $4g$-gons, and their areas vary monotonically from $(4g-2)\pi$ to 0. The interval angle varies continuously from 0 to β_{min} s.t. $(4g-2)\pi = 4g\beta_{min}$, and

$$\frac{\pi}{2g} \in (0, \beta_{min})$$

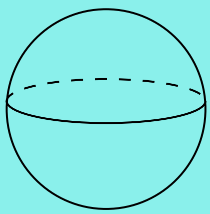
Theorem. For each $g \geq 2$, $\exists$ an abstract Riemannian metric on the compact surface of genus g with curvature $\kappa \equiv -1$ (locally isometricto $\mathbb{H}^2$). Recall:



$g = 3$

genus g =number of holes

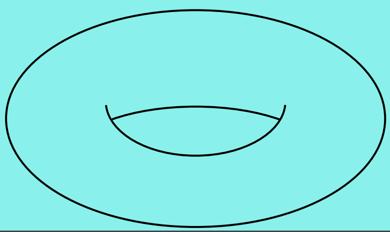
$g = 0$



round sphere

$\kappa \equiv 1$

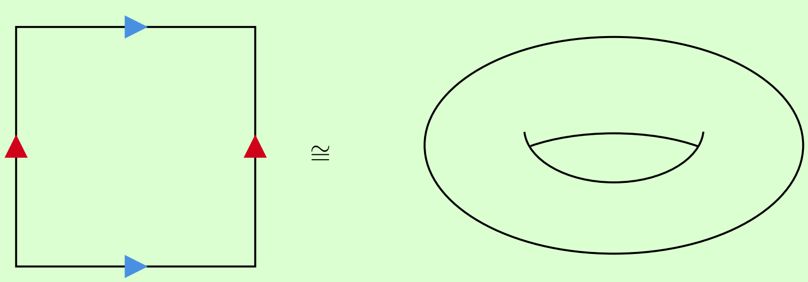
$g = 1$



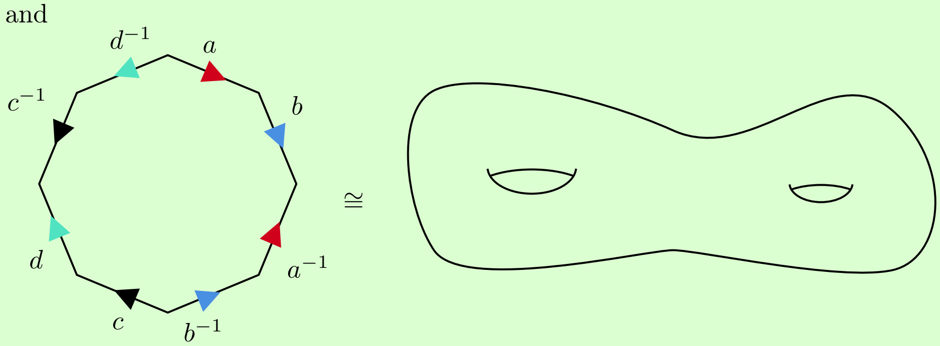
torus with $\kappa \equiv 0$

$= \mathbb{R}^2/\mathbb{Z}^2$

Proof. Recall



and



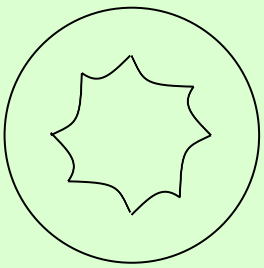
Proof (continued)**.** Analagously, a $4g$-gon with side labels

$$a_1, b_1, a_1^{-1}, b_1^{-1}, a_2, b_2, a_2^{-1}, b_2^{-1}, \ldots, a_g, b_g, a_g^{-1}, b_g^{-1}$$

would give on gluing an orientable compact surface of genus g. Observation: let's say a flag comprises

(i) an oriented hyperbolic line
(ii) a point on that line
(iii) a choice of side to the line

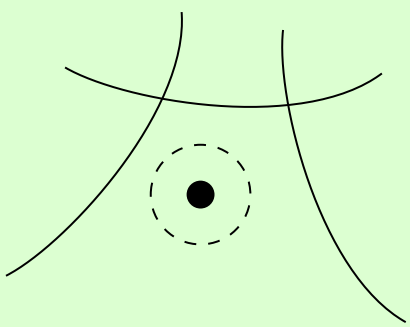
Given 2 such, there is a hyperbolic isometry taking one to the other. (We can "swap sides" using inversions)



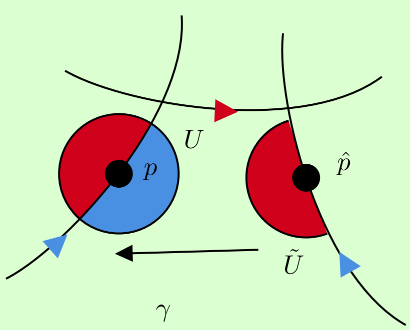
Take our regular $4g$-gon with internal angle $\pi/2g$. For each paired set of 2 edges, there is a hyperbolic isometry taking one to the other (respecting orientations) and taking the "inside" of polygon at e_1 to the "outside" at its twin e_2.

Now we'll give an atlas Σ_g as follows:

- if $p \in$ interior(Polygon), just take a small disc contained in interior(P) and include it into D ($\subseteq \mathbb{R}^2$)



- if $p \in$ edge(P), say e_1, and $\hat{p} \in e_2$ on the paired edge, we have an isometry γ from e_1 to e_2 exchanging sides (as above)



$$[p] = [\hat{p}] \in \Sigma = (\text{Polygon})/\sim$$

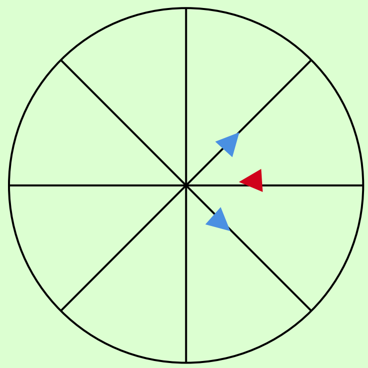
Define $U \cup \tilde{U} \to D$ (hyperbolic disc) to be inclusion on U and γ on $\tilde{U}$. These descend to maps on

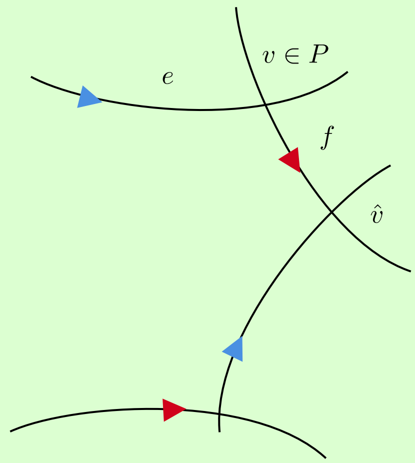
$$[U] \subseteq \Sigma, \quad [\tilde{U}] \subset \Sigma$$

which agree on the set $[U \cap \tilde{U}]$ (projection to Σ_g)

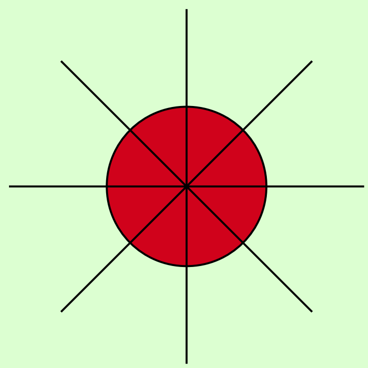
Proof (continued). • In our gluing pattern, all $4g$ vertices are identified to one point of Σ_g and we want a chart there.

Imaging putting a vertex of P at $0 \in D$



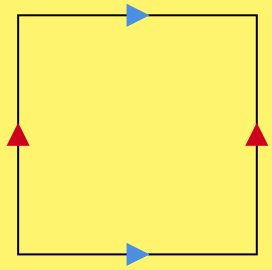


Condition that internal angles sum to 2π means that we have a neighboughood of $[v] \in \Sigma_g$



Define chart at $[v] \in \Sigma$ this way all our charts are obtained either from inclusion into D, or the composite of inclusion and some hyperbolic isometry. So the transition maps are hyperbolic isometries (so smooth)

Remark.

$T^2 =$  $\cong \mathbb{R}^2/\mathbb{Z}^2$

The second description was especially helpful for seeing the flat metric.
In fact, for Σ_g, we picked $2g$ hyperbolic isometries (which paired sides) so we have a group

$$\Gamma = \langle \gamma_1, \ldots, \gamma_{2g} \rangle \subseteq \text{ Isom}(\mathbb{H})$$

Part II Algebraic topology will construct

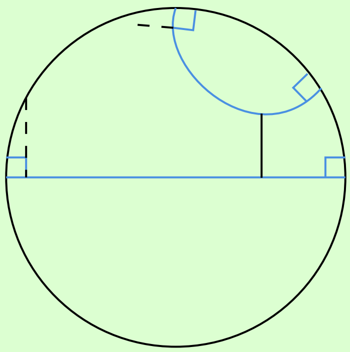
$$\Sigma_f = \mathbb{H}/\Gamma$$

A variant construction: we have another construction of metrics on Σ_g $(g \geq 2)$ which starts from polygons but is "more flexible".

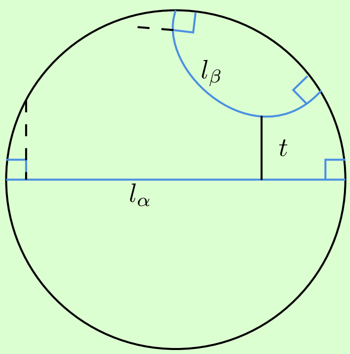
Lemma. For each $l_\alpha, l_\beta, l_\gamma \in \mathbb{R}_{>0}$ there is a right-angled hyperbolic hexagon with side lengths

$$l_\alpha,\ ?,\ l_\beta,\ ?,\ l_\gamma, ?$$

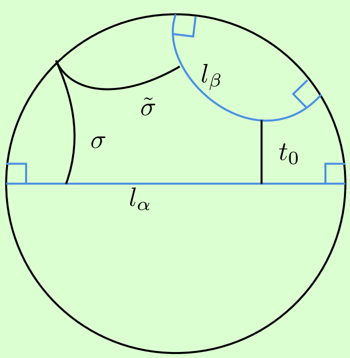
Proof.



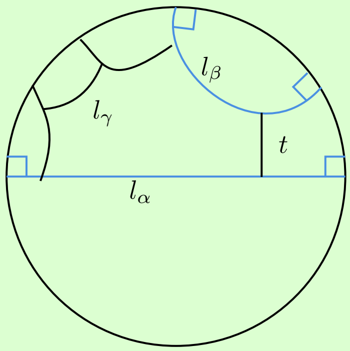
Take a pair of ultraparallel hyperbolic lines. ES4: $\exists$ a unique common perpendicular geodesic. Given $l_\alpha > 0$ and $l_\beta > 0$, we shoot off new geodesics orthogonal to the originals having travelled l_α, l_β from the common perpendicular. In fact, given $t > 0$, $\exists$ an original ultraparallel pair distance exactly t apart.



If $t >> 0$, the new geodesics will also be ultraparallel.
$\exists$ a threshold value t_0, by continuity when the new geodesics first become parallel:

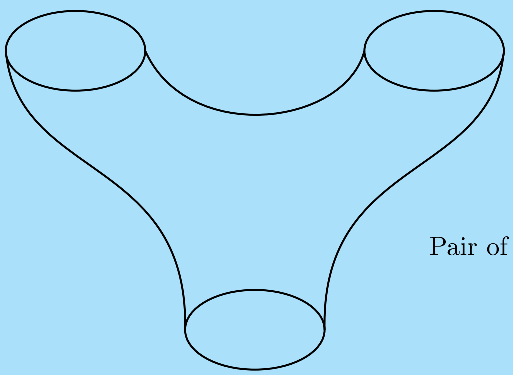


Proof (continued)**.** Now consider $t \in (t_0, \infty)$. Then $\sigma, \tilde{\sigma}$ are ultraparallel, so they have a unique common perpendicular. As we increase t, the length of that increases monotonically, so $\exists$ a value of $t > t_0$ s.t. the new common $\perp r$ has length l_γ

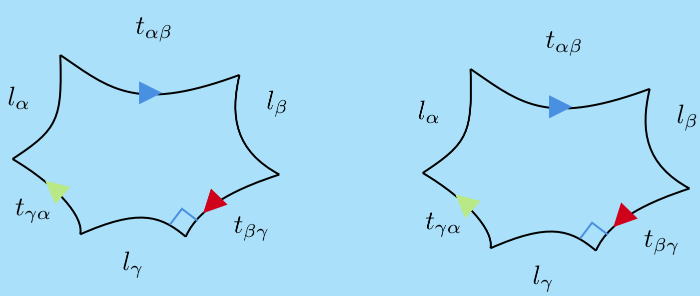


This is our right-angled hexagon

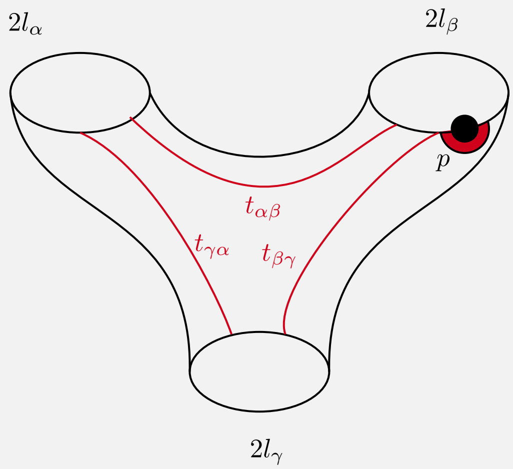
Definition. A **pair of pants** is any topological space homeomorphic to the complement of 3 open discs in S^2



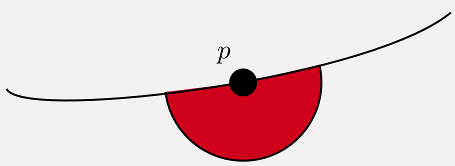
Pair of pants (a surface with boundary)



We take 2 copies of the $(l_\alpha, l_\beta, l_\gamma)$ hexagon. The original configuration of 2 ultraparallel geodesics distance t apart is unique upt oisometry (exercise). So our hexagon is unique.
We glue this pair of polygons as indicated
Since hexagon was right-angled, in the end we get a "hyperbolic" pair of pants



the boundary circles are geodesics in the sense that for any point on such, the local neighbourhood is like

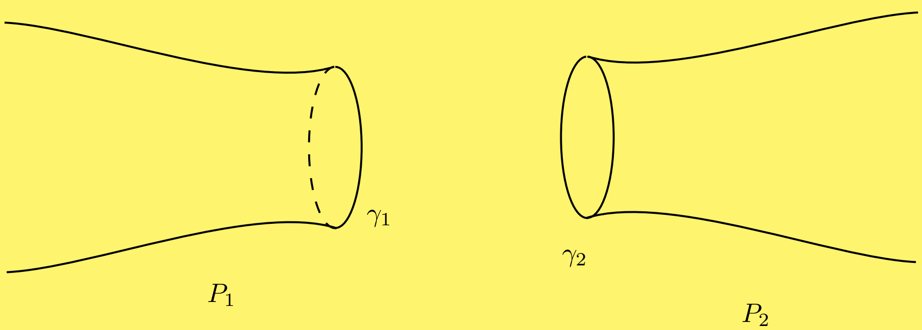


$\subseteq H$

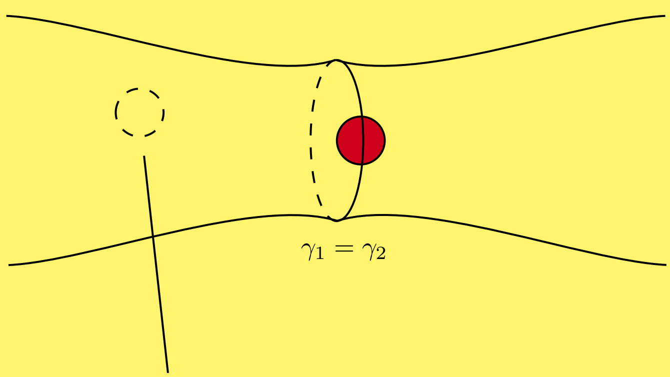
Remark. Using pairs-of-pants, we also obtain hyperbolic metrics on compact surfaces.
If P_1 and P_2 are two hyperbolic "surfaces" with geodesic boundary circles and if $\gamma_1 \subseteq P_1$ and $\gamma_2 \subseteq P_2$ are boundary circles of the same hyperbolic length, then

$$P_1 \cup_{\gamma_1 \sim \gamma_2} P_2$$

inherits a hyperbolic metric where we glue by an isometry of γ_1 and γ_2

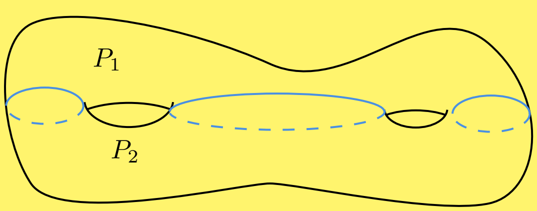


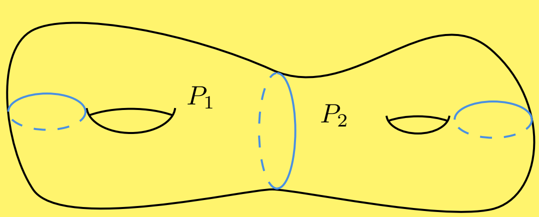
If $l(\gamma_1) = l(\gamma_1)$ (length in the hyperbolic metrics on P_i), then $P_1 \cup_{\gamma_1 \sim \gamma_2}$ is hyperbolic



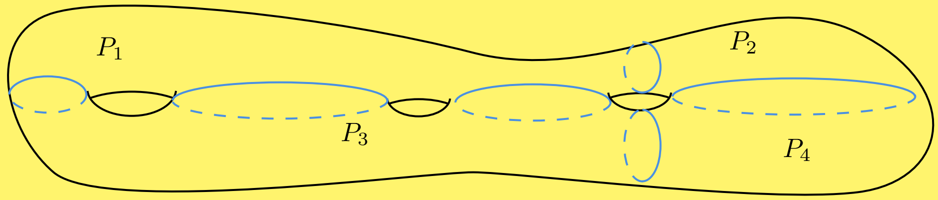
Open neighbourhood looks like a disc in $\mathbb{H}$ since P was hyperbolic

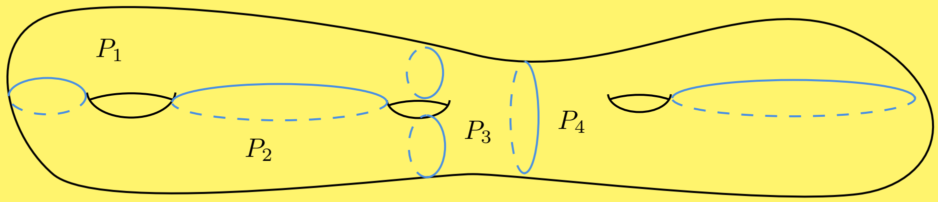
At $p \in \gamma_1 \sim \gamma_2$ we get a chart to a small disc in $\mathbb{H}$ using that the boundary circles were geodesic (cf charts near points $p \in \text{edge}(Q)$ for a hyperbolic polygon Q with side identifications).
Now every compact surface of genus $g \geq 2$ can be built from pairs of pants





Remark (continued)**.** For $g = 3$:





These are topological piictures, but we can use them as guides for gluing pairs-of-pants along common-length boundaries

Notes. We have many choices here

(i) lengths of circles coming from hyperbolic hexagons
(ii) Topologically different "pants" decompositions

Recall:

(i) In a spherical triangle with internal angles α, β, γ, we saw in ES2 area $\alpha + \beta + \gamma - \pi$ whilst a hyperbolic triangle with internal angles α, β, γ had area $\pi - \alpha - \beta - \gamma$
(ii) We also saw a convex Gauss-Bonnet theorem

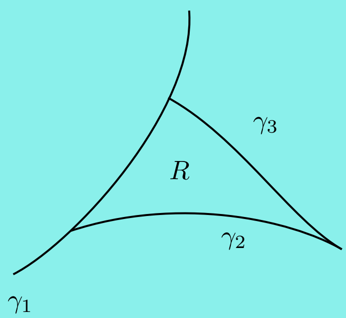
$$\int_\Sigma \kappa \, \mathrm{d}A = 4\pi$$

if Σ bounds a convex region in $\mathbb{R}^3$ and $\kappa_\Sigma > 0$

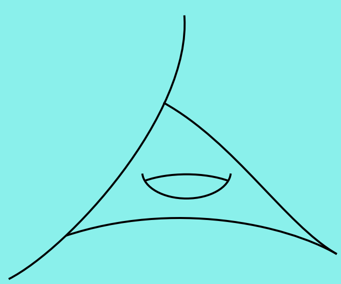
Theorem (Local Gauss-Bonnet)**.** Let Σ be an abstract smooth surface with abstract Riemannian metric g_Σ. Take a geodesic polygon R on Σ, i.e. a smooth disc whose boundary is decomposed into finitely many geodesic arcs. Then

$$\int_{R\subseteq\Sigma} \kappa_\Sigma \,\mathrm{d}A = \sum_{i=1}^{n} \alpha_i - (n-2)\pi$$

where $\{\alpha_i\}$ are the internal angles of the polygon R



Geodesic polygon



Not a geodesic polygon for our purposes

Theorem (Global Gauss-Bonnet)**.** If Σ is a compact smooth surface with abstract Riemannian metric g_Σ

$$\int_{\Sigma} \kappa_\Sigma \,\mathrm{d}A = 2\pi\chi(\Sigma)$$

Remarks.

(i) Gauss curvature, area and $\mathrm{d}A$ can be defined just using an abstract Riemannian metric

(ii) For our hyperbolic surfaces

- We glued Σ_g from a regular $4g$-gon with angules $\pi/2g$ so then total area of Σ

$$\begin{aligned}\int_\Sigma 1\,\mathrm{d}A &= \text{Area(Polygon)} \\ &= (4g-2)\pi - \sum_1^{4g} \frac{\pi}{2g} \\ &= (4g-4)\pi\end{aligned}$$

  and

$$\kappa_\Sigma \equiv -1, \quad \chi(\Sigma_g) = 2-2g$$

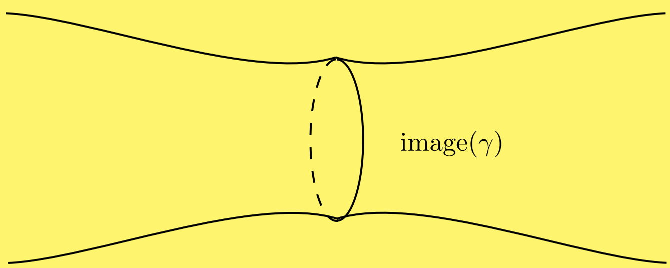
- A right-angled hyperbolic hexagon has area

$$4\pi - \sum_1^6 \frac{\pi}{2} = \pi$$

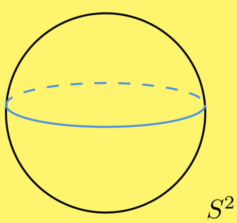
  Each pair of pants has 2-such, and a genus g surfaces uses $2g-2$ pants. So again this fits.

(iii) Shows $\chi(\Sigma)$ doesn't depend on choice of triangulation

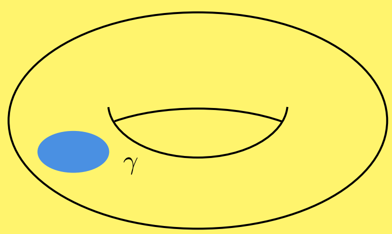
(iv) Suppose Σ is a flat surface, so $\kappa_\Sigma \equiv 0$ and γ is a closed geodesic, i.e. $\gamma : \mathbb{R} \to \Sigma$ is defined on all of$\mathbb{R}$ but $\exists T > 0$ s.t. $\gamma(t+T) = \gamma(t)\ \forall t$



Then γ cannot bound a (smooth) disc in Σ

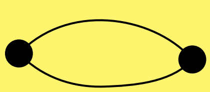


$\kappa > 1$



If $\kappa_\Sigma = 0$, γ not geodesic

Indeed,



Local Gauss-Bonnet

$$0 = \int_R \kappa_\Sigma\,\mathrm{d}A = \underbrace{\sum_1^n \alpha_i - (n-2)\pi}_{2\pi}$$

$n = 2 \implies \alpha_1 = \pi = \alpha_2$ ※

Global Gauss-Bonnet is deduced from local Gauss-Bonnet

Lemma. A compact smooth surface admits subdivisions into geodesic polygons (cf "exponential map" in Part II)

Given that lemma, take a subdivision on Σ and apply local Gauss-Bonnet

$$\sum_{Polygons} \int_P \kappa_\Sigma \, \mathrm{d}A = \int_\Sigma \kappa_\Sigma \, \mathrm{d}A$$

$$\sum_n \sum_{P \text{ an } n-\text{gon}} \left(\sum_{i=1}^n \alpha_i(P) - (n-2)\pi \right) = 2\pi V + 2\pi F - 2\pi E = 2\pi\chi(\Sigma)$$

where V, E, F are the numbers of vertices, edges and faces in the subdivision.
The local G-B theorem is ver closely related to Green's theorem in the plane

Non-examinable sketch of this:
Green's theorem says
Take a region $R \subseteq \mathbb{R}^2$ bound by piecewise smooth curve γ and take $P, Q : V \to \mathbb{R}$ smooth defined on open set $V \supseteq R$, then

$$\int_\gamma P \,\mathrm{d}u + Q \,\mathrm{d}v = \int_R (Q_u - P_v) \,\mathrm{d}y \,\mathrm{d}v$$

Consider a geodesic polygon on Σ lying wholly in the domain of a local parametrisation defined on some open $V \subseteq \mathbb{R}^2$.
We'll work with an orthonormal basis for $\mathbb{R}^2$ varying from point to point ("moving frames"). Specifically we take

$$e = \sigma_u$$
$$f = \frac{\sigma_v}{\sqrt{G}}$$

where we use geodesic normal co-ordinates u, v (s.t. $E = 1,\ F = 0,\ G = G(u, v)$). So $T_p\Sigma = \mathrm{Span}_{\mathbb{R}}(e, f)$ if $p \in \mathrm{image}(\sigma)$. We parametrise γ by arc-length and let

$$I := \int_\gamma \langle e, \dot{f} \rangle \,\mathrm{d}t$$

$\dot{f} = f_u \dot{u} + f_v \dot{v}$ so let $P = \langle e, f_u \rangle,\ Q = \langle e, f_V \rangle$ then

$$\begin{aligned} Q_v - P_v &= \langle e_v, f_v \rangle - \langle f_u, e_v \rangle + \langle e, f_{uv} \rangle - \langle e, f_{uv} \rangle \\ &= \langle e_v, f_v \rangle - \langle f_u, e_v \rangle \\ &= -\sqrt{G}_{uu} \text{ (ES3)} \\ {}^{=}\kappa\sqrt{G} \text{ but } \sqrt{G} = \sqrt{EG - F^2} \\ &= \kappa \,\mathrm{d}A \end{aligned}$$

so

$$\int_R (Q_u - P_v) \,\mathrm{d}u \,\mathrm{d}v = \int_R \kappa_\Sigma \,\mathrm{d}A$$

Let $\theta(t) = $ angle between $\dot{\gamma}(t)$ and $e(t)$ (function of $t \in \mathrm{Domain}(\gamma)$)
i.e. $\dot{\gamma}(t) = e \cos\theta(t) + f \sin\theta(t)$

$$\implies \ddot{\gamma}(t) = \dot{e}\cos\theta + \dot{f}\sin\theta + \eta\dot{\theta}$$

where $\eta = -e\sin\theta + f\cos\theta$. γ is a (piecewise smooth) geodesic so (if $\Sigma \subseteq \mathbb{R}^3$ was smooth in $\mathbb{R}^3$) then $\ddot{\gamma} \perp T_p\Sigma = \langle e, f \rangle_{\mathbb{R}-\mathrm{Span}}$ so

$$\langle \ddot{\gamma}, \eta \rangle = 0 \tag{†}$$

Expand this:

$$\langle \dot{e}\cos\theta + \dot{f}\sin\theta + \eta\dot{\theta}, -e\sin\theta + f\cos\theta \rangle = 0$$

But $\langle e, e \rangle = 1 = \langle f, f \rangle,\ \langle e, f \rangle = 0$

$$\implies \langle e, \dot{e} \rangle = 0 = \langle f, \dot{f} \rangle$$

and

$$\langle e, \dot{f} \rangle + \langle \dot{e}, f \rangle = 0$$

Then $\langle \ddot{\gamma}, \eta \rangle = 0$ becomes $\dot{\theta} = \langle e, \dot{f} \rangle$ so

$$I = \int_\gamma \langle e, \dot{f} \rangle \,\mathrm{d}t = \int_\gamma \dot{\theta}(t) \,\mathrm{d}t$$

and

$$\int_\gamma \dot{\theta}(t) \,\mathrm{d}t = 2\pi - \sum \text{external angles of } R$$

this is RHS of local Gauss-Bonnet.

Remarks.

(i) For surfaces not in $\mathbb{R}^3$ need a little more technology

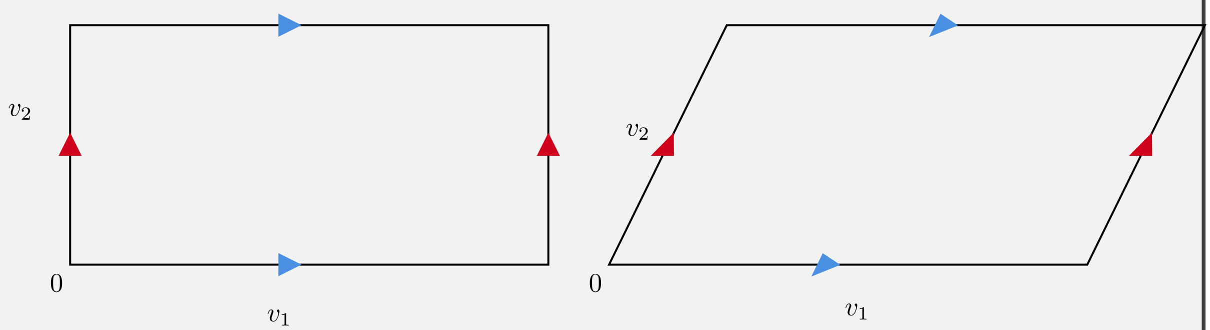
(ii) Green's theorem suggests one should ask about non-geodesic polygons too

2.2 Back To The Torus

We built a flat metric on

$$T^2 = \mathbb{R}^2/\mathbb{Z}^2 = [0,1]^2/\sim$$

The key to getting a smooth atlas s.t. the transition maps preserved g_{eucl} - Euclidian metric on $\mathbb{R}^2$ is that we could identify sides by translation. So any parallelogram $Q \subseteq \mathbb{R}^2$ deifnes a flat metric g_Q on T^2

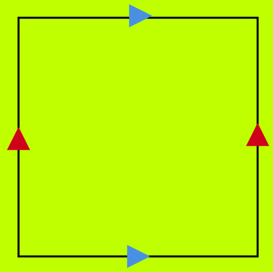


Remark. If we make one vertex $0 \in \mathbb{R}^2$ and label the edges by their endpoints v_1, v_2 then

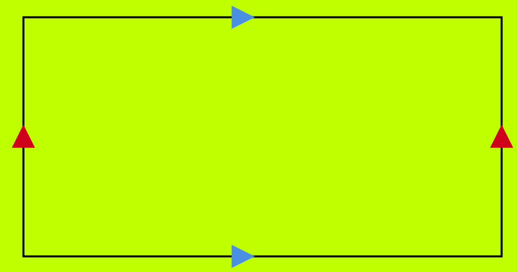
$$(T^2, g_Q) = \mathbb{R}^2/(\mathbb{Z}v1 \oplus \mathbb{Z}v_2)$$

where $\mathbb{Z}v1 \oplus \mathbb{Z}v_2$ is a subgroup of $\mathbb{R}^2$ of translations
Observation: $\text{Area}_{g_Q}(T^2) = \text{Area}_{Eucl}(Q)$
So if two quadrilaterals Q_1 and Q_2 have different Euclidean area, then the associated metrics g_{Q_1} and g_{Q_2} on T^2 are not globally isometric

Lemma.



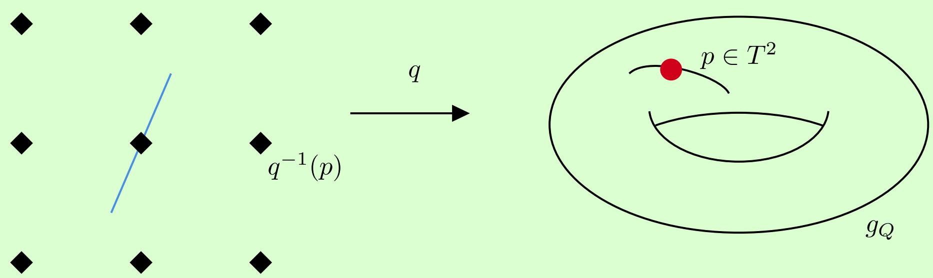
$Q = [0,1]^2$



$[0,10] \times \left[0, \frac{1}{10}\right] = \hat{Q}$

The metrics g_Q and $g_{\hat{Q}}$ are not isometric (both have total area 1)

Proof. Recall geodesics in flat T^2 are straight lines in a local isometry to $\mathbb{R}^2$



geodesic through p lifts to a straight arc

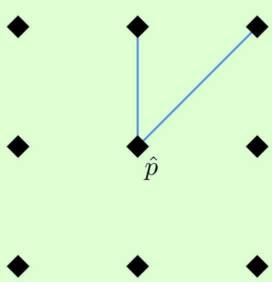
Picard: there is a unique geodesic through p in each possible direction.

We see in fact all geodescis through p are the images of straight lines in $\mathbb{R}^2$ through $\hat{p}$.

Recall a closed geodesic is one defined on all of $\mathbb{R}$ and periodic ($\gamma(t+T) = \gamma(t)$ $\forall t$ and fixed $T > 0$)

A geodesic in $\mathbb{R}^2$ through $\hat{p}$ defines a closed geodesic on T^2 through p exactly if the line passes through another lift of p, i.e. line has rational slope.

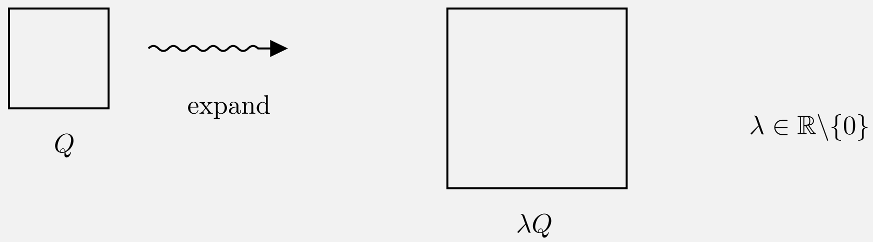
So the shortest closed geodesic on (T^2, g_Q) is length 1.



But in $g_{\hat{Q}}$ corresponding to rectangle $[0,10] \times [0, \frac{1}{10}]$ $\exists$ a clearly closed geodesic of length $1/10$

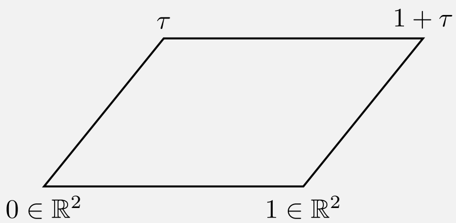
We would like to understand all flat metrics on T^2; up to various notions of equivalence

- global dilation



- translation, isometries of $Q \subseteq \mathbb{R}^2$ lead to essentially the same geometry on (T^2, g_Q).

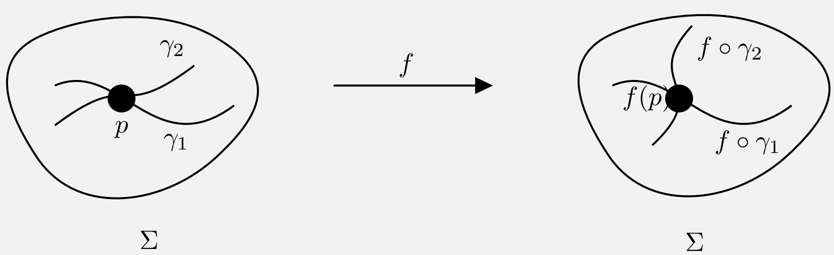
Using this, gien a quadtilateral, we can put vertices at $0 \in \mathbb{R}^2,\ 1 \in \mathbb{R}^2,\ \tau \in h$ upper-half plane and final vertex is $1 = +\tau$



This defines a map

$$h \to \{\text{flat metrics on } T^2\}/\text{Dilation}$$

“Recall” also one can pull back metrics by diffeomorphisms



Metrics let us measure lengths of curves, by integrating lengths of tangent vectors, so view metirc on g as an inner product on $T_p\Sigma,\ \forall p \in \Sigma$. Pullback metric f^*g was defined s.t.

$$\langle \dot\gamma_1, \dot\gamma_2 \rangle_{p, f^*g} := \langle (f \dot\circ \gamma_1), (f \dot\circ \gamma_2) \rangle_{f(p), g}$$

Note $SL(2,\mathbb{Z})$ acts on $\mathbb{R}^2$ preserving $\mathbb{Z}^2$ so it acts on $\mathbb{R}^2/\mathbb{Z}^2 = T^2$

Lemma. This is an action by diffeomorphism of the abstract smooth surface T^2

Proof. Clearly $A \in SL(2,\mathbb{Z})$ acts smoothly (linearly) on $\mathbb{R}^2$ and the charts for our smooth atlas are s.t. A then acts smoothly with our local charts

Note. Also $SL(2,\mathbb{Z}) \subseteq SL(2,\mathbb{R})$ acts on h via Möbius maps

Theorem. The map $h \to \{\text{Flat metrics on } T^2\}/\text{Dilation}$ descends to a map

$$h/SL(2,\mathbb{Z}) \to \frac{\{\text{Flat metrics on } T^2\}}{\text{Dilation and diffeomorphism}}$$

which is a bijection. We say that $h/SL(2,\mathbb{Z})$ is the **Moduli space** of flat metrics on T^2 (Our diffeomorphisms here preserve a choice of orientation)

Remark. (i) The LHS is naturally an object of hyperbolic geometry
(ii) The moduli space of hyperbolic metrics on Σ_g ($g \geq 2$) is perhaps the most studied space in all of geometry

What next?
(i) Algebraic topology: study spaces through algebraic invariants like Euler characteristic, and covering maps of surfaces like $S^2 \to \mathbb{RP}^2$, $\mathbb{R}^2 \to R^2$
(ii) Differential geometry: we studied $\det(DN)$, $N : \Sigma \to S^2$ the Gauss map. The tract is the mean-curvature, related to soap films
(iii) Riemann surfaces is about the fact that if $f : \mathbb{C} \to \mathbb{C}$ is holomorphic and $w \in \mathbb{C}$ $f(z+w)$ is holomorphic. $f : D \to D$ is holomorphic and $A \in \text{Möb}(D)$, $f \circ A$ is holomorphic
(iv) General Relativity is the study of geodesics