Vector Calculus

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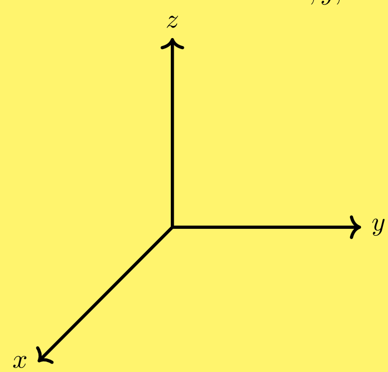
Contents

0 Notation

Notation. Throughout this course a column vector

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

is to be interpreted as the vector $\mathbf{x} = a\mathbf{e}_x + b\mathbf{e}_y + c\mathbf{e}_z$ where $\{\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$ are basis vectors aligned with the fixed Cartesian x, y, z axes in $\mathbb{R}^3$



$$\mathbf{e}_1 \equiv \mathbf{e}_x$$
$$\mathbf{e}_2 \equiv \mathbf{e}_y$$
$$\mathbf{e}_3 \equiv \mathbf{e}_z$$

i.e.

$$\mathbf{x} \equiv x_i \mathbf{e}_i \equiv \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

1 Differential Geometry of curves

1.1 Parametrised Curves and Arc Length

Definition. A **parametrised curve** C in $\mathbb{R}^3$ is just the image of a continuous map

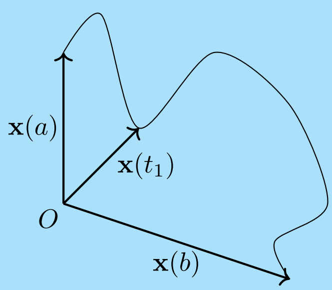
$$\mathbf{x} : [a, b] \to \mathbb{R}^3$$

in which

$$t \mapsto \mathbf{x}(t)$$

In caresian coordinates

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}$$



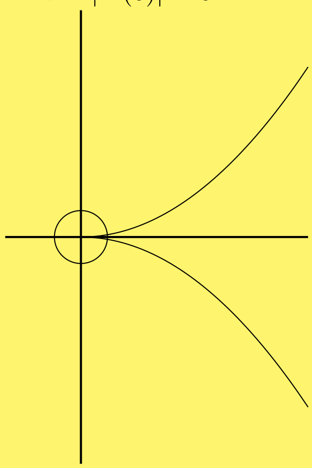
Definition. We say C is **differentiable** if each of the components $\{x_i(t)\}_{i=1}^3$ are differentiable.

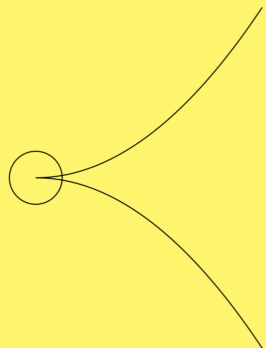
Definition. We say C is **regular** if $|\mathbf{x}'(t)| \neq 0$

Definition. If C is differentiable and regular say C is **smooth**

Remark. Why "regular" condition?
Consider $\mathbf{x}(t) = (t^2, t^3)$. Clearly differentiabble but $\mathbf{x}(t)$ has cusp at $t = 0$.

Note. $|\mathbf{x}'(0)| = 0$





Note. Recall that $x_i(t)$ is differentiable at t iff

$$x_i(t+h) = x_i(t) + x_i'(t)h + o(h)$$

where $o(h)$ represents function that obeys

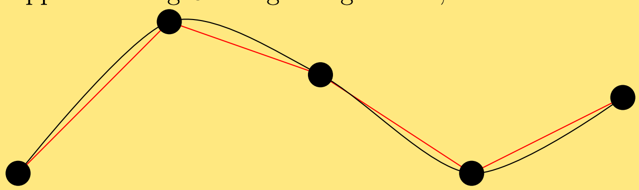
$$\frac{o(h)}{h} \to 0 \text{ as } h \to 0$$

In terms of vectors

$$\mathbf{x}(t+h) = \mathbf{x}(t) + \mathbf{x}'(t)h + o(h)$$

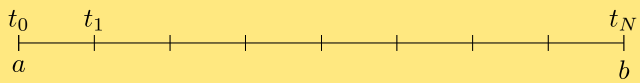
where $o(h)$ a vector for which $\frac{|o(h)|}{h} \to 0$

Method. Finding length of a curve C.
Approximating C using straight lines,



$C : t \mapsto \mathbf{x}(t),\ t \in [a, b]$
Introduce partition P of $[a, b]$ with $t_0 = a$, $t_N = b$ and

$$t_0 < t_1 < t_2 < \cdots < t_N$$



Set $\Delta t_i = t_{i+1} - t_i$ and $\Delta t = \max_i \Delta t_i$
Define length of C relative to P by

$$l(C, P) = \sum_{i=0}^{N-1} |\mathbf{x}(t_{i+1} - t_i)|$$

As Δt gets smaller, expect $l(C, P)$ to give better approximation to length of C, $l(C)$. Define length of C by:

$$\begin{aligned} l(C) &= \lim_{\Delta t \to 0} \sum_{i=0}^{N-1} |\mathbf{x}(t_{i+1} - t_i)| \\ &= \lim_{\Delta t \to 0} l(C, P) \end{aligned}$$

If limit doesn't exist, say curve is non-rectifiable.
Suppose C is differentiable. Then

$$\begin{aligned} \mathbf{x}(t_{i+1}) &= \mathbf{x}(t_i + t_{i+1} - t_i) \\ &= \mathbf{x}(t_i + \Delta t_i) \\ &= \mathbf{x}(t_i) + \mathbf{x}'(t_i)\Delta t_i + o(\Delta t_i) \end{aligned}$$

It follows

$$|\mathbf{x}(t_{i+1} - t_i)| = |\mathbf{x}'(t_i)||\Delta t_i + o(\Delta t_i)|$$

So if C is differentiable,

$$l(C, P) = \sum_{i=0}^{N-1} |\mathbf{x}'(t_i)||\Delta t_i + o(\Delta t_i)|$$

Method (continued)**.** Recall that $o(\Delta)t_i$ represents a dunction for which $\frac{o(\Delta t_i)}{\Delta t_i} \to 0$ as $\Delta t \to 0$. So for any $\varepsilon > 0$, if $\Delta t = \max_i \Delta t_i$ is sufficiently small, have

$$|o(\Delta t_i)| < \frac{\varepsilon}{b-a} \Delta t_i$$

for $i = 0, \dots, N-1$. So

$$|l(C,P) - \sum_{i=0}^{N-1} |\mathbf{x}'(t_i)| \Delta t_i| = |\sum_{i=0}^{N-1} o(\Delta t_i)| < \frac{\varepsilon}{b-a} \sum_{i=0}^{N-1} = \varepsilon$$

So the $LHS \to 0$ as $\Delta t \to 0$. Get

$$\begin{aligned} l(C) &= \lim_{\Delta t \to 0)} l(C,P) \\ &= \lim_{\Delta t \to 0)} \sum_{i=0}^{N-1} |\mathbf{x}'(t_i)| \Delta t_i \\ &= \int_a^b |\mathbf{x}'(t)| \, \mathrm{d}t \end{aligned}$$

Note. See Analysis I, definition of Reimann integral.
So in summary have equation below:

Equation. if $C : t \mapsto x(t),\ t \in [a,b]$

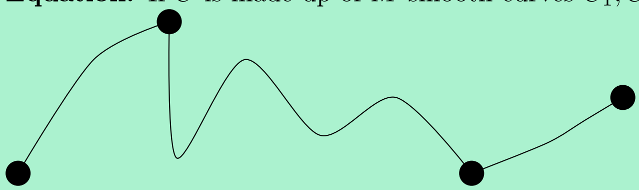
$$\begin{aligned} l(C) &= \int_a^b |\mathbf{x}'(t_i)| \, \mathrm{d}t \\ &= \int_C \mathrm{d}s \end{aligned}$$

$$\mathrm{d}s = |\mathbf{x}'(t_i)| \, \mathrm{d}t$$

s is the "arc-length element"
Similarly define

$$\int_C f(\mathbf{x}) \, \mathrm{d}s = \int_a^b f(\mathbf{x}(t)) |\mathbf{x}'(t_i)| \, \mathrm{d}t$$

Equation. If C is made up of M smooth curves $C_1, C_2, \ldots, C_M$



Wrtie $C = C_1 + C_2 + \cdots + C_M$ and define

$$\int_C f(\mathbf{x})\,\mathrm{d}s = \sum_{i=1}^{n} \int_{C_i} f(\mathbf{x})\,\mathrm{d}s$$

Note.

$$\mathrm{d}s = |\mathbf{x}'(t)|\,\mathrm{d}t = \sqrt{(\frac{\mathrm{d}x}{\mathrm{d}t})^2 + (\frac{\mathrm{d}y}{\mathrm{d}t})^2 + (\frac{\mathrm{d}z}{\mathrm{d}t})^2}\,\mathrm{d}t$$

i.e. $\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$

Example. Let C be circle of radius $r > 0$ in $\mathbb{R}^3$

$$\mathbf{x}(t) = \begin{bmatrix} r\cos t \\ r\sin t \\ 0 \end{bmatrix} \quad t \in [0, 2\pi]$$

So

$$\mathbf{x}'(t) = \begin{bmatrix} -r\sin t \\ r\cos t \\ 0 \end{bmatrix} \quad t \in [0, 2\pi]$$

$$\begin{aligned} \int_C \mathrm{d}s &= \int_0^{2\pi} \sqrt{r^2 \sin^2 t + r^2 \cos^2 t}\,\mathrm{d}t \\ &= \int_0^{2\pi} r\,\mathrm{d}t \\ &= 2\pi r \end{aligned}$$

Also

$$\begin{aligned} \int_C x^2 y\,\mathrm{d}s &= \int_0^{2\pi} (r\cos t)^2 (r\sin t) r\,\mathrm{d}t \\ &= 0 \end{aligned}$$

(as $r\,\mathrm{d}t = |\mathbf{x}'(t)|\,\mathrm{d}t$)

Remark. Does $l(C)$ depend on parametrisation? e.g.

$$\mathbf{x}(t) = \begin{bmatrix} r\cos t \\ r\sin t \\ 0 \end{bmatrix} \quad t \in [0, 2\pi]$$

$$\tilde{\mathbf{x}(t)} = \begin{bmatrix} r\cos(2t) \\ r\sin(2t) \\ 0 \end{bmatrix} \quad t \in [0, \pi]$$

Both give different parametrisation of circle of radius r
Suppose C has two different parametrisations

$$\mathbf{x} = \mathbf{x}_1(t), \ a \le t \le b$$

$$\mathbf{x} = \mathbf{x}_2(\tau), \ \alpha \le t \le \beta$$

Must have $\mathbf{x}_2(\tau) = \mathbf{x}_1(t(\tau))$ for some function $t(\tau)$. Assume $\frac{\mathrm{d}t}{\mathrm{d}\tau} \neq 0$ so map between t and τ invertible and differentiable. (see inverse function theorem in Analysis + Topology). Note

$$\begin{aligned} \mathbf{x}_2(\tau) &= \frac{\mathrm{d}}{\mathrm{d}\tau}\mathbf{x}_2(t) \\ &= \frac{\mathrm{d}}{\mathrm{d}\tau}\mathbf{x}_1(t(\tau)) \\ &= \frac{\mathrm{d}t}{\mathrm{d}\tau}\mathbf{x}_1'(t(\tau)) \end{aligned}$$

From definitions,

$$\int_C f(\mathbf{x}) \, \mathrm{d}s = \int_a^b f(\mathbf{x}(t)) |\mathbf{x}'(t_i)| \, \mathrm{d}t$$

Make substitution $t = t(\tau)$, and assume $\frac{\mathrm{d}t}{\mathrm{d}\tau} > 0$, latter integral becomes

$$\int_\alpha^\beta f(\mathbf{x}_2(\tau)) \underbrace{|\mathbf{x}_1'(t(\tau)| \frac{\mathrm{d}t}{\mathrm{d}\tau} \, \mathrm{d}\tau}_{|\mathbf{x}_2'(\tau)| \, \mathrm{d}\tau}$$

Which is precisely the same as $\int_C f(\mathbf{x}) \mathrm{d}s$ using $\mathbf{x}_2(\tau)$ parametrisation. Similar holds when $\frac{\mathrm{d}t}{\mathrm{d}\tau} < 0$ (exercise). So definition of $\int_C f(\mathbf{x}) \, \mathrm{d}s$ does not depend on choice of parametrisation of C.

Definition. The **arc-length function** for a curve $[a, b] \ni t \mapsto \mathbf{x}(t)$ by

$$s(t) = \int_a^t |\mathbf{x}'(\tau)| \, \mathrm{d}\tau$$

So $s(a) = 0$ and $s(b) = l(c)$.
Also:

$$\frac{\mathrm{d}s}{\mathrm{d}t} = |\mathbf{x}'(t)| \ge 0$$

Definition. For regular curves have $\frac{\mathrm{d}s}{\mathrm{d}t} > 0$, so can invert relationship between s and t to find

$$t = t(s)$$

So we can parametrise regular curves wrt arc-length, If we write $\mathbf{r}(s) = \mathbf{x}(t(s))$ where $0 \le s \le l(C)$, then by chain rule:

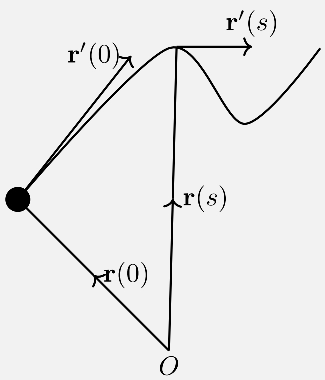
$$\frac{\mathrm{d}t}{\mathrm{d}s} = \frac{1}{\frac{\mathrm{d}s}{\mathrm{d}t}} = \frac{1}{|\mathbf{x}'(t(s))|}$$

So

$$\begin{aligned}\mathbf{r}'(s) &= \frac{\mathrm{d}}{\mathrm{d}s}\mathbf{x}(t(s)) \\ &= \frac{\mathrm{d}t}{\mathrm{d}s}\mathbf{x}'(t(s)) \\ &= \frac{\mathbf{x}'(t(s))}{|\mathbf{x}'(t(s))|}\end{aligned}$$

i.e. $|\mathbf{r}'(s)| = 1$. This (consistently) gives

$$l(C) = \int_0^{l(C)} |\mathbf{r}'(s)|\,\mathrm{d}s = \int_0^{l(C)} \mathrm{d}s\checkmark$$



1.2 Curvature and Torsion

Note. Throughout this section talk about generic regular curve C parametrised by arc-length, write $s \mapsto \mathbf{r}(s)$

Definition. Tangent vector

$$\mathbf{t}(s) = \mathbf{r}'(s)$$

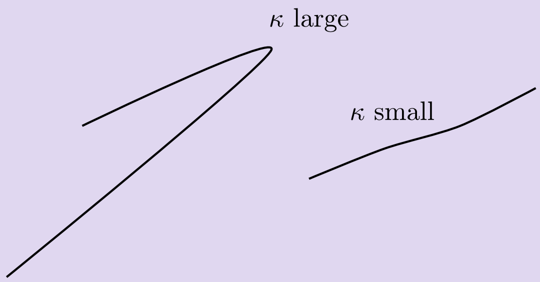
Already know $|\mathbf{t}(s)| = 1$. Since $|\mathbf{t}(s)|$ doesn't change, the second dervative $\mathbf{r}''(s) = \mathbf{t}'(s)$ only measures change in direction

So intuitively, if $|r''(s)|$ is large then curve rapidly changes direction, whereas if $|\mathbf{r}''(s)|$ is small, expect curve to be approximately flat.

Definition. The **curvature**

$$\kappa(s) = |\mathbf{r}''(s)| = |\mathbf{t}'(s)|$$

Example.



Since $\mathbf{t} = \mathbf{r}'(s)$ is a unit vector, differentiating $\mathbf{t} \cdot \mathbf{t} = 1$ gives $\mathbf{t} \cdot \mathbf{t}' = 0$.

Definition. The **principle normal** is defined by the formula

$$\mathbf{t}' = \kappa \mathbf{n}$$

$\mathbf{n}$ is the principle normal

Note. $\mathbf{n}$ is everywhere normal to C since

$$\mathbf{t} \cdot \mathbf{n} = 0$$

Definition. Can extend $\{\mathbf{t}, \mathbf{n}\}$ to orthonormal basis by defining the **binormal**

$$\mathbf{b} = \mathbf{t} \times \mathbf{n}$$

Since $|\mathbf{b}| = 1$ have $\mathbf{b}' \cdot \mathbf{b} = 0$. Also since $\mathbf{t} \cdot$ =0 and $\mathbf{n} \cdot \mathbf{b} = 0$

$$\begin{aligned} 0 = (\mathbf{t} \cdot \mathbf{b})' &= \mathbf{t}' \cdot \mathbf{b} + \mathbf{t} + \mathbf{t} \cdot \mathbf{b}' \\ &= \underbrace{\kappa \mathbf{n} \cdot \mathbf{b}}_{=0} + \mathbf{t} \cdot \mathbf{b}' \end{aligned}$$

So $\mathbf{b}'$ is orthogonal to both $\mathbf{t}$ and $\mathbf{b}$ i.e. it is parallel to $\mathbf{n}$.

Definition. The **torsion** of a curve is defined by the formula

$$\mathbf{b}' = -\tau \mathbf{n}$$

τ is the torsion

Have two equations

$$\mathbf{t}' = \kappa \mathbf{n}, \ \mathbf{b}' = -\tau \mathbf{n}$$

Prop. The curvature $\kappa(s)$ and torsion $\tau(s)$ define a curve up to translation/ orientation.

Proof. Since $\mathbf{n} = \mathbf{b} \times \mathbf{t}$, have

$$\mathbf{t}' = \kappa(\mathbf{b} \times \mathbf{t})$$

$$\mathbf{b}' = -\tau(\mathbf{b} \times \mathbf{t})$$

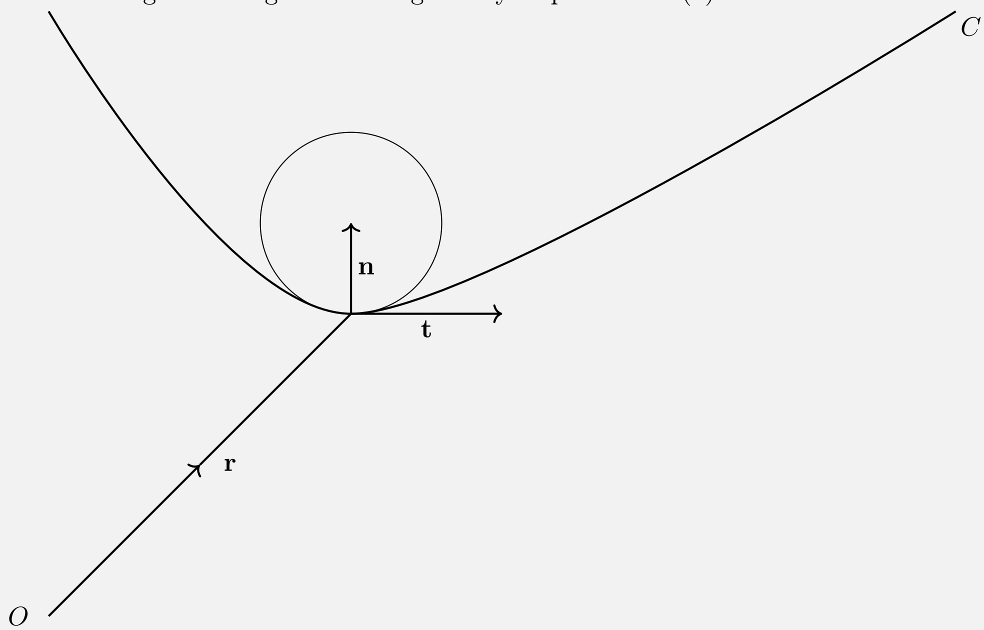
This gives six equations for six unknowns.
Given $\kappa(s)$, $\tau(s)$, $\mathbf{t}(0)$, $\mathbf{b}(0)$, can construct $\mathbf{t}(s)$, $\mathbf{b}(s)$ and hence $\mathbf{n} = \mathbf{b} \times \mathbf{t}$. Hence result $\square$

1.3 Radius of Curvature

Taylor expand a generic curve $s \mapsto \mathbf{r}(s)$ about $s = 0$. Write $\mathbf{t} = \mathbf{t}(0)$, $\mathbf{n} = \mathbf{n}(0)$ etc.

$$\begin{aligned}\mathbf{r}(s) &= \mathbf{r}(0) + s\mathbf{r}'(0) + \frac{1}{2}s^2\mathbf{r}''(0) + o(s^2) \\ &= \mathbf{r} + s\mathbf{t} + \frac{1}{2}s^2\kappa\mathbf{n} + o(s^2)\end{aligned}$$

Suppose, WLOG, that $\mathbf{t}$ is horizontal.
What circle goes through curve tangentially at point $\mathbf{r} = \mathbf{r}(0)$ is best fit?



Equation of circle

$$\mathbf{x}(\theta) = \mathbf{r} + R(1 - \cos\theta)\mathbf{n} + R\sin\theta\mathbf{t}$$

Expand for $|\theta|$ small

$$\mathbf{x}(\theta) = \mathbf{r} + R\theta\mathbf{t} + \frac{1}{2}R\theta^2\mathbf{n} + o(\theta^2)$$

Arc length on circle is $s = R\theta$. So

$$\mathbf{x}(\theta) = \mathbf{r} + s\mathbf{t} + \frac{1}{2}\frac{1}{R}s^2\mathbf{n} + o(s^2)$$

To match equation for curve up to scond order, would require

$$R = \frac{1}{\kappa}$$

Definition. We say $R(s) = \frac{1}{\kappa(s)}$ is the **radius of curvature** of curve $s \mapsto \mathbf{r}(s)$

1.4 Gaussian Curvature

Note. Non-examinable

Definition. The **Gaussian curvature**: $\kappa_G = \kappa_{\min}\kappa_{\max}$ Where κ varies over fixed point on surface curve in intersection of planes through normal rotating from $[0, 2\pi)$

Theorem (Remarkable Theorem)**.** Gaussian curvature of surface S is invariant if you bend the surface without stretching it.

2 Coordinates, Differentials + Gradients

2.1 Differentials + First Order Changes

Definition. The **differential** of f, written $\mathrm{d}f$, by

$$\mathrm{d}f = \frac{\partial f}{\partial u_i}\,\mathrm{d}u_i$$

Call $\{\mathrm{d}u_i\}$ **differential forms**. These are L.I. if $\{u_1, \dots, u_n\}$ are independent.
I.e. if $\alpha_i \mathrm{d}u_i = 0 \implies \alpha_i = 0$ for $i = 1, \dots, n$. Similarly, if $\mathbf{x} = \mathbf{x}(u_1, \dots, u_n$ we define

$$\mathrm{d}\mathbf{x} = \frac{\partial \mathbf{x}}{\partial u_i} u_i$$

Example. If $f(u, v, w) = u^2 + w\sin(v)$. Then

$$\mathrm{d}f = 2u\,\mathrm{d}u + w\cos(v)\,\mathrm{d}v + sin(v)\,\mathrm{d}w$$

If $\mathbf{x}(u, v, w) = \begin{bmatrix} u^2 - v^2 \\ w \\ e^v \end{bmatrix}$,

$$\mathrm{d}\mathbf{x} = \begin{bmatrix} 2u \\ 0 \\ 0 \end{bmatrix}\mathrm{d}u + \begin{bmatrix} -2v \\ 0 \\ e^v \end{bmatrix}\mathrm{d}v + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\mathrm{d}w$$

Note. Differenials encode info about how a function/ vector field changes when we "wobble" our coords. Indeed, by calculus:

$$f(u_1 + \delta u_1, \ldots, u_n + \delta u_n) - f(u_1, \ldots, u_n) = \frac{\partial f}{\partial u_i}\delta u_i + o(\delta \mathbf{u})$$

$(\delta \mathbf{u} = (\delta u_1, \ldots, \delta u_n)$
$\frac{o(\delta \mathbf{u})}{|\delta \mathbf{u}|} \to 0$ as $|\delta \mathbf{u}| \to 0$
So if δf denotes change in $f(u_1, \ldots, u_n)$ under perturbation of coords

$$(u_1, \ldots, u_n) \mapsto (u_1 + \delta u_1, \ldots, u_n + \delta u_n)$$

We have, to first order,

$$\delta f \simeq \frac{\partial f}{\partial u_i}\,\delta u_i$$

Similarly for vector fields

$$\delta \mathbf{x} \simeq \frac{\partial \mathbf{x}}{\partial u_i}\delta u_i$$

(this gives us the chain rule for free, see Ashton's notes)

2.2 Coordinates and Line Elements

Already seen at least two different sets of coords for $\mathbb{R}^2$: Cartesian coordinates (x, y) and polar coordinates (r, θ). Have invertible relationship:

$$x = r\cos\theta$$

$$y = r\cos\theta$$

A general set of coords (u, v) on $\mathbb{R}^2$ can be specified by its relationship to (x, y), i.e. specify smooth functions

$$x = x(u, v)$$

$$y = y(u, v)$$

which can be inverted to give smooth functions

$$u = u(x, y)$$

$$v = v(x, y)$$

Similarly for $\mathbb{R}^3$, have (u, v, w) coords by specifying

$$x = x(u, v, w)$$

$$y = y(u, v, w)$$

$$z = z(u, v, w)$$

Definition. Standard **Cartesian coords**

$$\mathbf{x}(x,y) = \begin{bmatrix} x \\ y \end{bmatrix} = x\mathbf{e}_x + y\mathbf{e}_y$$

$\{\mathbf{e}_x, \mathbf{e}_y\}$ are orthonormal vectors. $\mathbf{e}_x$ points in the direction of changing x with y fixed.

Said differently,

$$\mathbf{e}_x = \frac{\frac{\partial}{\partial x}\mathbf{x}(x,y)}{\frac{\partial}{\partial x}\mathbf{x}(x,y)}, \ \mathbf{e}_y = \frac{\frac{\partial}{\partial y}\mathbf{x}(x,y)}{\frac{\partial}{\partial y}\mathbf{x}(x,y)}$$

Feature of Cartesian coords:

$$\begin{aligned} \mathrm{d}\mathbf{x} &= \frac{\partial \mathbf{x}}{\partial x}\,\mathrm{d}x + \frac{\partial \mathbf{x}}{\partial y}\,\mathrm{d}y \\ &= \mathrm{d}x\,\mathbf{e}_x + \mathrm{d}y\,\mathbf{e}_y \end{aligned}$$

i.e. changing coord $x \mapsto x + \delta x$, then the vector changes (to first order) by $\mathbf{x} \mapsto \mathbf{x} + \delta x\,\mathbf{e}_x$. We call $\mathrm{d}\mathbf{x}$ the line element

Definition. The **line element** is:

$$\mathrm{d}\mathbf{x} = \frac{\partial \mathbf{x}}{\partial u_1}\,\mathrm{d}u_1 + \frac{\partial \mathbf{x}}{\partial u_2}\,\mathrm{d}u_2$$

It tells us how small changes in coord produce changes in position vectors.

For polars (r,θ)

$$\mathbf{x}(r,\theta) = \begin{bmatrix} r\cos\theta \\ r\sin\theta \end{bmatrix} \equiv r\mathbf{e}_r$$

where we have used basis vectors $\{\mathbf{e}_2, \mathbf{e}_\theta\}$

$$\mathbf{e}_r = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}, \ \mathbf{e}_\theta = \{-\sin\theta \cos\theta\}$$

Warning. $\{\mathbf{e}_r, \mathbf{e}_\theta\}$ are orthonormal at each (r,θ), but NOT the same for each (r,θ)

Note. As before,

$$\mathbf{e}_r = \frac{\frac{\partial}{\partial r}\mathbf{x}(r,\theta)}{|\frac{\partial}{\partial r}\mathbf{x}(r,\theta)|}, \ \mathbf{e}_\theta = \frac{\frac{\partial}{\partial \theta}\mathbf{x}(r,\theta)}{|\frac{\partial}{\partial \theta}\mathbf{x}(r,\theta)|}$$

Since $\{\mathbf{e}_r, \mathbf{e}_\theta\}$ are orthogonal, makes sense to call (r,θ) orthogonal curvilinear coordinates.

For polars, have line element

$$\begin{aligned} \mathrm{d}\mathbf{x} &= \frac{\partial \mathbf{x}}{\partial r}\,\mathrm{d}r + \frac{\partial \mathbf{x}}{\partial \theta}\,\mathrm{d}\theta \\ &= \mathbf{e}_r\,\mathrm{d}r + r\,\mathrm{d}\theta\,\mathbf{e}_\theta \end{aligned}$$

See that a change $\theta \mapsto \theta + \delta\theta$ produces a (first order) change

$$\mathbf{x} \mapsto \mathbf{x} + r\delta\theta\,\mathbf{e}_\theta$$

Warning. NOT $\mathbf{x} \mapsto \mathbf{x} + \delta\theta\,\mathbf{e}_\theta$

2.2.1 Orthogonal Curvilinear Coordinates

Definition. We say that (u, v, w) are a **set of orthogonal curvilinear coords** if the vectors

$$\mathbf{e}_u = \frac{\frac{\partial \mathbf{x}}{\partial u}}{\left|\frac{\partial \mathbf{x}}{\partial u}\right|},\ \mathbf{e}_v = \frac{\frac{\partial \mathbf{x}}{\partial v}}{\left|\frac{\partial \mathbf{x}}{\partial v}\right|,\ \mathbf{e}_w = \frac{\frac{\partial \mathbf{x}}{\partial w}}{\left|\frac{\partial \mathbf{x}}{\partial w}\right|}}$$

form a right-handed handed basis for each (u, v, w)

Note. Right handed means $\mathbf{e}_u \times \mathbf{e}_v = \mathbf{e}_w$

Warning. Just as with polar coordinates, $\{\mathbf{e}_u, \mathbf{e}_v, \mathbf{e}_w\}$ form orthonormal basis for $\mathbb{R}^3$ at each (u, v, w), but NOT necessarily the same basis at each point.

Notation. It is standard to write

$$h_u = \left|\frac{\partial \mathbf{x}}{\partial u}\right|, h_v = \left|\frac{\partial \mathbf{x}}{\partial v}\right|, h_w = \left|\frac{\partial \mathbf{x}}{\partial w}\right|$$

Definition. Call $\{h_u, h_v, h_w\}$ **scale factors**

Note. Line element is

$$\begin{aligned} \mathrm{d}\mathbf{x} &= \frac{\partial \mathbf{x}}{\partial u}\,\mathrm{d}u + \frac{\partial \mathbf{x}}{\partial v}\,\mathrm{d}v + \frac{\partial \mathbf{x}}{\partial w}\,\mathrm{d}w \\ &= h_u\mathbf{e}_u\,\mathrm{d}u + h_v\mathbf{e}_v\,\mathrm{d}v + h_w\mathbf{e}_w\,\mathrm{d}w \end{aligned}$$

Tells us how sall changes in coords “scale-up” to changes in position $\mathbf{x}$

2.2.2 Cylindrical Polar Coords

Definition. **Cyclindrical polars** (ρ, ϕ, z) defined by:

$$\mathbf{x}(\rho, \phi, z) = \begin{bmatrix} \rho \cos\phi \\ \rho \sin\phi \\ z \end{bmatrix}$$

with:

$$0 \leq \rho < \infty$$

$$0 \leq \phi < 2\pi$$

$$-\infty < z < \infty$$

Find

$$\mathbf{e}_\rho = \begin{bmatrix} \cos\phi \\ \sin\phi \\ 0 \end{bmatrix}, \ \mathbf{e}_\phi \begin{bmatrix} -\sin\phi \\ \cos\phi \\ 0 \end{bmatrix}$$

$$\mathbf{e}_z = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$h_\rho = 1, \ h_\phi = \rho, \ h_z = 1$$

$$\mathrm{d}\mathbf{x} = \mathrm{d}\rho\, \mathbf{e}_\rho + \rho\, \mathrm{d}\phi\, \mathbf{e}_\phi + \mathrm{d}z\, \mathbf{e}_z$$

Note.

$$\mathbf{x} = \begin{bmatrix} \rho \cos\phi \\ \rho \sin\phi \\ z \end{bmatrix} = \rho \begin{bmatrix} \cos\phi \\ \sin\phi \\ 0 \end{bmatrix} + z \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$
$$= \rho\, \mathbf{e}_\rho + z\, \mathbf{e}_z$$

Warning. STILL DEPENDENT ON ϕ AS $\mathbf{e}_\rho$ DEPENDS ON ϕ

2.2.3 Spherical Polar Coordinates

Definition. **Spherical polars** (r, θ, ϕ) defined by:

$$\mathbf{x}(r, \theta, \phi) = \begin{bmatrix} r \cos\phi \sin\theta \\ r \sin\phi \sin\theta \\ r \cos\theta \end{bmatrix}$$

with:

$$0 \leq r < \infty$$

$$0 \leq \theta \leq \pi$$

$$0 \leq \phi < 2\pi$$

$$\mathbf{e}_r = \begin{bmatrix} \cos\phi\sin\theta \\ \sin\phi\sin\theta \\ \cos\theta \end{bmatrix}, \ \mathbf{e}_\theta - \begin{bmatrix} \cos\phi\cos\theta \\ \sin\phi\cos\theta \\ -\sin\theta \end{bmatrix}$$

$$\mathbf{e}_\phi = \begin{bmatrix} -\sin\phi \\ \cos\phi \\ 0 \end{bmatrix}$$

$$h_r = 1,\ h_\theta = r,\ h_\phi = r\sin\theta$$

i.e.

$$\mathrm{d}\mathbf{x} = \mathrm{d}r\,\mathbf{e}_r + r\,\mathrm{d}\theta\,\mathbf{e}_\theta + r\sin\theta\,\mathrm{d}\phi\,\mathbf{e}_\phi$$

Note.

$$\mathbf{x} = r\begin{bmatrix} r\cos\phi\sin\theta \\ r\sin\phi\sin\theta \\ r\cos\theta \end{bmatrix} = r\,\mathbf{e}_r$$

Warning. STILL DEPENDENT ON ϕ, θ AS $\mathbf{e}_r$ DEPENDS ON ϕ, θ

2.3 Gradient Operator

Definition. For $f : \mathbb{R}^3 \to \mathbb{R}$, define **gradient** of f, written ∇f, by

$$f(\mathbf{x}+\mathbf{h}) = f(\mathbf{x}) + \nabla f(\mathbf{x})\cdot\mathbf{h} + o(\mathbf{h}) \qquad (*)$$

Definition. **Directional derivative** of f in direction $\mathbf{v}$, denoted by $D_\mathbf{v} f$ or $\frac{\partial f}{\partial \mathbf{v}}$, is defined by

$$D_\mathbf{v} f(\mathbf{x}) = \lim_{t\to 0}\frac{f(\mathbf{x}+t\mathbf{v}) - f(\mathbf{x})}{t}$$

I.e.

$$f(\mathbf{x}+t\mathbf{v}) = f(\mathbf{x}) + tD_\mathbf{v} f(\mathbf{x}) + o(t) \qquad (**)$$

Equation. Setting $\mathbf{h} = t\mathbf{v}$ in (*)

$$f(\mathbf{x}+t\mathbf{v}) = f(\mathbf{x}) + t\nabla f(\mathbf{x})\cdot\mathbf{v} + o(t)$$

Comparing to previous equation (**), we have:

$$D_\mathbf{v} = \mathbf{v}\cdot\nabla f$$

Note. By Cauchy-Schwarz know that $\mathbf{a}\cdot\mathbf{b}$ is maximised when $\mathbf{a}$ points in same direction as $\mathbf{b}$.

So ∇f points in direction of greatest increase of f
Similarly,
$-\nabla f$ points in direction of greatest decrease of f

Example. Suppose $f(\mathbf{x}) = \frac{1}{2}|\mathbf{x}|^2$. Then

$$\begin{aligned} f(\mathbf{x}+\mathbf{h}) &= \frac{1}{2}(\mathbf{x}+\mathbf{h})\cdot(\mathbf{x}+\mathbf{h}) \\ &= \frac{1}{2}|\mathbf{x}|^2 + \frac{1}{2}(2\mathbf{x}\cdot\mathbf{h}) + \frac{1}{2}|\mathbf{h}|^2 \\ &= f(\mathbf{x}) + \mathbf{x}\cdot\mathbf{h} + o(\mathbf{h}) \end{aligned}$$

$$\implies \nabla f(\mathbf{x}) = \mathbf{x}$$

Method. Suppose we have a curve $t \mapsto \mathbf{x}(t)$. How does f change as we move along this curve. Write

$$F(t) = f(\mathbf{x}(t))$$

$$\delta\mathbf{x} = \mathbf{x}(t+\delta t) - \mathbf{x}(t)$$

$$\begin{aligned} F(t+\delta t) &= f(\mathbf{x}(t+\delta t)) \\ &= f(\mathbf{x}(t) + \delta\mathbf{x}) \\ &= f(\mathbf{x}(t)) + \nabla f(\mathbf{x}(t))\cdot\delta\mathbf{x} + o(\delta\mathbf{x}) \end{aligned}$$

Since $\delta\mathbf{x} = \mathbf{x}'(t)\delta t + o(\delta t)$,

$$F(t+\delta t) = F(t) + \mathbf{x}'(t)\cdot\nabla f(\mathbf{x}(t))\delta t + i(\delta t)$$

I.e.

$$\frac{\mathrm{d}F}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}f(\mathbf{x}(t)) = \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}\cdot\nabla f(\mathbf{x}(t))$$

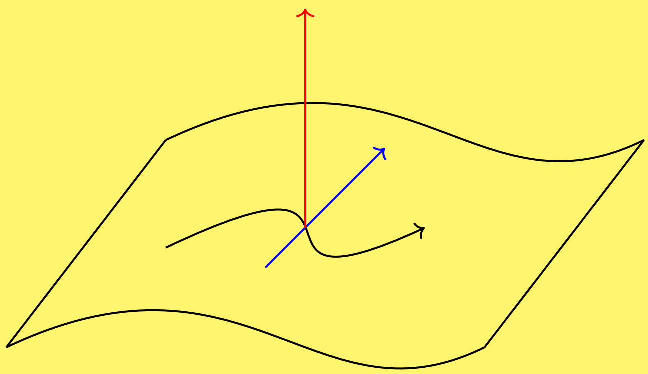
Note. Suppose surface S is defined implicitly

$$S = \{\mathbf{x} \in \mathbb{R}^3 : f(\mathbf{x}) = 0\}$$

If $t \mapsto \mathbf{x}(t)$ is ANY curve in S, then $f(\mathbf{x}(t)) = 0$ identically. So

$$0 = \frac{\mathrm{d}}{\mathrm{d}t} f(\mathbf{x}(t)) = \nabla f(\mathbf{x}(t)) \cdot \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}$$

So ∇f is orthogonal to tangent vector of ANY curve in S.
I.e. $\nabla f(\mathbf{x})$ is normal to surface at $\mathbf{x}$



2.4 Computing the gradient

Equation. If working with orthogonal curbilinear coordinates (O.C.C), (u, v, w), not clear how to compute ∇f, not clear how to change (u, v, w) so that $\mathbf{x} = \mathbf{x}(u, v, w)$ changes to $\mathbf{x} + \mathbf{h}$.
In cartesian coordinates, life is easy: to get change

$$\mathbf{x} \mapsto \mathbf{x} + \mathbf{h}$$

just

$$\begin{aligned} &\mapsto x + h_1 \\ y &\mapsto y + h_2 \\ z &\mapsto z + h_3 \end{aligned}$$

$$\begin{aligned} f(\mathbf{x} + \mathbf{h}) &= f(x + h_1 + y + h_2 + z + h_3) \\ &= f(\mathbf{x}) + \frac{\partial f}{\partial x} h_1 + \frac{\partial f}{\partial y} h_2 + \frac{\partial f}{\partial z} h_3 + o(\mathbf{h}) \\ &= f(\mathbf{x}) + \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{bmatrix} \cdot h + o(\mathbf{h}) \end{aligned}$$

i.e.

$$\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{bmatrix}$$

Or, using suffix notation

$$\nabla f = \mathbf{e}_i \frac{\partial f}{\partial x_i}, \text{ or } [\nabla f]_i = \frac{\partial f}{\partial x_i}$$

See that ∇ is a kind of vector differential operator.In Cartesian coordinates

$$\begin{aligned} \nabla &= \mathbf{e}_x \frac{\partial}{\partial x} + \mathbf{e}_y \frac{\partial}{\partial y} + \mathbf{e}_z \frac{\partial}{\partial z} \\ &\equiv \mathbf{e}_i \frac{\partial}{\partial x_i} \end{aligned}$$

Example.

$$f = \frac{1}{2}(x^2 + y^2 + z^2) = \frac{1}{2}|\mathbf{x}|^2$$

Then

$$\begin{aligned} \nabla f]_i &= \frac{\partial}{\partial x_i} \left[\frac{1}{2} x_j x_j \right] \\ &= \frac{1}{2}[\delta_{ij} x_j + d_j \delta_{ij}] \\ &= x_i \end{aligned}$$

[So $\nabla f = \mathbf{e}_i x_i = \mathbf{x}$ as expected]

Equation. Recall, in Cartesian Coordinates,

$$\begin{aligned} \mathrm{d}\mathbf{x} &= \mathrm{d}x\mathbf{e}_x + \mathrm{d}y\mathbf{e}_y + \mathrm{d}z\mathbf{e}_z \\ &= \mathrm{d}x_i\mathbf{e}_i \end{aligned}$$

Also $f = f(x, y, z)$ has differential

$$\mathrm{d}f = \frac{\partial f}{\partial x_i}\mathrm{d}x_i$$

Then

$$\begin{aligned} \nabla f \cdot \mathrm{d}\mathbf{x} &= \left(\mathbf{e}_i \frac{\partial f}{\partial i}\right) \cdot (\mathbf{e}_j \mathrm{d}x_j) \\ &= \frac{\partial f}{\partial x_i} (\mathbf{e}_i \cdot \mathbf{e}_j)\, \mathrm{d}x_i \\ &= \mathrm{d}f \end{aligned}$$

$$\nabla f \cdot \mathrm{d}\mathbf{x} = \mathrm{d}f$$

Note. Coordinate independent statement!

Remark. Have been abusing notation.
Jumped from writing

$$f(\mathbf{x}) \to f(x, y, z)$$

Really, we should write

$$F(x, y, z) = f(\mathbf{x}(x, y, z))$$

Seems over the top in Cartesians, but would be more proper to write

$$F(u, v, w) - f(\mathbf{x}(u, v, w))$$

We should do so as otherwise could have:

$$p(\mathbf{x}) = p(x, y, z) \text{ pressure}$$

$$p(\mathbf{x}) = \tilde{p}(r, \theta, \phi) \text{ pressure}$$

$$p(\mathbf{x}) = \tilde{\tilde{p}}(x, y, z) \text{ pressure}$$

Too many different coordinate systems to choose from. Yuck!

Prop. If (u, v, w) are O.C.C and $f = f(u, v, w)$,

$$\nabla f = \frac{1}{h_u}\frac{\partial f}{\partial u} + \mathbf{e}_v + \frac{1}{h_v}\frac{\partial f}{\partial v}\mathbf{e}_w + \frac{1}{h_w}\frac{\partial f}{\partial u}\mathbf{e}_w$$

Proof. If $f = f(u, v, w)$ and $\mathbf{x} = \mathbf{x}(u, v, w)$

$$\mathrm{d}f = \frac{\partial f}{\partial u}\mathrm{d}u + \cdots + \frac{\partial f}{\partial w}\mathrm{d}w,\ \mathrm{d}\mathbf{x} = h_u \mathrm{d}u\,\mathbf{e}_u + \cdots + h_1 \mathrm{d}w\,\mathbf{e}_w$$

Using $\mathrm{d}f = \nabla f \cdot \mathrm{d}x$, and writing

$$\nabla f = (\nabla f)_u \mathbf{e}_u + \cdots + (\nabla f)_w \mathbf{e}_w$$

We find

$$\frac{\partial f}{\partial u}\mathrm{d}u + \frac{\partial f}{\partial v}\mathrm{d}v + \frac{\partial f}{\partial w}\mathrm{d}w = h_u(\nabla f)_u\,\mathrm{d}u + \cdots + h_w(\nabla f)_w\,\mathrm{d}w$$

Since $\{\mathrm{d}u, \mathrm{d}v.\mathrm{d}w\}$ are linearly independent,

$$(\nabla f)_u = \frac{1}{h_u}\frac{\partial f}{\partial u}$$

$$\vdots$$

$$(\nabla f)_w = \frac{1}{h_w}\frac{\partial f}{\partial w}\square$$

Equation. In cyclindrical polars (ρ, ϕ, z), $h_\rho = 1$, $h_\phi = \rho$, $h_z = 1$

$$\nabla f = \frac{\partial f}{\partial \rho}\mathbf{e}_\rho + \frac{1}{\rho}\frac{\partial f}{\partial \phi}\mathbf{e}_\phi + \frac{\partial f}{\partial z}\mathbf{e}_z$$

In spherical polars (r, θ, ϕ), $\ h_r = 1,\ h_\theta = r,\ h_\phi = r\sin\theta$,

$$\nabla f = \frac{\partial f}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\mathbf{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\mathbf{e}_\phi$$

Example. Let $f(\mathbf{x}) = \frac{1}{2}|\mathbf{x}|^2$. Then

$$f = \begin{cases} \frac{1}{2}(x^2 + y^2 + z^2) & \text{Cartesians} \\ \frac{1}{2}(\rho^2 + z^2) & \text{Cylindrical} \\ \frac{1}{2}r^2 & \text{Spherical} \end{cases}$$

$$\implies \nabla f = \begin{cases} x\mathbf{e}_x + y\mathbf{e}_y + z\mathbf{e}_z & \text{Cartesians} \\ \rho\mathbf{e}_\rho + z\mathbf{e}_z & \text{Cylindrical} \\ r\mathbf{e}_r & \text{Spherical} \end{cases}$$

$$= \mathbf{x}$$

Note. Answer is same in each coord system.

3 Integration over lines, surfaces and volumes

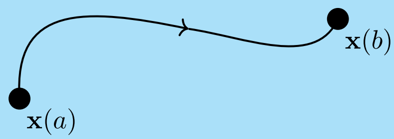
3.1 Line Integrals

Definition. For a vector field $\mathbf{F} = \mathbf{F}(\mathbf{x})$ and piecewise smooth parametrised curve

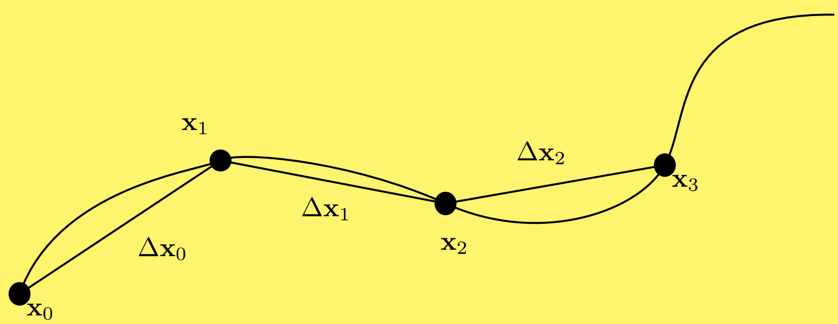
$$C : [a, b] \ni t \mapsto \mathbf{x}(t)$$

We define **line integral**

$$\int_X \mathbf{F} \cdot \mathrm{d}\mathbf{x} = \int_a^b \mathbf{F}(\mathbf{x}(t)) \cdot \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} \, \mathrm{d}t$$



Remark. If we want to integrate in opposit direction, write $\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{x}$. Can interpret as work fone by particle moving along C in presence of force $\mathbf{F}$.



$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{x} \simeq \sum_i \mathbf{F}(\mathbf{x}_i) \cdot \Delta\mathbf{x}_i$$

$$\Delta\mathbf{x}_i = \mathbf{x}_{i+1} - \mathbf{x}_i$$

Example. Consider

$$\mathbf{F} = \begin{bmatrix} x^2y \\ y^2 \\ 2zx \end{bmatrix}$$

Consider two courves connecting origin to $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$$C_{:}[0,1] \ni t \mapsto \begin{bmatrix} t \\ t \\ t \end{bmatrix}, \; C_2 : [0,1] \ni t \mapsto \begin{bmatrix} t \\ t \\ t^2 \end{bmatrix}$$

So

$$\int_{C_1} \mathbf{F} \cdot \mathrm{d}\mathbf{x} = \int_0^1 \begin{bmatrix} t^3 \\ t^2 \\ 2t^2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \, \mathrm{d}t = \frac{5}{4}$$

$$\int_{C_2} \mathbf{F} \cdot \mathrm{d}\mathbf{x} = \int_0^1 \begin{bmatrix} t^3 \\ t^3 \\ 2t^3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 2t \end{bmatrix} \, \mathrm{d}t = \frac{13}{10}$$

See that, in general, line integral between two points depends on path taken

Example. A particle at $\mathbf{x}$ experiences force in cylindrical polars

$$\mathbf{F}(\mathbf{x}) = z\rho\mathbf{e}_\phi$$

Calculate work done by travelling along

$$C : [0, 2\pi] \ni t \mapsto \begin{bmatrix} a\cos t \\ a\sin t \\ t \end{bmatrix} \quad (a > 0)$$

Recall line element in cylindrical polars

$$\mathrm{d}\mathbf{x} = \mathrm{d}\rho\,\mathbf{e}_\rho + \rho\,\mathrm{d}\phi\,\mathbf{e}_\phi + \mathrm{d}z\,\mathbf{e}_z$$

So

$$\mathbf{F} \cdot \mathrm{d}\mathbf{x} = z^2\rho^2\,\mathrm{d}\phi$$

Also, on path

$$\begin{aligned} &(\rho, \phi, z) = (a, t, t) \\ \implies &(\mathrm{d}\rho, \mathrm{d}\phi, \mathrm{d}z) = (0, \mathrm{d}t, \mathrm{d}t) \\ \implies &\mathbf{F} \cdot \mathrm{d}\mathbf{x} = a^2 t\,\mathrm{d}t \end{aligned}$$

Finally then

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{x} = a^2 \int_0^{2\pi} t\,\mathrm{d}t = 2\pi^2 a^2$$

Definition. We say a curve

$$[a,b] \ni t \mapsto \mathbf{x}(t)$$

is **closed** if $\mathbf{x}(a) = \mathbf{x}(b)$.
In this case, write

$$\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{x}$$

Sometimes call integrals of this form the circulation of $\mathbf{F}$ about C

Example. Take one before previous example with $C = C_1 - C_2$

$$\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{x} = \int_{C_1} \mathbf{F} \cdot \mathrm{d}x - \int_{C_2} \mathbf{F} \cdot \mathrm{d}\mathbf{x} = -\frac{1}{20}$$

3.2 Conservative Forces and Exact Differentials

We've seen how to interpret things like $\mathbf{F} \cdot \mathrm{d}\mathbf{x}$ when they're inside an integral. This is another differential form i.e. in coords (u, v, w)

$$\mathbf{F} \cdot \mathrm{d}\mathbf{x} = (\ )\mathrm{d}u + (\ )\mathrm{d}v + (\ )\mathrm{d}w$$

Definition. We say that $\mathbf{F} \cdot \mathrm{d}\mathbf{x}$ is **exact** if

$$\mathbf{F} \cdot \mathrm{d}\mathbf{x} = \mathrm{d}f$$

for some scalar f. Recall that

$$\mathrm{d}f = \nabla f \cdot \mathrm{d}\mathbf{x}$$

So $\mathbf{F} \cdot \mathrm{d}\mathbf{x}$ is exact iff $\mathbf{F} = \nabla f$ for some scalar f. Call such vector fields conservative.

Claim. So we have

$$\mathbf{F} \cdot \mathrm{d}\mathbf{x} \text{ is exact} \iff \mathbf{F} \text{ is conservative.}$$

Remark. Using properties $\mathrm{d}(\alpha f + \beta g) = \alpha \mathrm{d}f + \beta \mathrm{d}g$ (α, β) constant, $\mathrm{d}(fg) = g\mathrm{d}f + f\mathrm{d}g$ etc. usually easy to see if form $\mathbf{F} \cdot \mathrm{d}\mathbf{x}$ is exact

Prop. If θ is exact differential form then

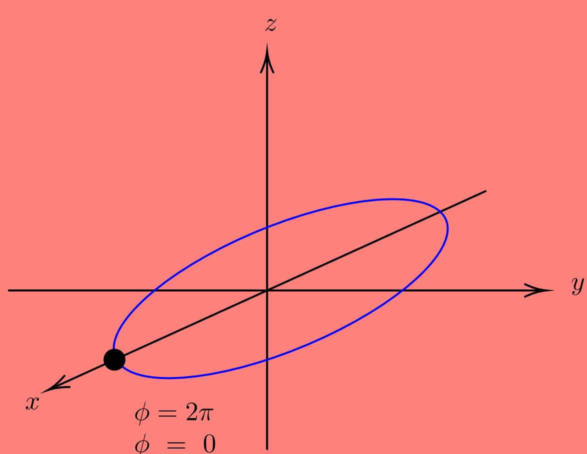
$$\oint_C \theta = 0$$

for any closed curve C

Proof. By previous, if θ exact, then $\theta = \nabla f \cdot \mathrm{d}\mathbf{x}$ for some scalar f. If C is $[a, b] \ni t \mapsto \mathbf{x}(t)$

$$\begin{aligned}\oint_C \theta = \oint \nabla f \cdot \mathrm{d}\mathbf{x} &= \int_a^b \nabla(\mathbf{x}(t)) \cdot \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} \\ &= \int_a^b \frac{\mathrm{d}}{\mathrm{d}t}[f(\mathbf{x}(t))]\,\mathrm{d}t \\ &= f(\mathbf{x}(a)) - d(\mathbf{x}(b)) \\ &= 0 \text{ if } \mathbf{x}(a) = \mathbf{x}(b)\end{aligned}$$

Warning. Might think e.g. in cylindrical polars, that $f(\rho, \phi, z) = \phi$ is a nice "function" on $\mathbb{R}^3$



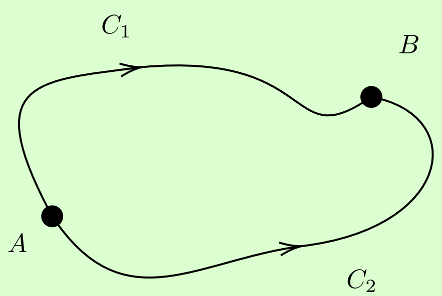
It is multi-valued at a given position

Prop. Equivalently, if $\mathbf{F}$ is conservative then circulation of $\mathbf{F}$ around any closed loop curve C vanishes

$$\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{x} = 0$$

If $\mathbf{F}$ conservative ($\mathbf{F} \cdot \mathrm{d}\mathbf{x}$ exact), then line integral between points $A = \mathbf{x}(a)$ and $B = \mathbf{x}(b)$ is independent of path

Proof.



If $C = C_1 - C_2$,

$$\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{x} = 0$$
$$\iff \int_{C_1} \mathbf{F} \cdot \mathrm{d}\mathbf{x} = \int_{C_2} \mathbf{F} \cdot \mathrm{d}\mathbf{x}$$

Claim. Let $(u_1, u_2, u_3) \equiv (u, v, w)$ be set of OCC. Let

$$\begin{aligned}\mathbf{F} \cdot \mathrm{d}\mathbf{x} = \theta &= \underbrace{A(u,v,w)}_{\theta_1} \mathrm{d}u + \underbrace{B(u,v,w)}_{\theta_2} \mathrm{d}v + \underbrace{C(u,v,w)}_{\theta_3} \mathrm{d}w \\ &= \theta_i \, \mathrm{d}u_i\end{aligned}$$

A necessary condition for θ to be exact is

$$\frac{\partial \theta_i}{\partial u_j} = \frac{\partial \theta_j}{\partial u_i} \text{ each } i, j \tag{†}$$

Proof. Indeed, if θ exact, then $\theta = \mathrm{d}f$, so

$$\theta = \frac{\partial f}{\partial u_i} \mathrm{d}u_i \iff \theta_i = \frac{\partial f}{\partial u_i}$$

and so

$$\frac{\partial \theta_i}{\partial u_j} = \frac{\partial^2 f}{\partial u_j \partial u_i} = \frac{\partial^2 f}{\partial u_i \partial u_j} = \frac{\partial \theta_j}{\partial u_i}$$

Definition. Call differential forms $\theta = \theta_i$ that obey (†) **closed**. So

$$\theta \text{ exact} \implies \theta \text{ closed}$$

Note. The reverse implication is true if the domain $\Omega \subseteq \mathbb{R}^3$ on which θ is defined is simply-connected.

Definition (Non-examinable)**.** Ω **simply connected** means all closed loops in Ω can be continously shrunk to any point insider Ω without leaving it

Look at de Rham Cohomology.

Example. (i)

$$\theta = y\,\mathrm{d}x - x\,\mathrm{d}y$$

Is it exact?
Check:is it closed

$$1 \neq -1$$

So

$$\frac{\partial}{\partial y} \neq \frac{\partial}{\partial x}$$

(ii) Compute line integral

$$\oint 3x^2 y\,\mathrm{d}x + x^3\,\mathrm{d}y$$

$$C : [\alpha_1, \alpha_{100}] \ni t \mapsto \begin{bmatrix} \cos[\mathrm{Im}[\zeta(\frac{1}{2} + it)]] \\ \sin[\mathrm{Im}[\zeta(\frac{1}{2} + it)]] \end{bmatrix}$$

where α_1 and α_{100} are the 1^{st} and 100^{th} zero of $\zeta(\frac{1}{2} + it)$ i.e.

$$\zeta\left(\frac{1}{2} + i\alpha_1\right) = \zeta\left(\frac{1}{2} + i\alpha_{100}\right) = 0$$

$$\oint_C 3x^2 y\,\mathrm{d}x + x^3 y\,\mathrm{d}y = 0$$

As

$$3x^2 y\,\mathrm{d}x + x^3\,\mathrm{d}y = \mathrm{d}\big(x^3 y\big)$$

Example.

$$\begin{aligned} \text{Work done } &= \int_C \mathbf{F} \cdot \mathrm{d}\mathbf{x} \\ &= m \int_a^b \ddot{\mathbf{x}} \cdot \dot{\mathbf{x}}\,\mathrm{d}t \\ &= \frac{1}{2} m |\dot{\mathbf{x}}|^2 \Big|_a^b \end{aligned}$$

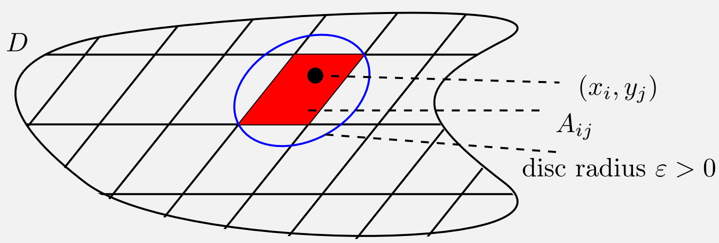
If $\mathbf{F} = -\nabla V$, i.e. $\mathbf{F}$ conservative,

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{x} = -\int_C \nabla V \cdot \mathrm{d}\mathbf{x} = V(\mathbf{x}(a)) - V(\mathbf{x}(b))$$

$$\left(\frac{1}{2} m |\dot{\mathbf{x}}|^2 + V(\mathbf{x}(t))\right)\Big|_{t=a} = \left(\frac{1}{2} m |\dot{\mathbf{x}}|^2 + V(\mathbf{x}(t))\right)\Big|_{t=b}$$

3.3 Integration in $\mathbb{R}^2$

Want to integrate over bounded region $D \subset \mathbb{R}^2$.
To do this: cover D with small disjoint sets A_{ij}, each with area δ_{ij}, each contained in a disc of radius $\varepsilon > 0$. Let (x_i, y_j) be points contained in each A_{ij}

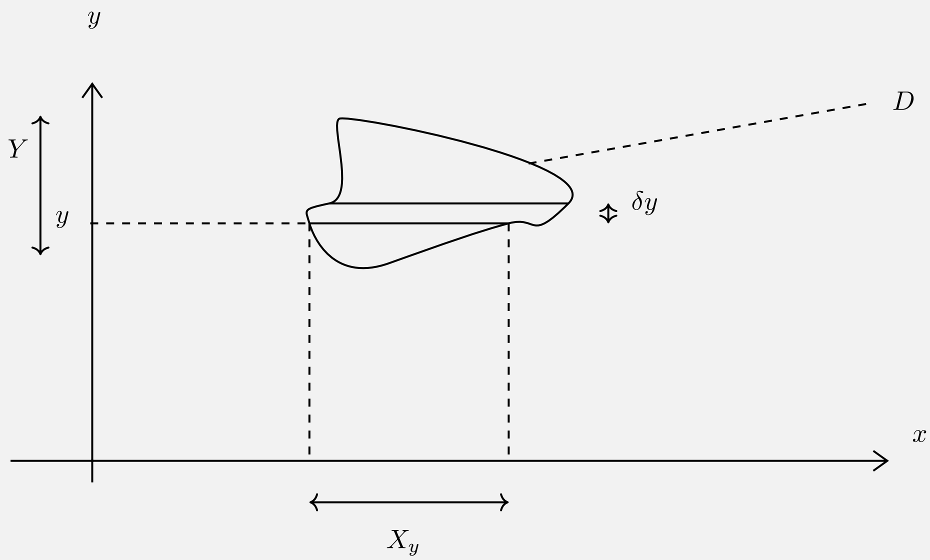


Now define

$$\int_D f(\mathbf{x}) \, \mathrm{d}A = \lim_{\varepsilon \to 0} \sum_{i,j} f(x_i y_j) \delta A_{ij}$$

Say the integral exists if it is independent of choice A_{ij} and choice (x_i, y_j)

Obvious choice of partition would be to use rectangles with $\delta A_{ij} = \delta x \delta y$



(*)

Sum over horizontal strips of width δy, then take limit as $\delta x \to 0$

$$\delta y \int_{X_y} f(x,y)\,\mathrm{d}x \text{ (where } X_y = \{x : (x,y) \in D\})$$

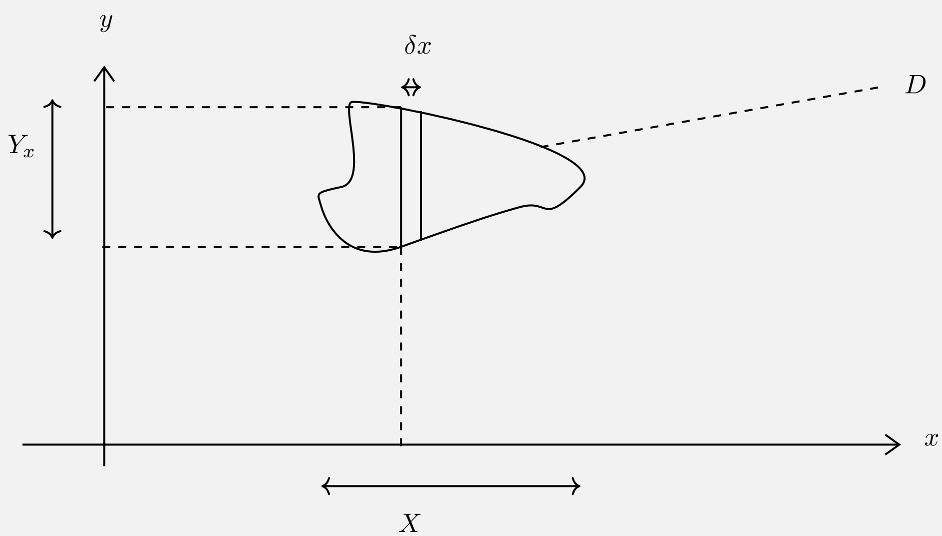
Summing over each such strip, taking $\delta y \to 0$ we get

$$\int_D f(c,y)\,\mathrm{d}A = \int_Y \left(\int_{X_y} f(x,y)\,\mathrm{d}x \right) \mathrm{d}y$$

where Y is as (*).

If we instead sum over vertical strips, get

$$\int_D f(x,y)\,\mathrm{d}A = \int_X \left(\int_{Y_x} f(x,y)\,\mathrm{d}y\right)\,\mathrm{d}x$$



More concisely, we have

$$\mathrm{d}A = \mathrm{d}x\,\mathrm{d}y = \mathrm{d}y\,\mathrm{d}x$$

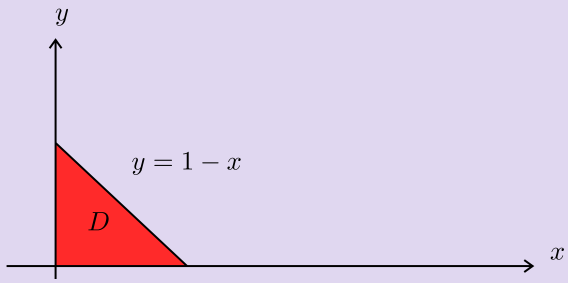
Note. See Fubini's theorem in Part II Probability and Measure:
If

$$\int_D |f(x,y)|\,\mathrm{d}A < \infty$$

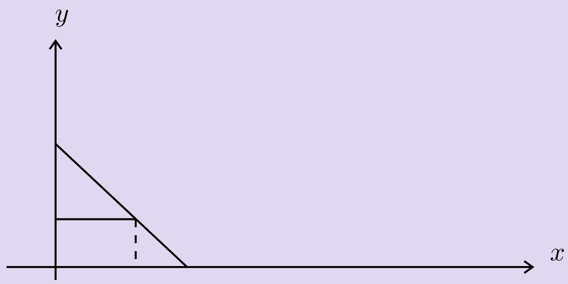
Then

$$\int \left(\int f(x,y)\,\mathrm{d}x\right)\,\mathrm{d}y = \int \left(\int f(x,y)\,\mathrm{d}y\right)\,\mathrm{d}x = \int_D \int f(x,y)\,\mathrm{d}A$$

Example. Let D be a triangle with vertices $(0,0),(1,0),(0,1)$

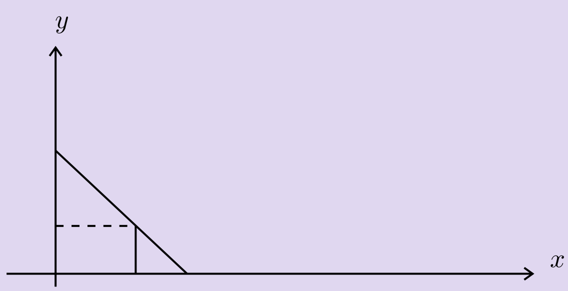


If $f(x,y) = xy^2$ then if we integrate over horizontal strips



$$\begin{aligned}\int_D f \,\mathrm{d}A &= \int_0^1 \left(\int_0^{1-y} xy^2 \,\mathrm{d}x\right) \mathrm{d}y \\ &= \int_0^1 y^2 \left[\frac{1}{2}x^2\right]_0^{1-y} \mathrm{d}y \\ &= \int_0^1 \frac{1}{2}y^2(1-y)^2 \,\mathrm{d}y = \frac{1}{60}\end{aligned}$$

With vertical:



$$\begin{aligned}\int_D f \,\mathrm{d}A &= \int_0^1 \left(\int_0^{1-x} xy^2 \,\mathrm{d}y\right) \mathrm{d}x \\ &= \int_0^1 x \left[\frac{1}{3}y^3\right]_0^{1-x} \mathrm{d}x \\ &= \frac{1}{60}\end{aligned}$$

Method. If $f(x,y) = g(x)h(y)$ and D is a rectangle

$$D = \{(x,y) : a \le x \le b,\ c \le y \le d\}$$

Then

$$\int_A f(x,y)\,\mathrm{d}A = \left(\int_a^b g(x)\,\mathrm{d}x\right)\left(\int_c^d h(y)\,\mathrm{d}y\right)$$

Method. Often useful to introduce change of variables to compute

$$\int_a^b f(x)\,\mathrm{d}x$$

If we introduce $x = x(u)$ with $x(\alpha) = a$ and $x(\beta) = b$ then:

$$\int_a^b f(x)\,\mathrm{d}x = \begin{cases} +\int_\alpha^\beta f(x(u))\frac{\mathrm{d}x}{\mathrm{d}u}\,\mathrm{d}u\ (\beta > \alpha,\ \frac{\mathrm{d}x}{\mathrm{d}u} > 0) \\ \\ -\int_\beta^\alpha f(x(u))\frac{\mathrm{d}x}{\mathrm{d}u}\,\mathrm{d}u\ (\alpha > \beta, \frac{\mathrm{d}x}{\mathrm{d}u} < 0) \end{cases}$$

If $I = [a,b]$ and $I' = x(I)$

$$\int_I f(x)\,\mathrm{d}x = \int_{I'} f(x(u))\left|\frac{\mathrm{d}x}{\mathrm{d}u}\right|\,\mathrm{d}u$$

Note. Similar formula in 2D

Prop. Let $x = x(u, v)$ and $y = y(u, v)$ be a smooth, invertible transformation with smooth inverse that maps the region D' in the (u, v) plane to the region D in the (x, y)-plane. Write $x = x(u, v)$, then

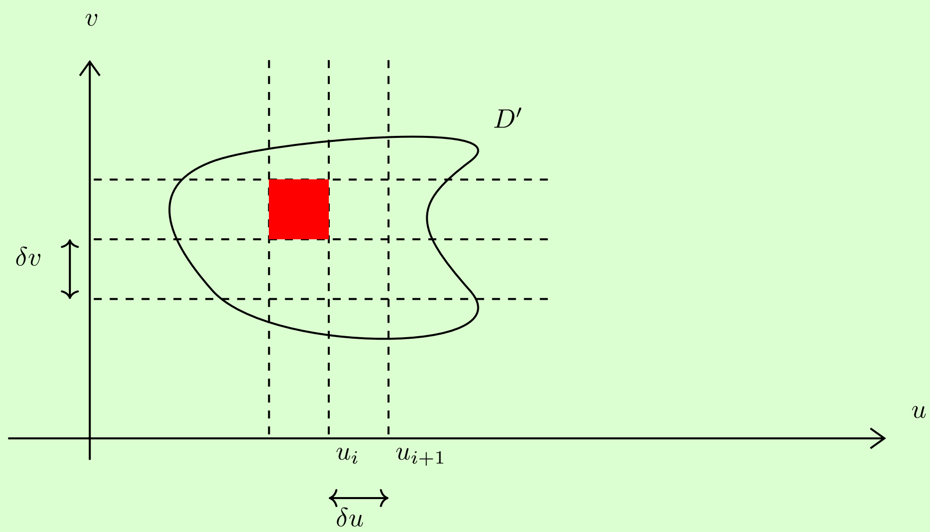
$$\int_D \int f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_{D'} \int f(x(u,v), y(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| \mathrm{d}u\,\mathrm{d}v$$

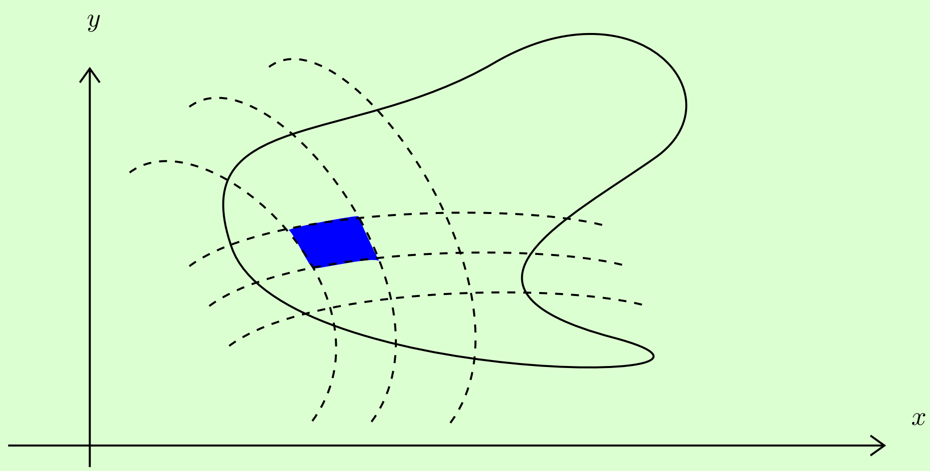
Where

$$\frac{\partial(x,y)}{\partial(u,v)} = \det \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} = \det \left[\frac{\partial \mathbf{x}}{\partial u} \middle| \frac{\partial \mathbf{x}}{\partial v} \right]$$

is the Jacobian, often denoted by J. Short version is $\mathrm{d}x\,\mathrm{d}y = |J|\,\mathrm{d}u\,\mathrm{d}v$

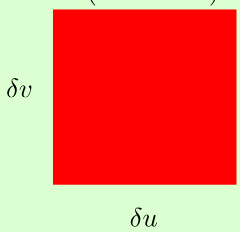
Proof. form a partiton of D using the image of a rectangular partition of D'

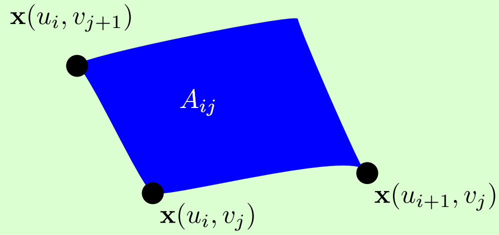




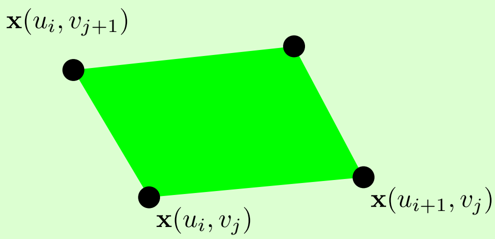
Prop (continued).

Proof (continued).





Note $\delta A_{ij} \simeq \text{Area}(B)$
$B =$

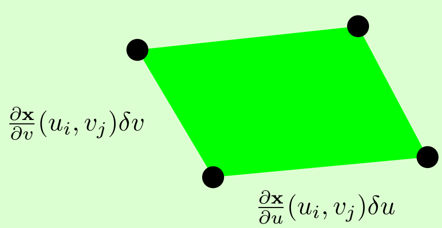


$$\mathbf{x}(u_{i+1}, v_j) - \mathbf{x}(u_i, v_j) \simeq \frac{\partial \mathbf{x}}{\partial u}\delta u$$

$$\mathbf{x}(u_i, v_{j+1}) - \mathbf{x}(u_i, v_j) \simeq \frac{\partial \mathbf{x}}{\partial v}\delta v$$

$$\text{Area}(B) \simeq \text{Area}(C)$$

$C =$



$$\begin{aligned}\text{Area}(C) &= \left|\det\left(\frac{\partial \mathbf{x}}{\partial u}\delta u \,\middle|\, \frac{\partial \mathbf{x}}{\partial v}\delta v\right)\right| \\ &= \underbrace{|J(u_i, v_j)|\delta u\, \delta v}_{A_{ij}}\end{aligned}$$

Prop (continued)**.**

Proof (continued)**.**

$$\begin{aligned}\int_D f \, \mathrm{d}A &= \lim_{\varepsilon \to 0} \sum_{i,j} f(x_i, y_j) \delta A_{ij} \\ &= \lim_{\varepsilon \to 0} \sum_{i,j} f(x(u_i, v_j), y(u_i, v_j)) |J(u_i, v_j)| \delta u \, \delta v \\ &= \int_{D'} \int f(x(u,v), y(u,v)) |J(u,v)| \, \mathrm{d}u \, \mathrm{d}v \\ &= \int_D \int f(x,y) \, \mathrm{d}x \, \mathrm{d}y\end{aligned}$$

Giving us

$$\mathrm{d}x \, \mathrm{d}y = |J| \, \mathrm{d}u \, \mathrm{d}v$$

Equation.

$$\mathrm{d}x \, \mathrm{d}y = |J| \, \mathrm{d}u \, \mathrm{d}v$$

Example. Use polar coords (ρ, ϕ)

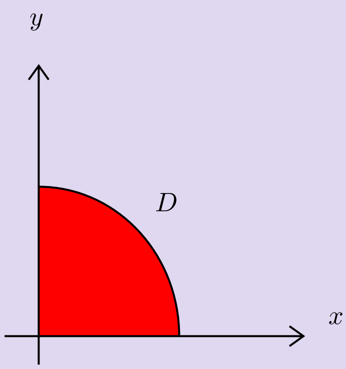
$$x(\rho, \phi) = \rho \cos \phi$$
$$y(\rho, \phi) = \rho \sin \phi$$

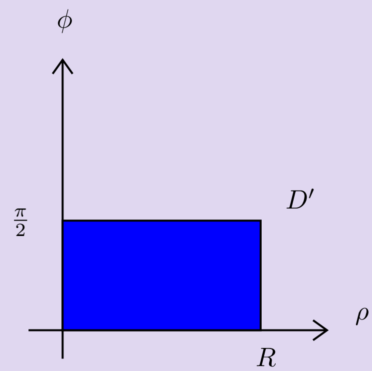
Hence

$$\begin{aligned} |J| &= \left| \det \begin{bmatrix} \cos \phi & -\rho \sin \phi \\ \sin \phi & \rho \cos \phi \end{bmatrix} \right| \\ &= |\rho| \\ &= \rho \end{aligned}$$

If

$$D' = \{(x, y) : x > 0,\ y > 0,\ x^2 + y^2 < R^2\}$$





$$D' = \{(\rho, \phi) : 0 < \rho < R,\ 0 < \phi, \frac{\pi}{2}\}$$

$$\int_D \int f(x, y) \,\mathrm{d}x \,\mathrm{d}y = \int_{D'} \int f(\rho \cos \phi, \rho \sin \phi) \rho \,\mathrm{d}\rho \,\mathrm{d}\phi$$

i.e.

$$\mathrm{d}x \,\mathrm{d}y - \rho \,\mathrm{d}\rho \,\mathrm{d}\phi$$

Take $R \to \infty$

$$\int_{x=0}^{\infty} \int_{y=0}^{\infty} f(x, y) \,\mathrm{d}y = \int_{\phi=0}^{\pi/2} \int_{\rho=0}^{\infty} f(\rho \cos \phi, \rho \sin \phi) \rho \,\mathrm{d}\rho \,\mathrm{d}\phi$$

Consider

$$I = \int_0^{\infty} e^{-x^2} \,\mathrm{d}x$$

Have

$$\begin{aligned} I^2 &= \int_0^{\infty} e^{-x^2} \,\mathrm{d}x \cdot \int_0^{\infty} e^{-y^2} \,\mathrm{d}y \\ &= \int_{x=0}^{\infty} \int_{y=0}^{\infty} e^{-x^2 - y^2} \,\mathrm{d}x \,\mathrm{d}y \\ &= \int_{\phi=0}^{\frac{\pi}{2}} \left(\int_{\rho=0}^{\infty} e^{-\rho^2} \rho \,\mathrm{d}\rho \right) \mathrm{d}\phi \\ &= \frac{\pi}{2} \int_0^{\infty} \frac{\mathrm{d}}{\mathrm{d}\rho} \left(-\frac{1}{2} e^{-\rho^2} \right) \mathrm{d}\rho = \frac{\pi}{4} \\ \implies I &= \frac{\sqrt{\pi}}{2} \end{aligned}$$

3.4 Integration in $\mathbb{R}^3$

Method. to integrate over regions V in $\mathbb{R}^3$, use similar ideas to those in section 3.3. Let

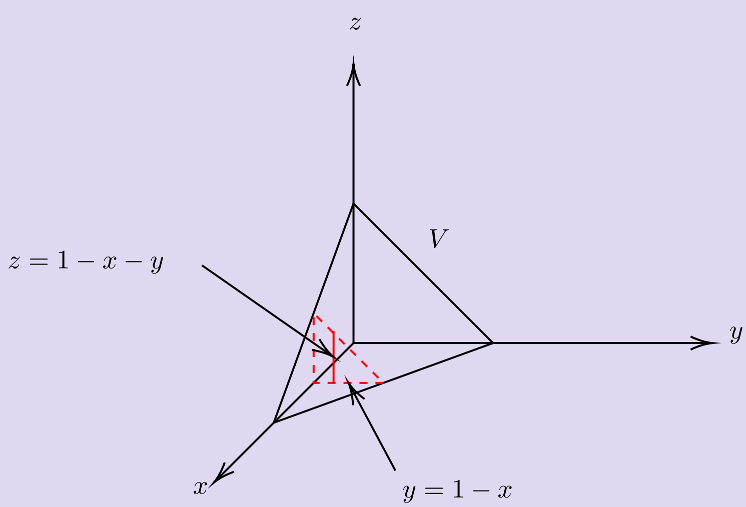
$$\int_V f(\mathbf{x})\,\mathrm{d}V = \lim_{\varepsilon\to 0}\sum_{i,j,k} f(x_i, y_i, z_i)\,\delta V_{ijk}$$

In this case the volume element satisfies

$$\mathrm{d}V = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

Note. Can do integrals in any order.

Example.



V bounded by plane $x+y+z=1$ and the three planes $x=0, y=0$ and $z=0$

$$\begin{aligned}\int_V \mathrm{d}V &= \int_{x=0}^{1}\int_{y=0}^{1-x}\int_{z=0}^{1-x-y}\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x\\ &= \int_{x=0}^{1}\mathrm{d}x\int_{y=0}^{1-x}(1-x-y)\,\mathrm{d}y\\ &= \frac{1}{6}\end{aligned}$$

Could compute CoM of V, assuming density $\rho = 1$

$$\mathbf{X} = \frac{1}{M}\int_V \rho\mathbf{x}\,\mathrm{d}V = \frac{1}{4}\begin{bmatrix}1\\1\\1\end{bmatrix}$$

Prop. Let $x = x(u, v, w)$, $y = y(u, v, w)$, $z = z(u, v, w)$ be a continuously differentiable bijection with continuously differentiable inverse that maps the volume V' to the volume V.

$$\int\int_V\int f(x, y, z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \int\int_{V'}\int f(x(u, v, w), y(u, v, w), z(u, v, w))|J|\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}w$$

Where

$$J = \frac{\partial(x, y, z)}{\partial(u, v, w)} = \det\left[\frac{\partial\mathbf{x}}{\partial u}\middle|\frac{\partial\mathbf{x}}{\partial v}\middle|\frac{\partial\mathbf{x}}{\partial w}\right]$$

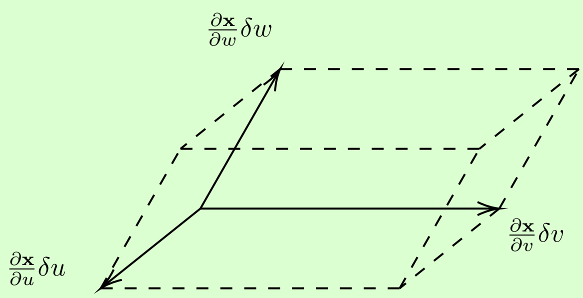
and

$$\mathbf{x} = \begin{bmatrix} x(u, v, w) \\ \vdots \\ z(u, v, w) \end{bmatrix}$$

Short version:

$$\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = |J|\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}w$$

Proof. Jacobian comes from fact that volume of a parallelepiped generated by



is

$$\det\left[\frac{\partial\mathbf{x}}{\partial u}\middle|\frac{\partial\mathbf{x}}{\partial v}\middle|\frac{\partial\mathbf{x}}{\partial w}\right]\delta u\,\delta v\,\delta w$$

The rest is (almost) same as 2D case.

Example. Find in cylindrical polars $(u, v, w) = (\rho, \phi, z)$

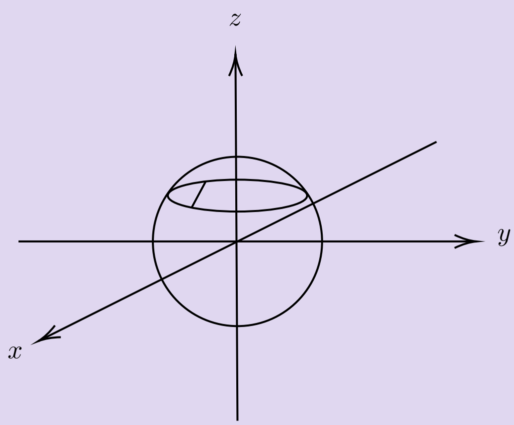
$$\mathrm{d}V = \rho\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}z \quad |J| = \rho$$

In spherical polars $(u, v, w) = (r, \theta, \phi)$

$$\mathrm{d}V = r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi \quad |J| = r^2\sin\theta$$

Example. Calculate volume of ball of radius R

$$V = \{(x, y, z) : x^2 + y^2 + z^2 \le R^2\}$$



$$\begin{aligned}
\int_V \mathrm{d}V &= \int_{z=-R}^{R} \int_{y=-\sqrt{R^2-z^2}}^{\sqrt{R^2-z^2}} \int_{x=-\sqrt{R^2-z^2-y^2}}^{\sqrt{R^2-z^2-y^2}} \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \\
&= \int_{z=-R}^{R} \left[\int_{y=-\sqrt{R^2-z^2}}^{\sqrt{R^2-z^2}} 2\sqrt{R^2 - z^2 - y^2}\,\mathrm{d}y \right] \mathrm{d}z \\
&= \int_{z=-R}^{R} \left[y\sqrt{R^2 - z^2 - y^2} + (R^2 - z^2)\tan^{-1}\left[\frac{y}{\sqrt{R^2 - z^2 - y^2}} \right] \right]_{y=-\sqrt{R^2-z^2}}^{\sqrt{R^2-z^2}} \mathrm{d}z \\
&= \int_{-R}^{R} \pi(R^2 - z^2)\,\mathrm{d}z \\
&= \frac{4\pi R^3}{3}
\end{aligned}$$

Alternatively, use spherical polars

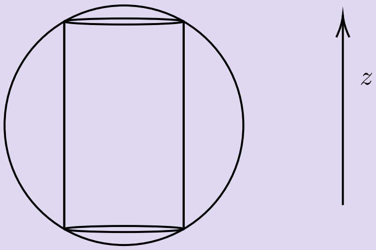
$$V' = \{(r, \theta, \phi) : 0 \le r \le R,\ 0 \le \theta \le \pi,\ 0 \le \phi \le 2\pi\}$$

So

$$\begin{aligned}
\text{Volume } &= \int_{\phi=0}^{2\pi} \left[\int_{\theta=0}^{\pi} \left[\int_{r=0}^{R} r^2 \sin\theta\,\mathrm{d}r \right] \mathrm{d}\theta \right] \mathrm{d}\phi \\
&= \int_{\theta=0}^{\pi} \frac{2\pi R^3}{3} \sin\theta\,\mathrm{d}\theta \\
&= \frac{4\pi R^3}{3}
\end{aligned}$$

MUCH NICER COMPUTATION

Example. Consider ball of radius a with cylinder of radius $b < a$ removed



Symmetry suggests use of cylindrical polars

$$V = \{(x, y, z) : x^+ y^2 + z^2 \leq a^2,\ x^2 + y^2 \geq b^2\}$$

Or in cylindrical polars

$$\{(\rho, \phi, z) : b \leq \rho \leq a,\ 0 \leq z^2 + \rho^2 \leq a^2,\ 0 \leq \phi < 2\pi\}$$

$$\begin{aligned}\int_V \mathrm{d}V &= \int_{\rho=b}^{a} \left[\int_{\phi=0}^{2\pi} \left[\int_{z=-\sqrt{a^2-\rho^2}}^{\sqrt{a^2-\rho^2}} \mathrm{d}z\right] \mathrm{d}\phi\right] \underbrace{\rho}_{=|J|} \mathrm{d}\rho \\ &= 2\pi \int_b^a 2\rho\sqrt{a^2 - \rho^2}\, \mathrm{d}\rho \\ &= \frac{4\pi}{3}(a^2 - b^2)^{3/2}\end{aligned}$$

3.5 Integration over surfaces

Remark. A two dimensional in $\mathbb{R}^3$ can be defined implicitly using a function $f : \mathbb{R}^3 \to \mathbb{R}$

$$S = \{\mathbf{x} \in \mathbb{R}^3 : f(\mathbf{x}) = 0\}$$

Normal to S at $\mathbf{x}$ is parallel to $\nabla f(\mathbf{x})$.
Call surface regular if $\nabla f(\mathbf{x}) \neq 0$ for $\mathbf{x} \in S$

Example.

$$S = \{(x, y, z) : x^2 + y^2 + z^2 - 1 = 0\}$$

So

$$\nabla f(\mathbf{x}) = \begin{bmatrix} 2x \\ 2y \\ 2z \end{bmatrix} = 2\mathbf{x}$$

which is normal to S at $\mathbf{x}$
Some surfaces have a boundary, e.g.

$$S = \{(x, y, z) : x^2 + y^2 + z^2 = 1,\ z \geq 0\}$$

Label the boundary by ∂S

$$\partial S = \{(x, y, z) : x^2 + y^2 = 1,\ z = 0\}$$

In this course, a surface S will either have no boundary ($\partial S = \varnothing$), or it will have boundary made of piecewise smooth curves. If S has no boundary, say S is a closed surface.

Moral. It is often useful to parametrise a surface using some coordinates (u, v)

$$S = \{\mathbf{x} = \mathbf{x}(u, v),\ (u, v) \in D\}$$

D some region in (u, v)-plane

Example. For hemisphere, use spherical polars

$$S = \{\mathbf{x} = \mathbf{x}(\theta, \phi) = \begin{bmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{bmatrix},\ 0 \leq < 2\pi\}$$

Definition. Call parametrisation of S **regular** if

$$\frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v} \neq 0 \text{ on } S$$

In this case, we can define normal

$$\mathbf{n} = \frac{\frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v}}{\left|\frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v}\right|}$$

Note. This normal will vary smoothly wrt (u, v).
Choosing a normal consistently over S gives us a way of orientating the boundary ∂S: make the convention that normal vectors in your immediate vicinity should be on your left as you traverse ∂S

Method. How should we compute area of

$$S = \{\mathbf{x} = \mathbf{x}(u, v),\ \ )u, v) \in D\}$$

Might think that it would be

$$\iint_D \mathrm{d}u\, \mathrm{d}v \text{ (WRONG)}$$

Patch of area $\delta u \delta v$ in D will not in general correspond to patch of area $\delta u \delta v$ on S
Note small changes $u \mapsto u + \delta u$ produces

$$\mathbf{x}(u + \delta u, v) - \mathbf{x}(u, v) \simeq \frac{\partial \mathbf{x}}{\partial u} \delta u$$

Similarly, $v \mapsto v + \delta v$ produces change

$$\mathbf{x}(u, v + \delta v) - \mathbf{x}(u, v) \simeq \frac{\partial \mathbf{x}}{\partial v} \delta v$$

So the patch of area $\delta u \delta v$ in D corresponds (to first order) to a parallelogram of area

$$\text{area(parallelogram)} \ = \left| \frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v} \right| \delta u \delta v$$

Definition. This leads us to define the **scalar area element** and **vector area element**

$$\mathrm{d}S = \left| \frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v} \right| \mathrm{d}u\, \mathrm{d}v$$

$$\mathrm{d}\mathbf{S} = \frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v} \mathrm{d}u\, \mathrm{d}v = \mathbf{n}\, \mathrm{d}S$$

Equation. Gives area of S:

$$\text{area}(S) = \int_S \mathrm{d}S = \iint_D \left| \frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v} \right| \mathrm{d}u\, \mathrm{d}v$$

and

$$\int_S f\, \mathrm{d}S = \iint_D f(\mathbf{x}(u, v)) \left| \frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v} \right| \mathrm{d}u\, \mathrm{d}v$$

Example. Consider hemisphere of radius R

$$S = \{\mathbf{x}(\theta,\phi) = \begin{bmatrix} R\sin\theta\cos\phi \\ R\sin\theta\sin\phi \\ R\cos\theta \end{bmatrix} \equiv R\mathbf{e}_r,\ 0 \le \theta \le \frac{\pi}{2},\ 0 \le \phi < 2\pi\}$$

So

$$\frac{\partial \mathbf{x}}{\partial \theta} = \begin{bmatrix} R\cos\theta\cos\phi \\ R\cos\theta\sin\phi \\ -R\sin\theta \end{bmatrix} = R\mathbf{e}_\theta$$

$$\frac{\partial \mathbf{x}}{\partial \phi} = \begin{bmatrix} -R\sin\theta\sin\phi \\ R\sin\theta\cos\phi \\ 0 \end{bmatrix} = R\sin\theta\mathbf{e}_\phi$$

$$\begin{aligned} \implies \mathrm{d}S &= R^2\sin\theta|\mathbf{e}_\theta \times \mathbf{e}_\phi|\,\mathrm{d}\theta\,\mathrm{d}\phi \\ &= R^2\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi \end{aligned}$$

$$\text{area}(S) = \int_{\theta=0}^{2\pi}\left(\int_{\phi=0}^{2\pi} R^2\sin\theta\,\mathrm{d}\phi\right)\mathrm{d}\theta = 2\pi R^2$$

Example. Suppose velocity of fluid is written $\mathbf{u} = \mathbf{u}(\mathbf{x})$. Given S, how to calculate how much fluid passes through it per unit time? On small patch ∂S on S, fluid passing through would be $(\mathbf{u}\cdot\delta\mathbf{S})\delta t$ in time δt. So amount of fluid that passes over S in ∂t is

$$\delta t \int_S \mathbf{u}\cdot\,\mathrm{d}\mathbf{S}$$

This is the rate at which fluid passes through surface S times δt.
Called "flux" integrals.

Are these surface integrals dependant on choice of parametrisation of S?
Let $\mathbf{x} = \mathbf{x}(u,v)$ and $\mathbf{x} = \tilde{\mathbf{x}}(\tilde{u},\tilde{v})$ be two different parametrisations of S with $(u,v) \in D$ and $(\tilde{u},\tilde{v}) \in \tilde{D}$.
Must have relationship

$$\mathbf{x}(u,v) = \tilde{\mathbf{x}}((\tilde{u}(u,v),\tilde{v}(u,v))$$

$$\begin{aligned} \frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v} &= \left(\frac{\partial \tilde{\mathbf{x}}}{\partial \tilde{u}}\frac{\partial \tilde{u}}{\partial u} + \frac{\partial \tilde{\mathbf{x}}}{\partial \tilde{v}}\frac{\partial \tilde{v}}{\partial u}\right) \times \left(\frac{\partial \tilde{\mathbf{x}}}{\partial \tilde{u}}\frac{\partial \tilde{u}}{\partial v} + \frac{\partial \tilde{\mathbf{x}}}{\partial \tilde{v}}\frac{\partial \tilde{v}}{\partial v}\right) \\ &= \left(\frac{\partial \tilde{u}}{\partial u}\frac{\partial \tilde{v}}{\partial v} - \frac{\partial \tilde{u}}{\partial v}\frac{\partial \tilde{v}}{\partial u}\right)\frac{\partial \tilde{\mathbf{x}}}{\partial \tilde{u}} \times \frac{\partial \tilde{\mathbf{x}}}{\partial \tilde{v}} \\ &= \frac{\partial(\tilde{u},\tilde{v})}{\partial(u,v)}\frac{\partial \tilde{\mathbf{x}}}{\partial \tilde{u}} \times \frac{\partial \tilde{\mathbf{x}}}{\partial \tilde{v}} \end{aligned}$$

Note.

$$\int_S f \,\mathrm{d}S = \iint\limits_{\tilde{D}} f(\tilde{\mathbf{x}}(\tilde{u},\tilde{v})) \left| \frac{\partial \tilde{\mathbf{x}}}{\partial \tilde{u}} \times \frac{\partial \tilde{\mathbf{x}}}{\partial \tilde{v}} \right| \mathrm{d}\tilde{u}\,\mathrm{d}\tilde{v}$$

Change of variables $\tilde{u} = \tilde{u}(u,v)$ and $\tilde{v} = \tilde{v}(u,v)$

$$\begin{aligned} \int_S f \,\mathrm{d}S &= \iint\limits_{D} f(\mathbf{x}(u,v)) \left| \frac{\partial \tilde{\mathbf{x}}}{\partial \tilde{u}} \times \frac{\partial \tilde{\mathbf{x}}}{\partial \tilde{v}} \right| \left| \frac{\partial(\tilde{u},\tilde{v})}{\partial(u,v)} \right| \mathrm{d}u\,\mathrm{d}v \\ &= \iint\limits_{D} f(\mathbf{x}(u,v)) \left| \frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v} \right| \mathrm{d}u\,\mathrm{d}v \end{aligned}$$

So $\int_S d \,\mathrm{d}S$ indep of parametrisation of S

4 Divergence, Curl and Laplacians

4.1 Definitions

Seen gradient operator ∇, acts on functions $f : \mathbb{R}^3 \to \mathbb{R}$. In Cartesians,

$$\nabla = \mathbf{e}_i \frac{\partial}{\partial x_i}$$

Definition. For a vector field $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$, define **divergence** of $\mathbf{F}$ by

$$\mathrm{div}(\mathbf{F}) = \nabla \cdot \mathbf{F}$$

Equation. So in Cartesians,

$$\begin{aligned}\nabla \cdot \mathbf{F} &= \left(\mathbf{e}_i \frac{\partial}{\partial x_i}\right) \cdot (F_j \mathbf{e}_j) \\ &= \mathbf{e}_i \cdot \left[\frac{\partial}{\partial x_i} (F_j \mathbf{e}_j)\right] \\ &= \underbrace{(\mathbf{e}_i \cdot \mathbf{e}_j)}_{\delta_{ij}} \frac{\partial F_j}{\partial x_i} \\ &= \frac{\partial F_i}{\partial x_i}\end{aligned}$$

Note. Divergence of a vector field is a scalar field.

Definition. For a vector field $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$, define **curl** of $\mathbf{F}$ by

$$\mathrm{curl}(\mathbf{F}) = \nabla \times \mathbf{F}$$

Equation. So in Cartesians

$$\begin{aligned}\nabla \times \mathbf{F} &= \left(\mathbf{e}_j \frac{\partial}{\partial x_j}\right) \times (F_k \mathbf{e}_k) \\ &= \mathbf{e}_j \times \left[\frac{\partial}{\partial x_j} (F_k \mathbf{e}_k)\right] \\ &= \underbrace{(\mathbf{e}_j \times \mathbf{e}_k)}_{\varepsilon_{ijk}\mathbf{e}_i} \frac{\partial F_k}{\partial x_j} \\ &= \left(\varepsilon_{ijk} \frac{\partial F_k}{\partial x_j}\right) \mathbf{e}_i\end{aligned}$$

So in Cartesians,

$$[\nabla \times \mathbf{F}]_i = \varepsilon_{ijk} \frac{\partial}{\partial x_j} F_k$$

Note. Curl of vector field is another vector field. In terms of a "formal" determinant

$$\nabla \times \mathbf{F} = \det \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ F_1 & F_2 & F_3 \end{bmatrix}$$

Definition. For scalar field $f : \mathbb{R}^3 \to \mathbb{R}$, define **Laplacian** of f

$$\nabla^2 f = \nabla \cdot \nabla f \ (= \text{div}(\text{grad } f))$$

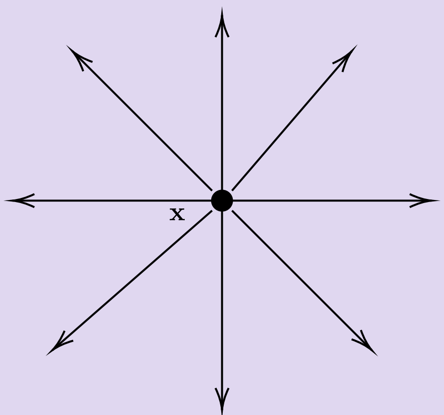
In Cartesians, $[\nabla f] = \frac{\partial f}{\partial x_i}$, so

$$\nabla^2 f = \frac{\partial^2 f}{\partial x_i \partial x_i}$$

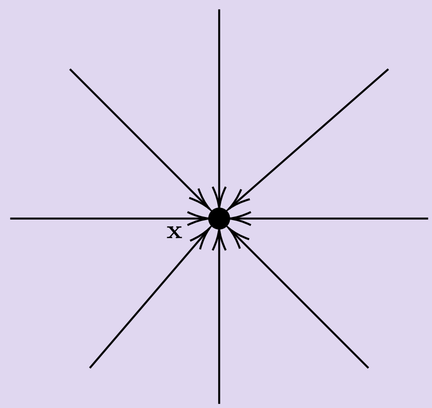
Example. Consider $\mathbf{F}(\mathbf{x}) = \mathbf{x}$. Then using Cartesians

$$\nabla \cdot \mathbf{F} = \frac{\partial}{\partial x_i} x_i = \delta_{ii} = 3$$

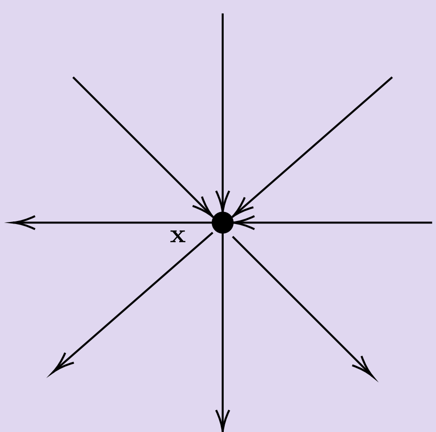
$$\begin{aligned}[\nabla \times \mathbf{F}]_i &= \varepsilon_{ijk} \frac{\partial}{\partial x_j} x_k \\ &= \varepsilon_{ijk} \delta_{kj} \\ &= \varepsilon_{ijj} \\ &= 0\end{aligned}$$



$\nabla \cdot \mathbf{F}(\mathbf{x}) > 0$

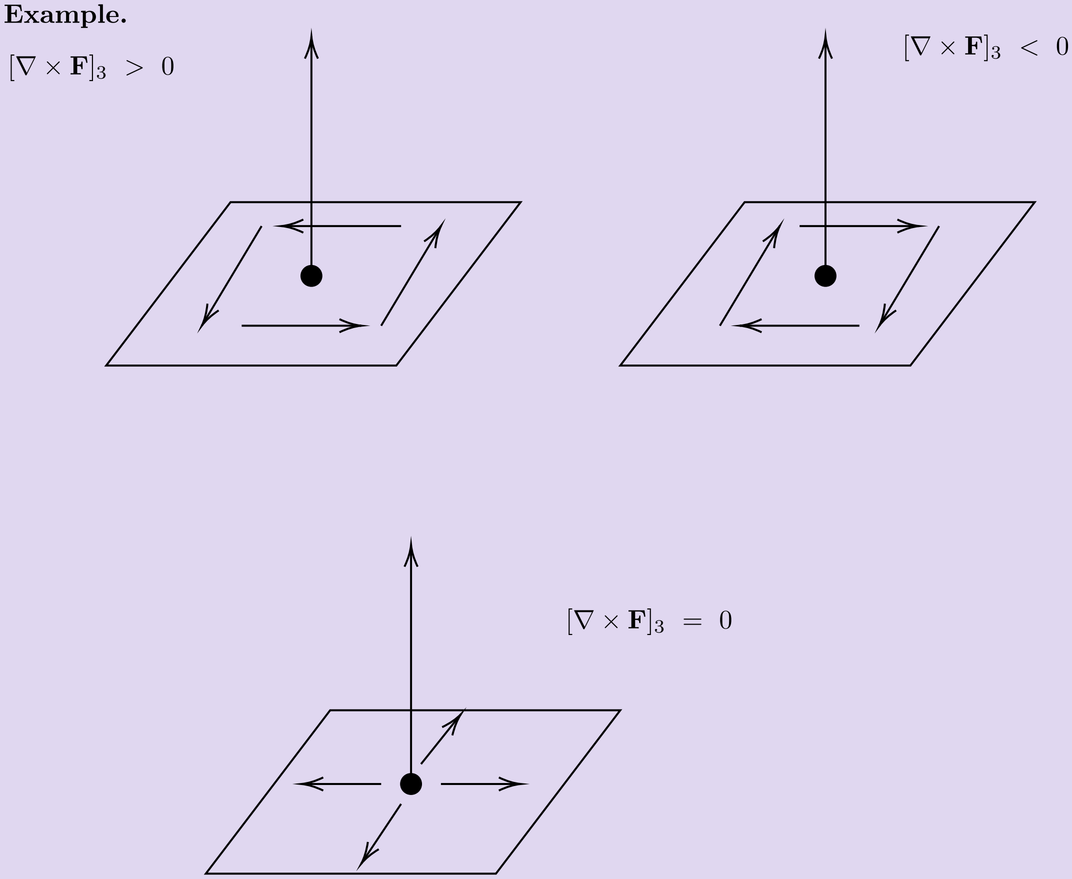


$\nabla \cdot \mathbf{F}(\mathbf{x}) < 0$



$\nabla \cdot \mathbf{F}(\mathbf{x}) = 0$

Example.



Prop. For f, g scalar fields, $\mathbf{F}, \mathbf{G}$ vector fields

$$\begin{aligned}
\nabla \cdot (fg) &= \nabla f)g + (\nabla g)f \\
\nabla \cdot (f\mathbf{F}) &= (\nabla f) \cdot \mathbf{F} + f(\nabla \cdot \mathbf{F}) \\
\nabla \times (f\mathbf{F}) &= (\nabla f) \times \mathbf{F} + f(\nabla \times \mathbf{F}) \\
\nabla(\mathbf{F} \cdot \mathbf{G}) &= \mathbf{F} \times (\nabla \times \mathbf{G}) + \mathbf{G} \times (\nabla \times \mathbf{F}) + (\mathbf{F} \cdot \nabla)\mathbf{G} + (\mathbf{G} \cdot \nabla)\mathbf{F} \\
\nabla \times (\mathbf{F} \times \mathbf{G}) &= \mathbf{F}(\nabla \cdot \mathbf{G}) - \mathbf{G}(\nabla \cdot \mathbf{F}) + (\mathbf{G} \cdot \nabla)\mathbf{F} - (\mathbf{F} \cdot \nabla)\mathbf{G} \\
\nabla \cdot (\mathbf{F} \times \mathbf{G}) &= (\nabla \times \mathbf{F}) \cdot \mathbf{G} - \mathbf{F} \cdot (\nabla \times \mathbf{G})
\end{aligned}$$

Proof.

Note.

$$\begin{aligned}
[(\mathbf{F} \cdot \nabla)\mathbf{G}]_i &= \left(F_j \frac{\partial}{\partial x_j}\right) G_i \\
&= F_j \frac{\partial G_i}{\partial x_j}
\end{aligned}$$

All similar so we only prove the 5^{th}, leave rest as exercise.
By definitions, LHS is

$$\begin{aligned}
[\nabla \times (\mathbf{F} \times \mathbf{G})]_i &= \varepsilon_{ijk} \frac{\partial}{\partial x_j} (\mathbf{F} \times \mathbf{G})_k \\
&= \varepsilon_{ijk} \frac{\partial}{\partial x_j} (\varepsilon_{klm} F_l G_m) \\
&= \underbrace{\varepsilon_{ijk}\varepsilon_{klm}}_{\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}} \left[F_l \frac{\partial G_m}{\partial x_j} + G_m \frac{\partial F_l}{\partial x_j}\right] \\
&= F_i \frac{\partial G_j}{\partial x_j} - F_j \frac{\partial G_i}{\partial x_j} + G_j \frac{\partial G_i}{\partial x_j} - G_i \frac{\partial F_j}{\partial x_j} \\
&= [\mathbf{F}(\nabla \cdot \mathbf{G})]_i - [(\mathbf{F} \cdot \nabla)\mathbf{G}]_i + [(\mathbf{G} \cdot \nabla)\mathbf{G}]_i - [(\nabla \cdot \mathbf{F})\mathbf{G}]_i \square
\end{aligned}$$

Remark. These identities hold in ANY OCC, but are most easily established using Cartesians

Equation. For general OCC, divergence defined by same formula $\nabla \cdot \mathbf{F}$, i.e.

$$\left(\mathbf{e}_u \frac{1}{h_u}\frac{\partial}{\partial u} + \mathbf{e}_v \frac{1}{h_v}\frac{\partial}{\partial v} + \mathbf{e}_w \frac{1}{h_w}\frac{\partial}{\partial w}\right) \cdot (F_u \mathbf{e}_u + \cdots + F_w \mathbf{e}_w)$$

Would get terms like

$$\begin{aligned}\left(\mathbf{e}_u \frac{1}{h_u}\frac{\partial}{\partial u}\right) \cdot (F_v \mathbf{e}_v) &= \frac{1}{h_u}\mathbf{e}_u \cdot \left[\frac{\partial}{\partial u}(F_v \mathbf{e}_v)\right] \\ &= \frac{1}{h_u}\mathbf{e}_u \cdot \left[\frac{\partial F_v}{\partial u}\mathbf{e}_v + F_v \frac{\partial \mathbf{e}_v}{\partial u}\right] \\ &= \frac{F_v}{h_u}\left(\mathbf{e}_u \cdot \frac{\partial \mathbf{e}_v}{\partial u}\right)\end{aligned}$$

Remark. Gets quite messy as $\{\mathbf{e}_u, \mathbf{e}_v, \mathbf{e}_w\}$ will depend on (u, v, w). Just state results:

$$\nabla \cdot \mathbf{F} = \frac{1}{h_u h_v h_w}\left[\frac{\partial}{\partial u}(h_v h_w F_u) + \frac{\partial}{\partial v}(h_u h_w F_v) + \frac{\partial}{\partial w}(h_u h_v F_w)\right]$$

$$\begin{aligned}\nabla \times \mathbf{F} &= \frac{1}{h_v h_w}\left[\frac{\partial}{\partial v}(h_w F_w) - \frac{\partial}{\partial w}(h_v F_v)\right]\mathbf{e}_u + \text{ cyc. perms} \\ &= \frac{1}{h_u h_v h_w}\det\begin{bmatrix} h_u \mathbf{e}_u & h_v \mathbf{e}_v & h_w \mathbf{e}_w \\ \frac{\partial}{\partial u} & \frac{\partial}{\partial v} & \frac{\partial}{\partial w} \\ h_u F_u & h_v F_v & h_w F_w \end{bmatrix}\end{aligned}$$

AND

$$\nabla^2 f = \frac{1}{h_u h_v h_w}\left[\frac{\partial}{\partial u}\left(\frac{h_v h_w}{h_u}\frac{\partial f}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{h_u h_w}{h_v}\frac{\partial f}{\partial v}\right) + \frac{\partial}{\partial w}\left(\frac{h_u h_v}{h_w}\frac{\partial f}{\partial w}\right)\right]$$

Since

$$[\nabla f]_u = \frac{1}{h_u}\frac{\partial f}{\partial u} \text{ etc.}$$

Example. In cylindrical polars (ρ, ϕ, z),

$$(h_\rho, h_\phi, h_z) = (1, \rho, 1)$$

So

$$\begin{aligned}\nabla^2 f &= \frac{1}{\rho}\left[\frac{\partial}{\partial \rho}\left(\frac{\partial f}{\partial \rho}\right) + \frac{\partial}{\partial \phi}\left(\frac{1}{\rho}\frac{\partial f}{\partial \phi}\right)\frac{\partial}{\partial z}\left(\rho \frac{\partial f}{\partial z}\right)\right] \\ &= \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial f}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2}\end{aligned}$$

Remark. For Laplacian of vector field, might guess

$$? \ \nabla \cdot (\nabla \mathbf{F}) \ ?$$

But haven't defined $\nabla \mathbf{F}$. In Cartesians, it would make sense

$$\begin{aligned} \nabla^2 \mathbf{F} &= \nabla^2 (F_1 \mathbf{e}_i) \\ &= (\nabla^2 F_i) \mathbf{e}_i \end{aligned}$$

Using suffix notation, can show

$$\nabla^2 \mathbf{F} = \nabla(\nabla \cdot \mathbf{F}) - \nabla \times (\nabla \times \mathbf{F}) \qquad (\dagger)$$

i.e.

$$[\nabla(\nabla \cdot \mathbf{F}) - \nabla \times (\nabla \times \mathbf{F})]_i = \frac{\partial^2 f_i}{\partial x_j \partial x_j} = \nabla^2 F_i$$

Since RHS of (†) is well-defined in any OCC, use it as a definition

Definition.

$$\nabla^2 \mathbf{F} = \nabla(\nabla \cdot \mathbf{F}) - \nabla \times (\nabla \times \mathbf{F})$$

Remark. If f harmonic, i.e.

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0 (\text{ in } \mathbb{R}^2)$$

(elliptic) f analytic
i.e.

$$f(x, y) = \sum_{n,m} a_{nm} x^n y^m$$

But if

$$\frac{\partial^2 f}{\partial x^2} - \frac{\partial^2 f}{\partial y^2} = 0$$

(hyperbolic) can't say as much about nature

4.2 Relations between div, grad and curl

Prop. For a scalar field f and a vector field $\mathbf{F}$

$$\nabla \times \nabla f = 0$$

$$\nabla \cdot (\nabla \times \mathbf{F}) = 0$$

i.e. curl $\cdot$ grad $= 0$, div $\cdot$ curl $= 0$

Proof.

$$\begin{aligned}[\nabla \times \nabla f]_i &= \varepsilon_{ijk} \frac{\partial}{\partial x_j} \left(\frac{\partial f}{\partial x_k} \right)] \\ &= \varepsilon_{ijk} \frac{\partial^2 f}{\partial x_j \partial x_k} \\ &= 0\end{aligned}$$

$\varepsilon_{ijk}$is anti-symmetric in j, k but $\frac{\partial^2 f}{\partial x_j \partial x_k}$ is symmetric in j, k resulting in product being zero

$$\begin{aligned}\nabla \cdot (\nabla \times \mathbf{F}) &= \frac{\partial}{\partial x_i} \varepsilon_{ijk} \frac{\partial}{\partial x_j} F_k \\ &= \varepsilon_{ijk} \frac{\partial^2 F_k}{\partial x_i \partial x_j} \\ &= 0\end{aligned}$$

similarly.

Note. Recall $\mathbf{F}$ was conservative if $\mathbf{F} = \nabla f$.

Definition. Say $\mathbf{F}$ is **irrotational** if

$$\nabla \times \mathbf{F} = 0$$

Remark. So from proposition

$$\mathbf{F} \text{ conservative} \implies \mathbf{F} \text{ irrotational}$$

Reverse implication is true if domain of $\mathbf{F}$ is simply connected (or "1-connected")
e.g. $\mathbb{R}^3$ is 1-connected byt $\mathbb{R}^3 \backslash \{z\text{-axis}\}$ is not 1-connected

Remark. Similarly, if there exists a vector potential for $\mathbf{F}$ i.e.

$$\mathbf{F} = \nabla \times \mathbf{A}$$

then

$$\nabla \cdot \mathbf{F} = 0$$

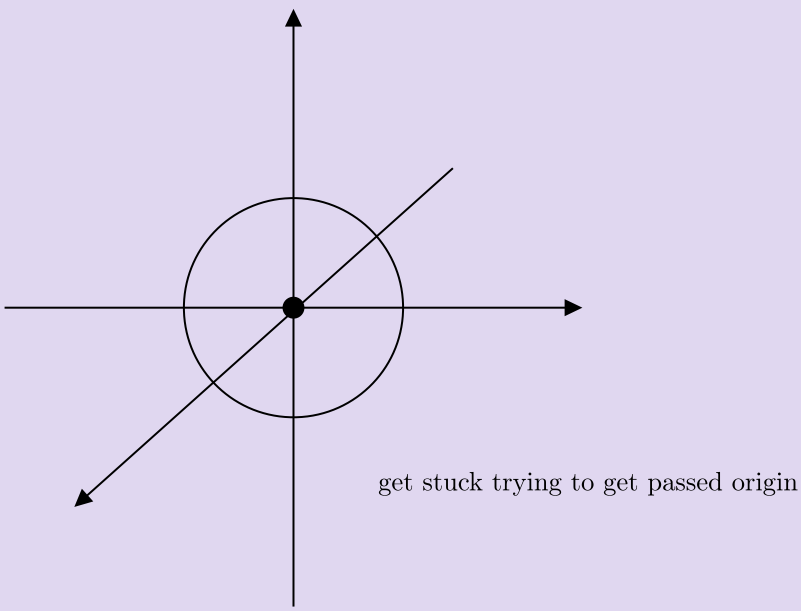
Here $\mathbf{A}$ is called the vector potential for $\mathbf{F}$

Definition. When $\nabla \cdot \mathbf{F} = 0$, say that $\mathbf{F}$ is **solenoidal**

Remark. So existence of vector potential for $\mathbf{F} \implies \mathbf{F}$ solenoidal
Reverse implication is true if domain of $\mathbf{F}$ is 2-connected.

Definition. Say $\Omega \subseteq \mathbb{R}^3$ is **2-connected** if it is 1-connected and every sphere in Ωcan be continuously shrunk to any point in Ω

Example. $\mathbb{R}^3$ is 2-connected. $\mathbb{R}^3 \backslash \{0\}$ is 1-connected, but not 2-connected



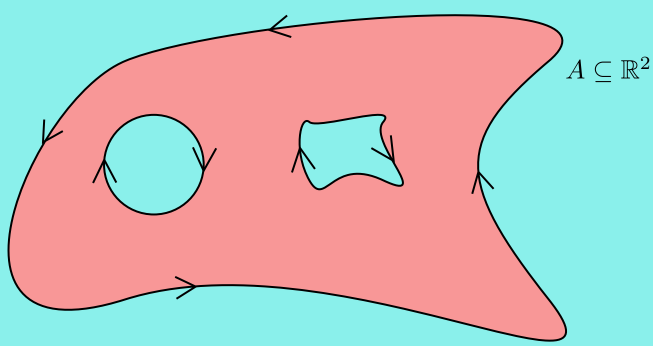
5 Integral Theorems

5.1 Greens Theorem: Statement and Examples

Theorem. If $P = P(x, y),\ Q = Q(x, y)$ are continuously differentiable functions on $A \cup \partial A$ and ∂A is piecewise smooth, then

$$\oint_{\partial A} P\,\mathrm{d}x + Q\,\mathrm{d}y = \int\!\!\int_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \mathrm{d}x\,\mathrm{d}y$$

Orientation of ∂A is suhc that A lies to your left as you traverse it.



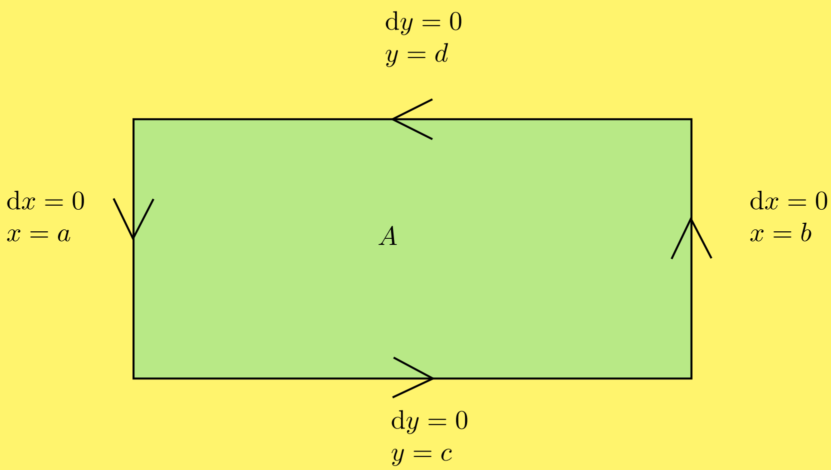
Proof. Proved later through other integral theorems

Note. It is easy to establish this result for

$$A = \{(x,y) : a \le x \le b, c \le y \le d\}$$

In this case, RHS is

$$\begin{aligned} &\int_c^d \left(\int_a^b \frac{\partial Q}{\partial x} \, \mathrm{d}x \right) \mathrm{d}y - \int_a^b \left(\int_c^d \frac{\partial P}{\partial y} \, \mathrm{d}y \right) \mathrm{d}x \\ &= \int_c^d [Q(b,y) - Q(a,y)] \, \mathrm{d}y + \int_a^b [P(x,c) - P(x,d)] \, \mathrm{d}x \\ &\equiv \oint_{\partial A} P \, \mathrm{d}x + Q \, \mathrm{d}y \end{aligned}$$



Example. Let $P = -\frac{1}{2}y, \ Q = \frac{1}{2}x$. Then:

$$\begin{aligned} \operatorname{area}(A) &= \iint_A \mathrm{d}x \, \mathrm{d}y \\ &= \iint_A \Bigg(\underbrace{\frac{1}{2}}_{= \frac{\partial Q}{\partial x}} + \underbrace{\frac{1}{2}}_{= -\frac{\partial P}{\partial y}} \Bigg) \, \mathrm{d}x \, \mathrm{d}y \\ &= \frac{1}{2} \oint_{\partial A} x \, \mathrm{d}y - y \, \mathrm{d}x \end{aligned}$$

If A is ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} \le 1$$

Then ∂A

$$[0, 2\pi] \ni t = \begin{bmatrix} a \cos t \\ b \sin t \end{bmatrix}$$

$$\begin{aligned} \operatorname{area}(A) &= \frac{1}{2} \int_0^{2\pi} (ab \cos^2 t + ab \sin^2 t) \, \mathrm{d}t \\ &= \pi ab \end{aligned}$$

5.2 Stoke's Theorem: Statement and Examples

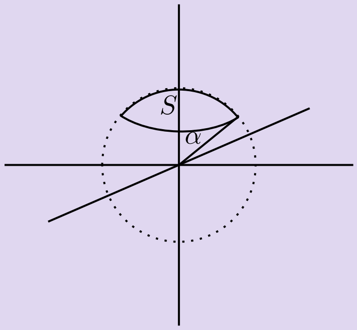
Theorem. If $\mathbf{F} = \mathbf{F}(\mathbf{x})$ is a continuously differentiable vector field and S is an orientable, piece-wise regular surface with piecewise smooth boundary ∂S then

$$\int_S (\nabla \times \mathbf{F}) \cdot \, \mathrm{d}\mathbf{S} = \oint_{\partial D} \mathbf{F} \cdot \, \mathrm{d}x$$

Note. Generalisation of FTC

Remark. The "orientable" bitmeans there's a consistent choice of normal vector at each point of S. I.e. S has "two sides".

Example. Let S be a cap of a a sphere



$$S = \{\mathbf{x}(\theta,\phi) = \begin{bmatrix}\sin\theta\cos\phi\\ \sin\theta\sin\phi\\ \cos\theta\end{bmatrix} = \mathbf{e}_r,\ 0 \le \theta \le \alpha,\ 0 \le \phi < 2\pi\}$$

$$\mathbf{F} = \begin{bmatrix}-x^2 y\\ 0\\ 0\end{bmatrix}$$

$$\implies \nabla \times \mathbf{F} = \begin{bmatrix}0\\ 0\\ x^2\end{bmatrix}$$

On S:

$$\begin{aligned}\mathrm{d}\mathbf{S} &= \frac{\partial \mathbf{x}}{\partial \theta} \times \frac{\partial \mathbf{x}}{\partial \phi}\,\mathrm{d}\theta\,\mathrm{d}\phi\\ &= \mathbf{e}_\theta(\sin\theta \mathbf{e}_\phi)\,\mathrm{d}\theta\,\mathrm{d}\phi\\ &= \mathbf{e}_r \sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi\end{aligned}$$

Note that since $(x^2\mathbf{e}_x \cdot \mathbf{e}_r) = (\sin\theta\cos\phi)^2\cos\theta$ on S:

$$\begin{aligned}\int_S \nabla \times \mathbf{F} \cdot \mathrm{d}\mathbf{D} = \int_{\phi=0}^{2\pi} \left(\int_{\theta=0}^{\alpha} \cos^2\phi \underbrace{\sin^3\theta\cos\theta}_{\frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}\theta}}\,\mathrm{d}\theta\right)\mathrm{d}\phi\\ = \frac{\pi 4}{\sin^4\alpha}\end{aligned}$$

∂S is described by

$$[0, 2\pi] \ni t \mapsto \begin{bmatrix}\sin\alpha\cos t\\ \sin\alpha\sin t\\ \cos\alpha\end{bmatrix}$$

$$\implies \mathrm{d}\mathbf{x} = \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}\,\mathrm{d}t = \sin\alpha \begin{bmatrix}-\sin t\\ \cos t\\ 0\end{bmatrix}\mathrm{d}t$$

And so

$$\begin{aligned}\oint_{\partial S} \mathbf{F} \cdot \mathrm{d}\mathbf{x} &= \sin^4\alpha \int_0^{2\pi} (-\cos^2 t \sin t)(-\sin t)\,\mathrm{d}t\\ &= \frac{\pi}{4}\sin^4\alpha\end{aligned}$$

Example. If S is an orientable, closed surface and $\mathbf{F}$ is continuously differentiable then

$$\int_S \nabla \times \mathbf{F} \cdot \, \mathrm{d}\mathbf{S} = 0$$

Prop. If $\mathbf{F}$ is continuously differentiable and for every loop X

$$\oint_C \mathbf{F} \cdot \, \mathrm{d}\mathbf{x} = 0$$

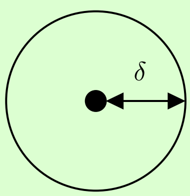
then $\nabla \times \mathbf{F} = 0$. So$\mathbf{F}$ irrotational $\impliedby$ $\mathbf{F}$ has zero circulation any closed loop.

Proof. Assume result is false i.e. $\exists$ unit vector is such that

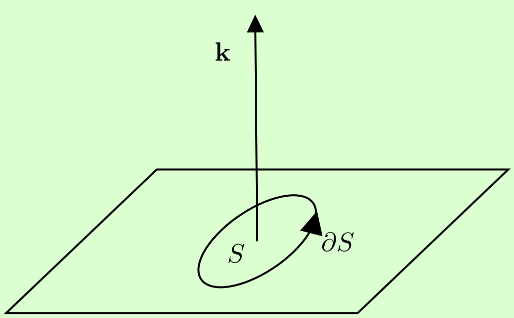
$$\mathbf{k} \cdot \underbrace{\nabla \times \mathbf{F}(\mathbf{x}_0)}_{\varepsilon} > 0$$

for some $\mathbf{x}$.
By continuity, for $\delta > 0$, sufficiently small so that, by continuity

$$\mathbf{k} \cdot \nabla \times \mathbf{F}(\mathbf{x}) > \frac{1}{2}\varepsilon \text{ for } |\mathbf{x} - \mathbf{x}_0| < \delta$$



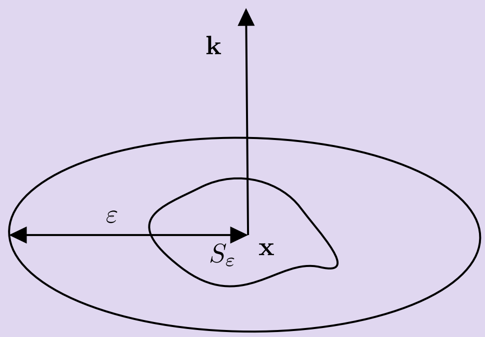
Take loop in this ball $\{\mathbf{x} : |\mathbf{x} - \mathbf{x}_0| < \delta\}$ that lies entirely in a plane with normal $\mathbf{k}$



Then:

$$\begin{aligned} 0 &= \oint_{\partial S} \mathbf{F} \cdot \, \mathrm{d}\mathbf{x} \\ &= \int_S \nabla \times \mathbf{F} \cdot \mathbf{k} \, \mathrm{d}S \\ &> \frac{1}{2}\varepsilon \int \, \mathrm{d}S \\ &> 0 ※ \end{aligned}$$

Example. Let S_ε denote a region contained inside a disc of radius $\varepsilon > 0$ centered at $\mathbf{x}$, with normal $\mathbf{k}$



$$\int_{S_\varepsilon} \nabla \times \mathbf{F} \cdot \mathrm{d}\mathbf{S} = \int_{S_\varepsilon} (\nabla \times \mathbf{F}(\mathbf{x}) - \nabla \times \mathbf{F}(\mathbf{x}_0)) \cdot \mathrm{d}\mathbf{S} + \underbrace{\int_{S_\varepsilon} \nabla \times \mathbf{F}(\mathbf{x}_0) \cdot \mathbf{k}\,\mathrm{d}S}_{\mathrm{area}(S_\varepsilon)\ \mathbf{k}\cdot\nabla\times\mathbf{F}(\mathbf{x}_0)}$$

$$= \mathrm{area}(S_\varepsilon)\ \mathbf{k} \cdot \nabla \times \mathbf{F}(\mathbf{x}_0) + \int_{S_\varepsilon} \nabla \times \mathbf{F} \cdot \mathrm{d}\mathbf{S} + \underbrace{\int_{S_\varepsilon} (\nabla \times \mathbf{F}(\mathbf{x}) - \nabla \times \mathbf{F}(\mathbf{x}_0)) \cdot \mathrm{d}\mathbf{S})}_{o(\mathrm{area}(S_\varepsilon))}$$

$$= \mathrm{area}(S_\varepsilon)\ \mathbf{k} \cdot \nabla \times \mathbf{F}(\mathbf{x}_0) + o(\mathrm{area}(S_\varepsilon))$$

$$\implies \mathbf{k} \cdot \nabla \times \mathbf{F}(\mathbf{x}_0) = \lim_{\varepsilon \to 0} \frac{1}{\mathrm{area}(S_\varepsilon)} \oint_{\partial S_\varepsilon} \mathbf{F} \cdot \mathrm{d}\mathbf{x}$$

So component of $\nabla \times \mathbf{F}(\mathbf{x}_0)$ in direction $\mathbf{k}$ is equal to infinitesimal circulation per unit area about k

5.3 Divergence Theorem: Statement and Examples (Gauss' Theorem)

Theorem. If $\mathbf{F} = \mathbf{F}(\mathbf{x})$ is continuously differentiable vector field and V is a volume with piecewise regular boundar ∂V then

$$\int_V \nabla \cdot \mathbf{F}\,\mathrm{d}V = \int_{\partial V} \mathbf{F} \cdot \mathrm{d}\mathbf{S}$$

where normal to ∂V points OUT of V

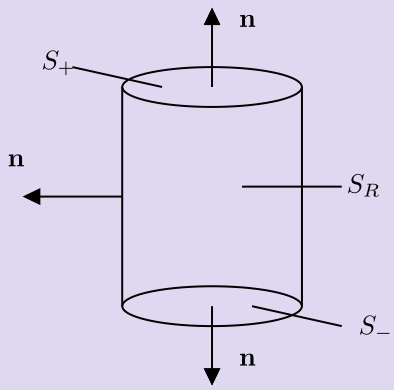
Prop. If $\mathbf{F} = \mathbf{F}(\mathbf{x})$ is continuously differentiable and $D \subseteq \mathbb{R}^2$ is a planar region with pievewise sooth boundary ∂D then

$$\int_D = \nabla \cdot \mathbf{F}\,\mathrm{d}A = \oint_{\partial D} \mathbf{F} \cdot \mathbf{n}\,\mathrm{d}s$$

(s arc-length)
again $\mathbf{n}$ points OUT of D.

Example. Let V be a cylinder. In cylindrical polars (ρ, ϕ, z):

$$V = \{(\rho, \phi, z) : \ 0 \leq \rho \leq R, \ -h \leq z \leq h, \ 0 \leq \phi \leq 2\pi\}$$



Consider $\mathbf{F} = \mathbf{x}$. So

$$\nabla \cdot \mathbf{F} = 3$$

$$\int_V \nabla \cdot \mathbf{F} \, \mathrm{d}V = 3 \int_v \mathrm{d}V = 6\pi R^2 h$$

Alternatively use Divergence Theorem. ∂V is made from

$$S_R = \{(\rho, \phi, z) : \ 0 \leq \rho \leq R, \ -h \leq z \leq h, \ 0 \leq \phi \leq 2\pi\}$$

$$S_\pm = \{(\rho, \phi, z) : \ 0 \leq \rho \leq R, \ z = \pm h, \ 0 \leq \phi \leq 2\pi\}$$

On S_R,

$$\mathrm{d}\mathbf{S} = \mathbf{e}_\rho R \, \mathrm{d}\phi \, \mathrm{d}z$$

and

$$\mathbf{x} \cdot \mathbf{e} + \rho = R$$

So

$$\int_{S_R} \mathbf{F} \cdot \, \mathrm{d}\mathbf{S} = \int_{z=-h}^{h} \left(\int_{\phi=0}^{2\pi} R^2 \, \mathrm{d}\phi \right) \, \mathrm{d}z = e\pi R^2 h$$

On $S_\pm$, find

$$\mathrm{d}\mathbf{S} = \pm \mathbf{e}_z \rho \, \mathrm{d}\rho \, \mathrm{d}\phi$$

and

$$\mathbf{x} \cdot \mathbf{e}_z = h$$

so

$$\int_{S_\pm} \mathbf{F} \cdot \, \mathrm{d}\mathbf{S} = \int_{\phi=0}^{2\pi} \left(\int_{\rho=0}^{R} h\rho \, \mathrm{d}\rho \right) \, \mathrm{d}\phi = \pi R^2 h$$

In summary

$$\begin{aligned} \int_{\partial V} \mathbf{F} \cdot \, \mathrm{d}\mathbf{S} &= \left(\int_{S_R} + \int_{S_+} + \int_{S_-} \right) \mathbf{F} \cdot \, \mathrm{d}\mathbf{S} \\ &= 4\pi R^2 h + \pi R^2 h + \pi R^2 h \\ &= 6\pi R^2 h \ \checkmark \end{aligned}$$

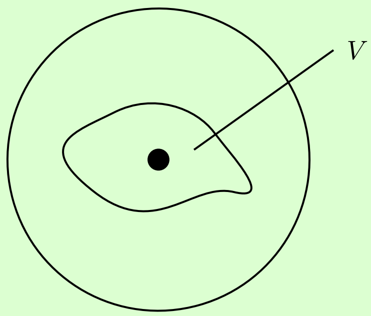
Prop. If $\mathbf{F} = \mathbf{F}(\mathbf{x})$ is continuously differentiable and for every closed surface S

$$\int_S \mathbf{F} \cdot \mathrm{d}\mathbf{S} = 0$$

then $\nabla \cdot \mathbf{F} = 0$

Proof. Suppose result is false. So $\nabla \cdot \mathbf{F} = \varepsilon > 0$. By continuity, for $\delta > 0$ sufficiently small

$$\nabla \cdot \mathbf{F}(\mathbf{x}) > \frac{1}{2}\varepsilon \; |\mathbf{x} - \mathbf{x}_0| < \delta$$

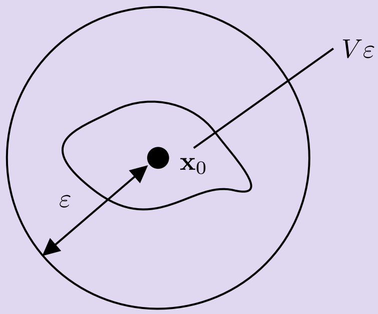


Choose a volume V inside ball $|\mathbf{x} - \mathbf{x}_0| < \delta$. Then by assumption

$$0 = \int_{\partial V} \mathbf{F} \cdot \mathrm{d}\mathbf{S} = \int_V \nabla \cdot \mathbf{F} \,\mathrm{d}V > \frac{1}{2}\varepsilon \int_V \mathrm{d}V > 0 \text{ ※}$$

Conclude that if vector field E has zero net flux through any closed surface then it is solenoidal $(\nabla \cdot \mathbf{F} = 0)$ $\square$

Example. Let V_ε be a volume in $\mathbb{R}^3$ contained inside a ball of radius $\varepsilon > 0$, centered at $\mathbf{x}_0$

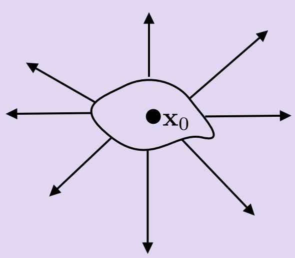


$$\int_{V_\varepsilon} \nabla \cdot \mathbf{F} \, \mathrm{d}V = \operatorname{vol}(V_\varepsilon) \nabla \cdot \mathbf{F}(\mathbf{x}_0) + \underbrace{\int_{V_\varepsilon} [\nabla \cdot \mathbf{F}(\mathbf{x}) - \nabla \cdot (\mathbf{F}(\mathbf{x}_0)] \, \mathrm{d}V}_{0(\operatorname{col}(V_\varepsilon))}$$

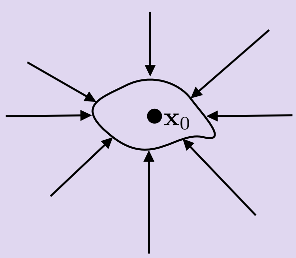
(can bound integral considering a max)
Dividing both sides by $\operatorname{vol}(V_\varepsilon)$, take $\varepsilon \to 0$, by Divergence Theorem

$$\nabla \cdot \mathbf{F}(\mathbf{x}_0) = \lim_{\varepsilon \to 0} \frac{1}{\operatorname{vol}(V_\varepsilon)} \int_{\partial V_\varepsilon} \mathbf{F} \cdot \, \mathrm{d}\mathbf{S}$$

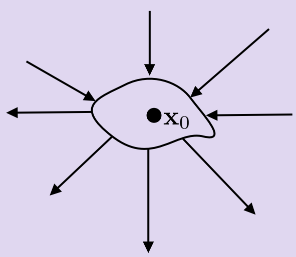
So $\nabla \cdot \mathbf{F}$ measures "infinitesimal flux per unit volume."



$\nabla \cdot \mathbf{F}(\mathbf{x}_0) > 0$



$\nabla \cdot \mathbf{F}(\mathbf{x}_0) < 0$



$\nabla \cdot \mathbf{F}(\mathbf{x}_0) = 0$

Example. Many equations in mathematical physics can be written in the form

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0 \tag{†}$$

Call these CONSERVATION LAWS.
Suppose both ρ and $|\mathbf{J}|$ decrease rapidly as $|\mathbf{x}| \to \infty$. ($\rho = (\rho(\mathbf{x}, t),\ \mathbf{J} = \mathbf{J}(\mathbf{x}, t)$. Define charge:

$$Q = \int_{\mathbb{R}^3} \rho(\mathbf{x}, t) \, \mathrm{d}V$$

We have conservation of charge:

$$\begin{aligned} \frac{\mathrm{d}Q}{\mathrm{d}t} &= \int_{\mathbb{R}^3} \frac{\partial \rho}{\partial t} \, \mathrm{d}V \\ &= -\int_{\mathbb{R}^3} \nabla \cdot \mathbf{J} \, \mathrm{d}V \\ &= -\lim_{R \to \infty} \int_{|\mathbf{x}| \le R} \nabla \cdot |\mathbf{J}| \, \mathrm{d}V \\ &= -\lim_{R \to \infty} \int_{|\mathbf{x}| = R} \mathbf{J} \cdot \, \mathrm{d}\mathbf{S} \\ &= 0 \end{aligned}$$

as $|J| \to 0$ rapidly as $|\mathbf{x}| \to \infty$
So (†) gives “conservation of charge”

5.4 Sketch Proofs

Prop. The divergence theorem is true

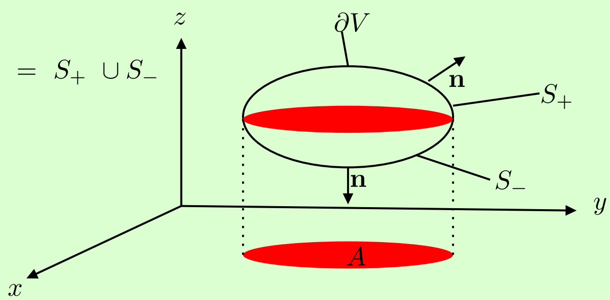
Proof. Suppose first that

$$\mathbf{F} = F_z(x,y,z)\mathbf{e}_z$$

Then divergence thm says

$$\int_V \frac{\partial F_z}{\partial z}\,\mathrm{d}V = \int_{\partial V} F_z\mathbf{e}_z \cdot \mathrm{d}\mathbf{S} \qquad (\dagger)$$

$\partial V = S_+ \cup S_-$



We write:

$$S_\pm = \{\mathbf{x}(x,y) = \begin{bmatrix} x \\ y \\ g_\pm(x,y) \end{bmatrix},\ (x,y) \in A\}$$

Then

$$\begin{aligned}\int_V \frac{\partial F_z}{\partial z}\,\mathrm{d}V &= \int\int_A \left[\int_{g_-(x,y)}^{g_+(x,y)} \frac{\partial F_z}{\partial z}\,\mathrm{d}z\right]\mathrm{d}x\,\mathrm{d}y \\ &= \int\int_A [F_z(x,y,g_+)x,y) - F_z(x,y,g_-(x,y))]\,\mathrm{d}x\,\mathrm{d}y\end{aligned}$$

To calculate RHS of ($\dagger$) over $S_\pm$

$$\mathrm{d}\mathbf{S} = \frac{\partial \mathbf{x}}{\partial x} \times \frac{\partial \mathbf{x}}{\partial y}\,\mathrm{d}x\,\mathrm{d}y = \begin{bmatrix} -\frac{\partial g_\pm}{\partial x} \\ -\frac{\partial g_\pm}{\partial y} \\ 1 \end{bmatrix}\mathrm{d}x\,\mathrm{d}y$$

Since we want $\mathbf{n}$ to point OUT of V, on $S_\pm$, we have

$$\mathrm{d}\mathbf{S}|_{S_\pm} = \pm\begin{bmatrix} -\frac{\partial g_\pm}{\partial x} \\ -\frac{\partial g_\pm}{\partial y} \\ 1 \end{bmatrix}\mathrm{d}x\,\mathrm{d}y$$

$$\begin{aligned}\implies \int_{\partial V} \mathbf{F}\cdot\mathrm{d}\mathbf{S} &= \left[\int_{S_+} + \int_{S_-}\right] F_z\mathbf{e}_z \cdot \mathrm{d}\mathbf{S} \\ &= \int\int_A F_z(x,y,g_+(x,y))\,\mathrm{d}x\,\mathrm{d}y - \int\int_A F_z(x,y,g_-(x,y))\,\mathrm{d}x\,\mathrm{d}y \\ &= \int_V \frac{\partial F_z}{\partial z}\,\mathrm{d}V\end{aligned}$$

Prop (cont.)**.**

Proof (cont.)**.** So (†) holds. In exactly the same way

$$\int_V \frac{\partial F_x}{\partial x}\,\mathrm{d}V = \int_{\partial V} F_x \mathbf{e}_x \cdot \mathrm{d}\mathbf{S}$$

$$\int_V \frac{\partial F_y}{\partial y}\,\mathrm{d}V = \int_{\partial V} F_y \mathbf{e}_y \cdot \mathrm{d}\mathbf{S}$$

Adding these three together

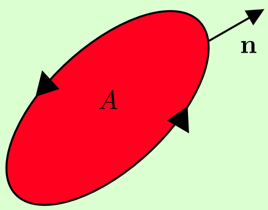
$$\int_V \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}\,\mathrm{d}V = \int_{\partial V} F_x \mathbf{e}_x + F_y \mathbf{e}_y + F_z \mathbf{e}_z \cdot \mathrm{d}\mathbf{S}$$

which is the divergence thm □

Prop. Div thm $\implies$ Green's thm

Proof. Use 2D div thm with $\mathbf{F} = \begin{bmatrix} Q \\ -P \end{bmatrix}$. Then

$$\int\int_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \mathrm{d}x\,\mathrm{d}y = \int_A \nabla \cdot \mathbf{F}\,\mathrm{d}A = \oint_{\partial A} \mathbf{F} \cdot \mathbf{x}\,\mathrm{d}s$$



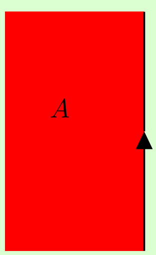
If ∂A is parametrised wrt arc length, so unit tangent vector is

$$\mathbf{t} = \begin{bmatrix} x'(s) \\ y'(s) \end{bmatrix}$$

Then the normal vector must be

$$\mathbf{n} = \begin{bmatrix} y'(s) \\ -x'(s) \end{bmatrix}$$

Check: if $\mathbf{t}$ points vertically upwards then A would be to our left:



$$\mathbf{t} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

And so

$$\begin{aligned} \mathbf{F} \cdot \mathbf{n}\,\mathrm{d}s &= \begin{bmatrix} Q \\ -P \end{bmatrix} \cdot \begin{bmatrix} y'(s) \\ -x'(s) \end{bmatrix} \mathrm{d}s \\ &= P\frac{\mathrm{d}x}{\mathrm{d}s}\,\mathrm{d}s + Q\frac{\mathrm{d}y}{\mathrm{d}s}\,\mathrm{d}s \\ &= P\,\mathrm{d}x + Q\,\mathrm{d}y \end{aligned}$$

i.e.

$$\int\int_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \mathrm{d}x\,\mathrm{d}y = \oint_{\partial A} \mathbf{F} \cdot \mathbf{x}\,\mathrm{d}s$$

Prop. Green's thm $\implies$ Stoke's thm

Proof. Consider regular surface

$$S = \{\mathbf{x} = \mathbf{x}(u, v) : (u, v) \in A\}$$

Then the boundary is

$$\partial S = \{\mathbf{x} = \mathbf{x}(u, v) : (u, v) \in \partial A\}$$

Green's thm gives

$$\oint_{\partial A} P \, \mathrm{d}u + Q \, \mathrm{d}v = \int\!\!\int_A \left(\frac{\partial Q}{\partial u} - \frac{\partial P}{\partial v} \right) \mathrm{d}u \, \mathrm{d}v$$

Make choices

$$P(x, y) = \mathbf{F}(\mathbf{x}(u, v)) \cdot \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}u}$$

$$Q(x, y) = \mathbf{F}(\mathbf{x}(u, v)) \cdot \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}v}$$

Then

$$\begin{aligned} P \, \mathrm{d}u + Q \, \mathrm{d}v &= \mathbf{F}(\mathbf{x}(u, v)) \cdot \left(\frac{\partial \mathbf{x}}{\partial u} \, \mathrm{d}u + \frac{\partial \mathbf{x}}{\partial v} \, \mathrm{d}v \right) \\ &= \mathbf{F}(\mathbf{x}(u, v)) \cdot \, \mathrm{d}\mathbf{x}(u, v) \end{aligned}$$

And so

$$\oint_{\partial A} P \, \mathrm{d}u + Q \, \mathrm{d}v = \oint_{\partial S} \mathbf{F} \cdot \, \mathrm{d}\mathbf{x}$$

Prop (cont.).

Proof (cont.). For the other side of Stokes'

$$\frac{\partial Q}{\partial u} = \frac{\partial x_j}{\partial u}\frac{\partial F_i}{\partial x_j}\frac{\partial x_i}{\partial v} + F_i\frac{\partial^2 x_i}{\partial v \partial u}$$

$$\frac{\partial P}{\partial v} = \frac{\partial x_j}{\partial v}\frac{\partial F_i}{\partial x_j}\frac{\partial x_i}{\partial u} + F_i\frac{\partial^2 x_i}{\partial u \partial v}$$

So:

$$\begin{aligned}
\frac{\partial Q}{\partial u} - \frac{\partial P}{\partial v} &= \left(\frac{\partial x_i}{\partial v}\frac{\partial x_j}{\partial u} - \frac{\partial x_i}{\partial u}\frac{\partial x_j}{\partial v}\right)\frac{\partial F_i}{\partial x_j} \\
&= (\delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp})\frac{\partial F_i}{\partial x_j}\frac{\partial x_p}{\partial v}\frac{\partial x_q}{\partial u} \\
&= \varepsilon_{ijk}\varepsilon_{pqk}\frac{\partial F_i}{\partial x_j}\frac{\partial x_p}{\partial u}\frac{\partial x_q}{\partial u} \\
&= [-\nabla \times \mathbf{F}]_k\left(-\frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v}\right)_k \\
&= (\nabla \times \mathbf{F}) \cdot \left(\frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v}\right)
\end{aligned}$$

So

$$\begin{aligned}
\int\int_A \left(\frac{\partial Q}{\partial u} - \frac{\partial P}{\partial v}\right) \mathrm{d}u\,\mathrm{d}v &= \int\int_A (\nabla \times \mathbf{F}) \cdot \left(\frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v}\right) \mathrm{d}u\,\mathrm{d}v \\
&= \int_S \nabla \times \mathbf{F} \cdot \mathrm{d}\mathbf{S}
\end{aligned}$$

This is Stokes' theorem. □

6 Maxwell's Equations

6.1 Brief Introduction to Electromagnetism

Notation. Denote by

$$\mathbf{B} = \mathbf{B}(\mathbf{x}, t)$$

the magnetic field and

$$\mathbf{E} = \mathbf{E}(\mathbf{x}, t)$$

electric field. These fields will depend on charge density

$$\rho = \rho(\mathbf{x}, t)$$

(electric charge per unit volume) and on current density

$$\mathbf{J} = \mathbf{J}(x, t)$$

(electric current per unit area)

Equation.

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} \tag{1}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{2}$$

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \tag{3}$$

$$\nabla \times \mathbf{B} - \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} = \mu_0 \mathbf{J} \tag{4}$$

The constants ε_0 and μ_0 are the permittivity and permeability of free space, which obey

$$\frac{1}{\mu_0 \varepsilon_0} = c^2$$

where $c = 299,792,458\,\mathrm{ms}^{-1}$ is the speed of light.

Method. Of we take div of (4), using $\nabla \cdot \nabla \times \mathbf{B} = 0$,

$$0 = \mu_0 \varepsilon_0 \frac{\partial}{\partial t} (\nabla \cdot \mathbf{E}) + \mu_0 \nabla \cdot \mathbf{J}$$

Use (1), $\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$, we get

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0$$

conservation law.
This gives rise to conservation of charge.
(Corresponds to "gauge symmetry")

6.2 Integral Formulations

Method. Integrating (1) over volume V and using divergence theorem,

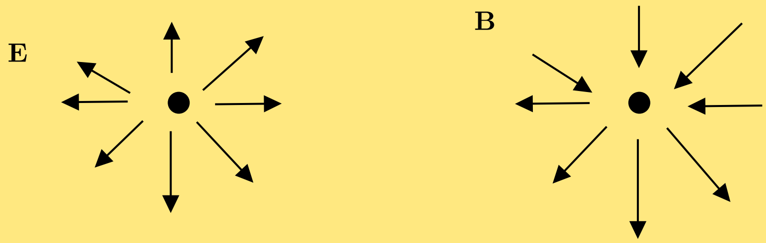
$$\int_{\partial V} \mathbf{E} \cdot \,\mathrm{d}\mathbf{S} = \frac{1}{\varepsilon_0} \int_V \rho \,\mathrm{d}V \equiv \frac{Q}{\varepsilon_0}$$

where Q is the "total charge in V"
This is called Gauss' Law.

Method. For magnetic fields, (2) gives

$$\int_{\partial V} \mathbf{B} \cdot \mathrm{d}\mathbf{S} = 0$$

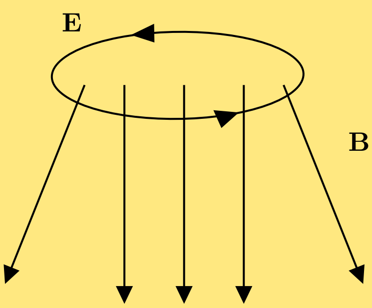
There is no net magnetic flux over any closed surface ∂V.



i.e. there are no magnetic monopoles

Method. Integrating (3) over surface S and use Stoke's theorem

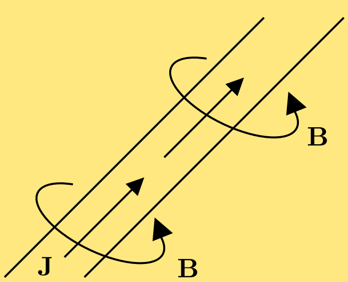
$$\oint_{\partial S} \mathbf{E} \cdot \mathrm{d}\mathbf{x} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot \mathrm{d}\mathbf{S} = -\frac{\mathrm{d}}{\mathrm{d}t} \int_S \mathbf{B} \cdot \mathrm{d}\mathbf{S}$$



The CHANGE in magnetic flux through S induces circulation in $\mathbf{E}$ about ∂S

Method. Integrate (4) over S and use Stokes

$$\oint_{\partial S} \mathbf{B} \cdot \mathrm{d}\mathbf{x} = \mu_0 \int_S \mathbf{J} \cdot \mathrm{d}\mathbf{S} + \mu_0 \varepsilon_0 \frac{\mathrm{d}}{\mathrm{d}t} \int_S \mathbf{E} \cdot \mathrm{d}\mathbf{S}$$



6.3 Electromagnetic Waves

Equation. In Empty space, $\rho = 0, \mathbf{J} = 0$, so (1) - (4) become

$$\nabla \cdot \mathbf{E} = 0 \tag{1}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{2}$$

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \tag{3}$$

$$\nabla \times \mathbf{B} - \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} = \mu_0 \mathbf{J} \tag{4}$$

Equation. Recall Laplacian of vector field $\mathbf{F}$

$$\nabla^2 \mathbf{F} = \nabla(\nabla \cdot \mathbf{F}) - \nabla \times (\nabla \times \mathbf{F})$$

Using (1),(3),(4)

$$\begin{aligned} \nabla^2 \mathbf{E} &= \nabla(0) - \nabla \times \left(-\frac{\partial \mathbf{B}}{\partial t}\right) \\ &= \frac{\partial}{\partial t} \\ &= \frac{\partial}{\partial t}\left(\mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}\right) \end{aligned}$$

Using

$$\mu_0 \varepsilon_0 = \frac{1}{c^2}$$

we get

$$\nabla^2 \mathbf{E} - \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0$$

(this is the wave equation in 3-D) So in vacuum, electric field travel at speed c.

Equation. Similarly, using (2), (3), (4)

$$\begin{aligned} \nabla^2 \mathbf{B} &= \nabla(0) - \nabla \times (\mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}) \\ &= -\mu_0 \varepsilon_0 \frac{\partial}{\partial t} \\ &= +\mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2} \end{aligned}$$

i.e.

$$\nabla^2 \mathbf{B} = -\frac{1}{c^2} \frac{\partial^2 \mathbf{B}}{\partial t^2} = 0$$

So electromagnetic waves always travel at speed c in a vacuum

6.4 Electrostatics + Magnetostatics

Equation. Suppose all fields and source terms are independent of t. Then Maxwell's equations decouple

$$(A)\begin{cases}\nabla\cdot\mathbf{E}=\frac{\rho}{\varepsilon_0}\\ \nabla\times\mathbf{E}=0\end{cases}$$

$$(B)\begin{cases}\nabla\cdot\mathbf{B}=\frac{\rho}{\varepsilon_0}\\ \nabla\times\mathbf{B}=\mu_0\mathbf{J}\end{cases}$$

If we are working on all of $\mathbb{R}^3$ (which is 2-connected), then $\nabla\times\mathbf{E}=0$ and $\nabla\cdot\mathbf{B}=0$ implies

$$\mathbf{E}=-\nabla\phi,\ \mathbf{B}=\nabla\times\mathbf{A}$$

Call ϕ the electric potential and $\mathbf{A}$ the magnetic potential.
Maxwell's equations (A) and (B) become

$$-\nabla^2\phi=\frac{\rho}{\varepsilon_0}$$

$$\nabla\times(\nabla\times\mathbf{A})=\mu_0\mathbf{J}$$

The first is called Poisson's equation, see section 7

6.5 Gauge Invariance (non-examinable)

Equation. The second of Maxwell's equations is

$$\nabla \cdot \mathbf{B} = 0$$

Assuming we are working on all of $\mathbf{R}^3$, can always write

$$\mathbf{B} = \nabla \times \mathbf{A}$$

$\mathbf{A}$ is not defined uniquely, can always change $\mathbf{A} \mapsto \mathbf{A} + \nabla\chi$ and $\mathbf{B}$ is unchanged since $\nabla \times \nabla\chi = 0$. Called gauge invariance, it gives rise to conservation of charge via Noether.
Using $\mathbf{B} = \nabla \times \mathbf{A}$ in (3)

$$\nabla \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t}\right) = 0$$

so we can write this term in brackers in terms of a scalar potential. So

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}$$

So Maxwell's equations reduce to

$$(1) \implies -\nabla^2\phi - \frac{\partial}{\partial t} = \frac{\rho}{\varepsilon_0}$$

$$(4) \implies \nabla \times (\nabla \times \mathbf{A}) + \mu_0\varepsilon_0\nabla\left(\frac{\partial\phi}{\partial t}\right) + \mu_0\varepsilon_0\frac{\partial^2\mathbf{A}}{\partial t^2} = \mu_0\mathbf{J}$$

Recall

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{a}) - \nabla^2\mathbf{A}$$

and

$$\mu_0\varepsilon_0 = \frac{1}{c^2}$$

So 2$^{\text{nd}}$ equation becomes

$$-\left(\nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2}\right) + \nabla\left(\nabla \cdot \mathbf{A} + \frac{1}{c^2}\frac{\partial\phi}{\partial t}\right) = \mu_0\mathbf{J}$$

Now exploit gauge freedom: change

$$\mathbf{A} \mapsto \mathbf{A} + \nabla\chi$$

in such a way that

$$\nabla \cdot \mathbf{A} + \frac{1}{c^2}\frac{\partial\phi}{\partial t} \to 0$$

So Maxwell's equations become

$$(1) \to -\nabla^2\phi + \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = \frac{\rho}{\varepsilon_0}$$

$$(4) \to -\nabla^2\mathbf{A} + \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} = \mu_0\mathbf{J}$$

Solve these to get

$$\mathbf{B} = \nabla \times \mathbf{A}$$

$$\mathbf{E} = -\nabla\phi - \frac{\partial\mathbf{A}}{\partial t}$$

7 Poisson's and Laplace Equations

7.1 The Boundary Value Problem

Remark. Many problems in mathematical physics can be reduced to the form

$$\nabla^2 \varphi = F$$

Called Poisson's Equation, or if $F \equiv 0$, call it Laplace's equation. We solve this equation on $\Omega = \mathbb{R}^n$ or $\Omega \subset \mathbb{R}^n,\ n = 2, 3$.
Physical problems involve boundary conditions,
i.e. φ will have prescribed behaviour on $\partial\Omega$ (or as $|x| \to \infty$ if $\Omega = \mathbb{R}^n$).

Example. The Dirichlet Problem is

$$\begin{cases} \nabla^2 \varphi = F \text{ in } \Omega \\ \varphi = f \text{ on } \partial\Omega \end{cases}$$

Example. The Neumann problem is

$$\begin{cases} \partial^2 \varphi = F \text{ in } \Omega \\ \frac{\partial \varphi}{\partial \mathbf{n}} = g \text{ on } \partial\Omega \end{cases}$$

where we have the normal derivative

$$\frac{\partial \varphi}{\partial \mathbf{n}} = \mathbf{n} \cdot \nabla \varphi$$

Must interpret boundary conditions in an appropriate manner: we assume that φ (or $\frac{\partial \varpi}{\partial \mathbf{n}}$ approaches the boundary data f (or g) continuously as $\mathbf{x} \to \partial\Omega$. That is to say, we assume φ and $\nabla\varphi$ are continuous on $\Omega \cup \partial\Omega$.

Warning. If $\nabla^2 \varphi = 0$ in Ω then φ needs to be well-defined on all of Ω. Don't fall into trap of assuming things like

$$\nabla^2 \left(\frac{1}{|\mathbf{x}|} \right) = 0$$

for all $\mathbf{x} \in \mathbb{R}^3$. It is only true for $\mathbf{x} \neq 0$

Example. As usual, let $r = |\mathbf{x}|$. Consider boundary value problem

$$\begin{cases} \nabla^2\varphi = r \text{ in } r < a \\ \varphi = 1 \text{ on } r = a \end{cases} \tag{†}$$

Guess solution of form $\varphi = \varphi(r)$. Using

$$\nabla^2\varphi = \frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}\varphi}{\mathrm{d}r}\right)$$

and subbing into (†)

$$\begin{cases} (r^2\varphi')' = r^3 \text{ in } r < a \\ \varphi(a) = 1 \end{cases}$$

General solution to (†)(a)

$$\varphi(r) = A + \underbrace{\frac{B}{r}}_{=0} + \frac{1}{12}r^3$$

MUST have $B \equiv 0$ or else φ not well-defined throughout $\Omega = \{r < a\}$. Using (†)(b)

$$1 = \varphi(a) = A + \frac{a^3}{12}$$

$$\implies A = 1 - \frac{a^3}{12}$$

So our solution is

$$\varphi(r) = 1 + \frac{1}{12}(r^3 - a^3)$$

Remark. Want solutions to be unique (or very almost unique)

Method. Consider generic linear problem

$$\begin{cases} L\varphi = F \text{ in } \Omega \\ B\varphi = f \text{ on } \partial\Omega \end{cases} \tag{††}$$

where L, B linear differential operators.
If φ_1 and φ_2 both solve (††), consider $\psi = \phi_1 - \phi_2$. By linearity

$$\begin{cases} L\psi = 0 \text{ in } \Omega \\ B\psi = 0 \text{ on } \partial\Omega \end{cases} \tag{†††}$$

If we can show that the ONLY solution to (†††) is $\psi = 0$, must conclude that $\varphi_1 = \varphi_2$, i.e. solution to (††) is unique.

Moral. The solution to a linear problem is unique iff the only solution to the homogenous problem is the zero solution

Prop. The solution of the Dirichlet problem is unique.
The solution to the Neumann problem is unique up ot the addition of a constant.

Proof. Let $\psi = \varphi_1 - \varphi_2$ be the difference of two solutions to Dirichlet or Neumann problem. so

$$\nabla^2 \psi = 0 \text{ in } \Omega$$

$$B\psi = 0 \text{ on } \partial\Omega$$

where $B\psi \equiv \psi$ (Dirichlet) or $B\psi \frac{\partial \psi}{\partial \mathbf{n}}$ (Neumann)
Consider the non-negative functional:

$$I[\psi] = \int_\Omega |\nabla \psi|^2 \, \mathrm{d}V \geq 0$$

Clearly $I[\psi] = 0$ if and only if $\nabla \psi = 0$ in Ω.
Note:

$$\begin{aligned}
I[\psi] &= \int_\Omega \nabla \psi \cdot \nabla \psi \, \mathrm{d}V \\
&= \int_\Omega \left(\nabla \cdot (\psi \nabla \psi) - \underbrace{\psi \nabla^2 \psi}_{=0} \right) \mathrm{d}V \text{ as } \nabla^2 \psi = 0 \text{ in } \Omega \\
&= \int_{\partial\Omega} (\psi \nabla \psi) \cdot \mathrm{d}\mathbf{S} \text{ (Div thm)} \\
&= \int_{\partial\Omega} \psi \frac{\partial \psi}{\partial \mathbf{n}} \, \mathrm{d}S \\
&= 0
\end{aligned}$$

using

$$\mathrm{d}\mathbf{S} = \mathbf{n} \, \mathrm{d}S, \; \mathbf{n} \cdot \nabla \psi = \frac{\partial \psi}{\partial \mathbf{n}}$$

Since $\psi = 0$ on $\partial\Omega$ (Dirichlet) or $\frac{\partial \psi}{\partial \mathbf{n}} = 0$ on $\partial\Omega$ (Neumann). Conclude that $\nabla \psi = 0$ throughout $\Omega \implies \psi = \text{const.}$ throughout Ω.

(i) For Dirichlet, $\psi = 0$ on $\partial\Omega$, so by continuity of ψ on $\Omega \cup \partial\Omega$, must have $\psi = 0$ everywhere. So solution to Dirichlet problem is unique.

(ii) From Neumann, only know $\frac{\mathrm{d}\psi}{\mathrm{d}\mathbf{n}} = 0$ on boundary so can't say any more, so since $\psi =$ const. deduce that

$$\varphi_1 = \varphi_2 + \text{ const.}$$

Any two solutions differ only by a constant. □

Example. From electrostatics, consider charge density

$$\rho(\mathbf{x}) = \begin{cases} 0 & r < a \\ F(r) & r \geq a \end{cases}$$

Claim. No electric field in $r < a$.

Proof. Indeed know that electric potential ϕ satisfies

$$\nabla^2 \phi = -\frac{\rho(\mathbf{x})}{\varepsilon_0} = 0 \; r < a$$

By spherical symmetry, $\phi = \phi(r)$. So

$$\phi = \phi(a) = \text{ const. on } r = a$$

Note that unique solution to

$$\begin{cases} \nabla^2 \phi = 0 & r < a \\ \phi = \text{ const.} & r = a \end{cases}$$

is $\phi =$ const throughout $r \leq a$ by proposition
$\implies$ $\mathbf{E} = -\nabla \phi = 0$ throughout $r < a$.
"Newton's Shell thm"

7.2 Gauss' Flux Method

Method. Suppose source term F is spherically symmetric, ie. $F = F(r)$, where $r = |\mathbf{x}|$. Write our problem as:

$$\nabla \cdot \nabla\varphi = F(r) \tag{*}$$

and assume $\Omega = \mathbb{R}^3$. Since RHS only depends on r, same is true of LHS. So assume that $\varphi = \varphi(r)$, in which case

$$\nabla\varphi = \varphi'(r)\mathbf{e}_r$$

Integrating (*) over region $|\mathbf{x}| < R$, and use divergence theorem

$$\int_{|\mathbf{x}|<R} \nabla \cdot \nabla\varphi \,\mathrm{d}V = \int_{|\mathbf{x}|<R} \nabla\varphi \cdot \,\mathrm{d}\mathbf{S} = \int_{|\mathbf{x}|<R} F(r)\,\mathrm{d}V$$

The RHS represents the amount of, e.g. mass, inside ball of radius $R > 0$. Set

$$\int_{|\mathbf{x}|<R} F \,\mathrm{d}V = Q(R)$$

where $Q(R)$ is "the amount of stuff inside ball $|\mathbf{x}| < R$"
So our equation is

$$\int_{|\mathbf{x}|<R} \nabla\varphi \cdot \,\mathrm{d}\mathbf{S} = Q(R)$$

Recall that on sphere of radius R

$$\mathrm{d}\mathbf{S} = \mathbf{e}_r R^2 \sin\theta \,\mathrm{d}\theta \,\mathrm{d}\phi$$

So on $|\mathbf{x}| = R$:

$$\nabla\varphi \cdot \,\mathrm{d}\mathbf{S} = \varphi'(r)\mathbf{e}_r \cdot (\,\mathbf{e}_r \underbrace{R^2 \sin\theta \,\mathrm{d}\theta\mathrm{d}\phi}_{\mathrm{d}S})\Big|_{|\mathbf{x}|=R} = \varphi'(R)\,\mathrm{d}S$$

So

$$Q(R) = \int_{|\mathbf{x}|<R} \varphi'(R)\,\mathrm{d}S = \varphi'(R) \underbrace{\int_{|\mathbf{x}|<R} \mathrm{d}S}_{4\pi R^2}$$

In summary

$$\varphi'(R) = \frac{Q(R)}{4\pi R^2} \;\forall R > 0$$

$$\implies \nabla\varphi = \frac{Q(R)}{4\pi r^2}\mathbf{e}_r$$

Example (Electrostatics)**.** Recall Maxwell's first equation

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}$$

If we use electric potential ϕ so

$$\mathbf{E} = -\nabla\phi$$

get

$$-\nabla^2\phi = \frac{\rho}{\varepsilon_0}$$

Consider charge density

$$\rho(r) = \begin{cases} \rho_0, & 0 \le r \le a \\ 0, & r > a \end{cases}$$

By previous result

$$\phi'(r) = -\frac{1}{4\pi\varepsilon_0}\frac{Q(r)}{r^2}$$

$$Q(r) = \int_{|\mathbf{x}|<r} \rho(R)\,\mathrm{d}V$$

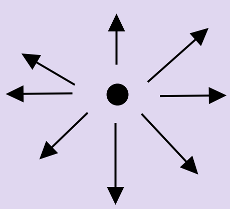
Note if $r > a$ then

$$Q(r) = Q(a) = Q$$

(the total charge)
So we find, using $\mathbf{E} = -\nabla\phi$:

$$\mathbf{E}(\mathbf{x}) = \begin{cases} \frac{1}{4\pi\varepsilon_0}\frac{Q(r)}{r^2}\mathbf{e}_r & r \le a \\ \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\mathbf{e}_r & r > a \end{cases}$$

$Q =$ total charge



Take $a \to 0$, keeping the total charge Q fixed (i.e. point charge)

$$\begin{aligned} \mathbf{E}(\mathbf{x}) &= \frac{Q}{4\pi\varepsilon_0}\frac{\mathbf{e}_r}{r^2} \\ &= \frac{Q}{4\pi\varepsilon_0}\frac{\mathbf{x}}{|\mathbf{x}|^3} \end{aligned}$$

The corresponding charge density $\rho(\mathbf{x}) = Q\delta(\mathbf{x})$

$$\int_{|\mathbf{x}|<R} \rho\,\mathrm{d}V = Q\ \forall R > 0$$

Method. What if our problem is symmetric about the z-axis i.e.

$$\nabla^2\varphi = F(\rho) \; \rho^2 = x^2 + y^2$$

Have "cylindrical symmetry". Integrate

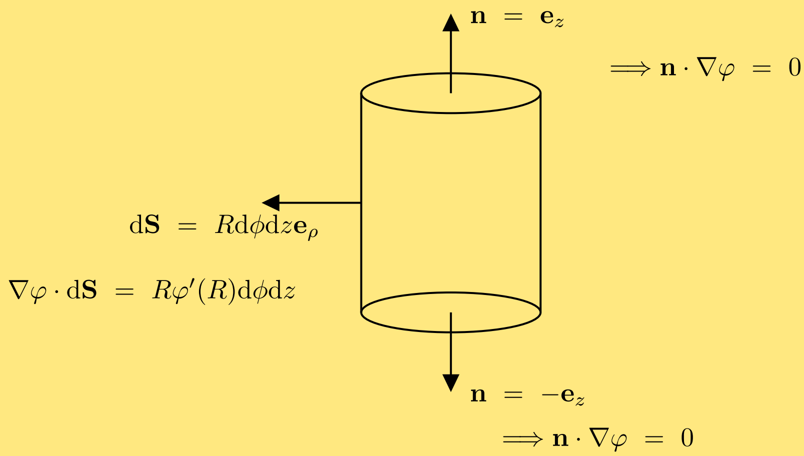
$$\nabla \cdot \nabla\varphi = F(\rho)$$

over cylinder of radius R, height a.
Assuming $\varphi = \varphi(\rho)$, have

$$\nabla\varphi = \varphi'(\rho)\,\mathbf{e}_\rho \text{ (cylindrical polars)}$$

$$\int_V \nabla \cdot \nabla\varphi \,\mathrm{d}V = \int_F (\rho)\,\mathrm{d}V$$

where V is cylinder



$$\begin{aligned} LHS &= \int_{\partial V} \nabla\varphi \cdot \,\mathrm{d}\mathbf{S} \\ &= \int_{\phi=0}^{2\pi}\int_{z=z_0}^{z_0+a} \varphi'(R)R\,\mathrm{d}\phi\,\mathrm{d}z \\ &= 2\pi a R\varphi'(R) \end{aligned}$$

so

$$\varphi'(R) = \frac{1}{R} \cdot \frac{1}{2\pi a} \underbrace{\int_V F(\rho)\,\mathrm{d}V}_{(\dagger)}$$

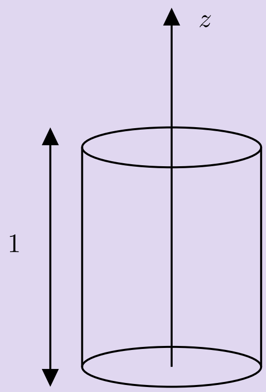
$$(\dagger) = \int_{z=z_0}^{z_0+a} \left(\int_{\phi=0}^{2\pi} \left(\int_{\rho=0}^{R} F(\rho)\rho\,\mathrm{d}\rho \right) \mathrm{d}\rho \right) \mathrm{d}\rho$$

$$= 2\pi a \int_0^R F(\rho)\rho\,\mathrm{d}\rho$$

In conclusion

$$\varphi'(\rho) = \frac{1}{\rho}\int_0^\rho sF(s)\,\mathrm{d}s$$

Example. How might we describe a line of charge density with constant charge density λ per unit length? Could proceed as before, consider cylinder of radius a, constant charge density. Take $a \to 0$ keep charge per unit length fixed.
Alternatively, let $F(\rho)$ be the desired charge density. So if we integrate over any cylinder of length 1



Should have total charge contained to be λ

$$\begin{aligned}\lambda &= \int_V F(\rho)\,\mathrm{d}V \\ &= \int_{z=z_0}^{z_0+1}\left(\int_{\phi=0}^{2\pi}\left(\int_{\rho=0}^{R} F(\rho\rho\,\mathrm{d}\rho\right)\,\mathrm{d}\phi\right)\,\mathrm{d}z \\ &= 2\pi\int_0^R \rho F(\rho)\,\mathrm{d}\rho\end{aligned}$$

So we see that choosing

$$F(\rho) = \frac{\lambda\delta(\rho)}{2\pi\rho}$$

corresponding electric potential would satisfy

$$\phi'(\rho) = -\frac{1}{\varepsilon_0}\frac{1}{\rho}\int_0^\rho \frac{\lambda}{2\pi}\delta(s)\,\mathrm{d}s = -\frac{\lambda}{2\pi\varepsilon_0}\frac{1}{\rho}$$

$$\implies \mathbf{E}(\mathbf{x}) = \frac{1}{2\pi\varepsilon_0}\frac{\mathbf{e}_\rho}{\rho}$$

7.3 Superposition Principle

Remark. Linear problems are relatively easy because of the following:

$$L\psi_n = F_n \; n = 1, 2, 3, \ldots$$

then

$$L\left(\sum_n \psi_n\right) = \sum_n F)n$$

We can superimpose solutions. Can often break up forcing term $F = \sum_n F_n$, solve each problem

$$L\psi_n = F_n$$

To get solution to $L\psi = F$, write $\psi = \sum_n \psi_n$

Example. Consider electric potential due to pair of point charges $Q_{\mathbf{a}}$ at $x = \mathbf{a}$, $Q_{\mathbf{b}}$ at $x = \mathbf{b}$. Charge density would be

$$\rho(\mathbf{x}) = Q_{\mathbf{a}}\delta(\mathbf{x} - \mathbf{a}) + Q_{\mathbf{b}}\delta(\mathbf{x} - \mathbf{b})$$

For one point charge, electric potential obeys

$$-\nabla^2\phi = \frac{Q_{\mathbf{a}}}{\varepsilon_0}\delta(\mathbf{x} - \mathbf{a})$$

Solution would be

$$\phi(\mathbf{x}) = \frac{Q_{\mathbf{a}}}{4\pi\varepsilon_0}\frac{1}{|\mathbf{x} - \mathbf{a}|}$$

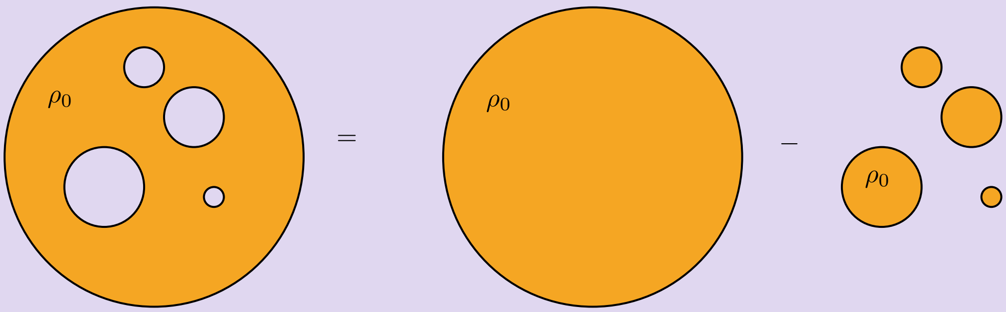
So by superposition principle, electric potential due to point charges at $\mathbf{x} = \mathbf{a}$ and $\mathbf{x} = \mathbf{b}$ is

$$\phi(\mathbf{x}) = \frac{Q_{\mathbf{a}}}{4\pi\varepsilon_0}\frac{1}{|\mathbf{x} - \mathbf{a}|} + \frac{Q_{\mathbf{a}}}{4\pi\varepsilon_0}\frac{1}{|\mathbf{x} - \mathbf{b}|}$$

Example. Consider electric potential outside ball of radius $|\mathbf{x}| < R$ of uniform charge density ρ_0, that has several balls removed from its interior

$$|\mathbf{x} - \mathbf{a}_i| < R_i \;\; i = 1, \dots, N$$

$$|\mathbf{a}_i| + R_i < R, \;\; |\mathbf{a}_i - \mathbf{a}_j| > R_i + R_j \text{ for each } i, j$$



Use superposition principle: represent each hole to be a ball of uniform charge density $-\rho_0$. Effective potential in $|\mathbf{x}| > R$ from each hole is

$$\phi(\mathbf{x}) = -\frac{1}{4\pi\varepsilon_0} \frac{Q_i}{|\mathbf{x} - \mathbf{a}_i}$$

using

$$Q = \left(\frac{4\pi R_i^3}{3}\right) \rho_0$$

by superposition principle

$$\phi(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0} \left[\frac{Q}{|\mathbf{x}|} - \sum_{i=1}^{N} \frac{Q_i}{|\mathbf{x} - \mathbf{a}_i|}\right]$$

7.4 Integral Solutions

We know electric potential due to point charge at $\mathbf{x} = \mathbf{a}$ is proportional to

$$\frac{1}{|\mathbf{x} - \mathbf{a}|}$$

or collection of point charges

$$\sum \frac{Q_i}{|\mathbf{x} - \mathbf{a}|}$$

This leads us to consider superpositions of form

$$\int_{\mathbb{R}^3} \frac{F(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|} \, \mathrm{d}V(\mathbf{y})$$

Prop. Assume $F \to 0$ rapidly as $|\mathbf{x} \to \infty$. The unique solution to the Dirichlet problem

$$\begin{cases} \nabla^2 \varphi = F \ \mathbf{x} \in \mathbb{R}^3 \\ |\varphi| \to 0 \ |\mathbf{x}| \to \infty \end{cases}$$

is given by

$$\varphi(\mathbf{x}) = -\frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{F(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|} \, \mathrm{d}V(\mathbf{y})$$

Proof. Note that for $r \neq 0$

$$\begin{aligned} \nabla^2 \left(\frac{1}{r}\right) &= \frac{\partial^2}{\partial x_i \partial x_i} \left(\frac{1}{r}\right) \\ &- \frac{\partial}{\partial x_i} \left(-\frac{x_i}{r^2}\right) \\ &= -\frac{\delta_{ii}}{r^3} + \frac{3x_i x_i}{r^5} \\ &= -\frac{3}{r^3} + \frac{3}{r^3} \\ &= 0 \end{aligned}$$

Certainly have

$$\nabla^2 \left(-\frac{1}{4\pi} \frac{1}{|\mathbf{x}|}\right) = \delta(\mathbf{x}) \ \mathbf{x} \neq 0$$

If we assume divergence thm works with delta function, on any ball $|\mathbf{x}| < R$

$$\begin{aligned} \int_{|\mathbf{x}|<R} \nabla^2 \left(\frac{1}{|\mathbf{x}|}\right) \mathrm{d}V &= \int_{\mathbf{x}=R} \nabla \left(\frac{1}{|\mathbf{x}|}\right) \cdot \mathrm{d}\mathbf{S} \\ &= \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \left(-\frac{\mathbf{e}_r}{R^2}\right) \cdot \mathbf{e}_r R^2 \sin\theta \, \mathrm{d}\phi \, \mathrm{d}\theta \\ &= -4\pi \end{aligned}$$

So for any $R > 0$

$$\int_{|\mathbf{x}|<R} \nabla^2 \left(\frac{1}{4\pi} \frac{1}{|\mathbf{x}|}\right) \mathrm{d}V = 1 = \int_{|\mathbf{x}|<R} \delta(\mathbf{x}) \, \mathrm{d}V$$

We conclude

$$\nabla^2 \left(-\frac{1}{4\pi} \frac{1}{|\mathbf{x}|}\right) = \delta(\mathbf{x})$$

so proposition follows.

Remark. This result is another way of saying

$$\nabla^2 \left(-\frac{1}{4\pi} \frac{1}{|\mathbf{x}|} \right) = \delta(\mathbf{x})$$

Since by differentiating under integral sign

$$\begin{aligned} \nabla^2 \left(-\frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{F(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|} \, \mathrm{d}V(\mathbf{y}) \right) &= -\frac{1}{4\pi} \int_{\mathbb{R}^3} F(\mathbf{y}) \nabla^2 \left(\frac{1}{|\mathbf{x}-\mathbf{y}|} \right) \mathrm{d}V(\mathbf{y}) \\ &= \int_{\mathbb{R}^3} F(\mathbf{y}) \delta(\mathbf{x}-\mathbf{y}) \, \mathrm{d}V(\mathbf{y}) \\ &= F(\mathbf{x}) \end{aligned}$$

7.5 Harmonic Functions

Definition. When the forcing term in Poisson's equation is identically zero, we call it **Laplace's equation**:

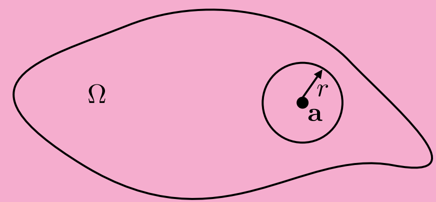
$$\nabla^2 \varphi = 0 \tag{\dagger}$$

Solutions to Laplace's equation are called harmonic functions

Prop. If φ harmonic on $\Omega \subseteq \mathbb{R}^3$, then

$$\varphi(\mathbf{a}) = \frac{1}{4\pi r^2} \int_{|\mathbf{x}-\mathbf{a}|=r} \varphi(\mathbf{x}) \, \mathrm{d}S \tag{*}$$

for $\mathbf{a} \in \Omega$ and r sufficiently small.



Proof. Let $F(r)$ denote RHS of (*). Then

$$\begin{aligned} F(r) &= \frac{1}{4\pi r^2} \int_{|\mathbf{x}|=r} \varphi(\mathbf{a}+\mathbf{x}) \, \mathrm{d}S \\ &= \frac{1}{4\pi r^2} \int_{\phi=0}^{2\pi} \left[\int_{\theta=0}^{\pi} \varphi(\mathbf{a} + r\mathbf{e}_r r^2 \sin\theta \, \mathrm{d}\theta \right] \mathrm{d}\phi \\ &= \frac{1}{4\pi} \int_{\phi=0}^{2\pi} \left[\int_{\theta=0}^{\pi} \varphi(\mathbf{a} + r\mathbf{e}_r \sin\theta \, \mathrm{d}\theta \right] \mathrm{d}\phi \end{aligned}$$

Computing $F'(r)$, using

$$\frac{\mathrm{d}}{\mathrm{d}r} \varphi(\mathbf{a} + r\mathbf{e}_r) = \mathbf{e}_r \cdot \nabla\varphi(\mathbf{a} + r\mathbf{e}_r)$$

as

$$\frac{\mathrm{d}}{\mathrm{d}t} f\mathbf{x}(t)) = \mathbf{x}'(t) \cdot \nabla f(\mathbf{x}(t))$$

$$\begin{aligned} F'(r) &= \frac{1}{4\pi r^2} \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \mathbf{e}_r \cdot \nabla\varphi(\mathbf{a} + r\mathbf{e}_r) r^2 \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi \\ &= \frac{1}{4\pi r^2} \int_{|\mathbf{x}|=r} \mathbf{e}_r \cdot \nabla\varphi(\mathbf{a} + r\mathbf{e}_r) \, \mathrm{d}S \\ &= \frac{1}{4\pi r^2} \int_{|\mathbf{x}|=r} \nabla\varphi(\mathbf{a}+\mathbf{x}) \cdot \, \mathrm{d}\mathbf{S} \\ &= \frac{1}{4\pi r^2} \int_{|\mathbf{x}-\mathbf{a}|=r} \nabla\varphi \cdot \, \mathrm{d}\mathbf{S} \\ &= \frac{1}{4\pi r^2} \int_{|\mathbf{x}-\mathbf{a}|<r} \nabla^2\varphi \cdot \, \mathrm{d}V \\ &= 0 \end{aligned}$$

So $F(r)$ is constant and we note from (†) that

$$\lim_{r\to 0} F(r) = \varphi(\mathbf{a})$$

So

$$F(r) = \varphi(\mathbf{a})$$

and result follows. □

Moral. Can use central idea in this proof to examine what the Laplacian helps us measure

Prop. For any smooth $\varphi : \mathbb{R}^3 \to \mathbb{R}$

$$\nabla^2 \varphi(\mathbf{a}) = \lim_{r \to 0} \frac{6}{r^2} \left[\frac{1}{4\pi r^2} \int_{|\mathbf{x}-\mathbf{a}|=r} \varphi(\mathbf{x}) \, \mathrm{d}S - \varphi(\mathbf{a}) \right]$$

In particular, if φ satisfies the MVP then it is harmonic.

Proof. Consider function $G(r)$ defined by

$$G(r) = \frac{1}{4\pi r^2} \int_{|\mathbf{x}-\mathbf{a}|=r} \varphi(\mathbf{x}) \, \mathrm{d}S - \varphi(\mathbf{a})$$

So G measures extent to which φ differs from its average. we have from previous proof

$$G'(r) = F'(r) = \frac{1}{4\pi r^2} \int_{|\mathbf{x}-\mathbf{a}|<r} \nabla^2 \varphi \, \mathrm{d}V$$

Obviously, this vanishes if φ harmonic. Note

$$\begin{aligned} \int_{|\mathbf{x}-\mathbf{a}|=r} &= \nabla^2 \varphi(\mathbf{a}) \int_{|\mathbf{x}-\mathbf{a}|<r} \mathrm{d}V + \int_{|\mathbf{x}-\mathbf{a}|<r} (\nabla^2 \varphi(\mathbf{x}) - \nabla^2 \varphi(\mathbf{a}) \, \mathrm{d}V \\ &= \frac{4\pi}{3} r^2 \nabla^2 \varphi(\mathbf{a}) + o(r^3) \ (r \to 0) \end{aligned}$$

So

$$\begin{aligned} G'(r) &= \frac{1}{4\pi r^2} \int_{|\mathbf{x}-\mathbf{a}|<r} \nabla^2 \varphi(\mathbf{a}) \, \mathrm{d}S \\ &= \frac{1}{4\pi r^2} \left[\frac{4\pi}{3} r^3 \nabla^2 \varphi(\mathbf{a}) + o(r^3) \right] \\ &= \frac{r}{3} \nabla^2 \varphi(\mathbf{a}) + o(r) \ (r \to 0) \end{aligned}$$

Compare this with Taylor expansion

$$G'(r) = G'(0) + rG''(0) + o(r) \ (r \to 0)$$

we deduce:

$$G'(0) = 0, \ G''(0) = \frac{1}{3} \nabla^2 \varphi(\mathbf{a})$$

So

$$\begin{aligned} G(r) &= \underbrace{G(0)}_{=0} + r \underbrace{G'(0)}_{=0} + \frac{r^2}{2} G''(0) + o(r^2) \\ &= \frac{1}{6} \nabla^2 \varphi(\mathbf{a}) r^2 + o(r^2) \ (r \to 0) \end{aligned}$$

$$\implies \nabla^2 \varphi(\mathbf{a}) = \lim_{r \to 0} \left[\frac{6}{r^2} G(r) \right] \implies \text{result } \square$$

Prop. If φ is harmonic on $\Omega \subseteq \mathbb{R}^3$ then cannot have a maximum at any interior point of Ω unless φ is constant.

Proof. Suppose $\mathbf{a} \in \Omega$ is such that

$$\varphi(\mathbf{a}) \geq \varphi(\mathbf{x})$$

for all $\mathbf{x} \in \Omega$. So certainly

$$\varphi(\mathbf{a}) \geq \varphi(\mathbf{x}) \text{ on } 0 < |\mathbf{x} - \mathbf{a}| \leq \varepsilon$$

for some $\varepsilon > 0$. But by mean value thm

$$\varphi(\mathbf{a}) = \frac{1}{4\pi\varepsilon^2} \int_{|\mathbf{x}-\mathbf{a}|=\varepsilon} \int \varphi(\mathbf{x}) \, \mathrm{d}S$$

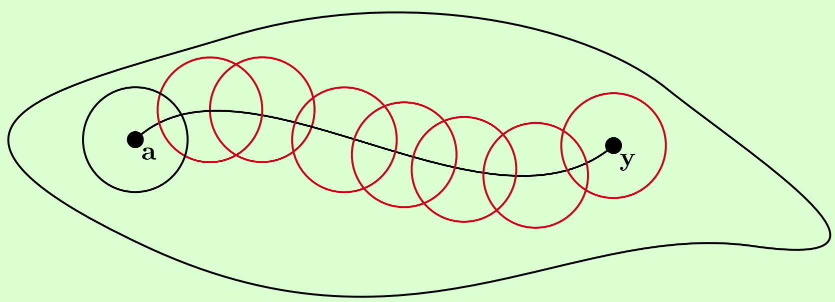
i.e.

$$0 = \frac{1}{4\pi\varepsilon^2} \int_{|\mathbf{x}-\mathbf{a}|=\varepsilon} \int \underbrace{\varphi(\mathbf{a}) - \varphi(\mathbf{x})}_{\geq 0} \, \mathrm{d}S$$

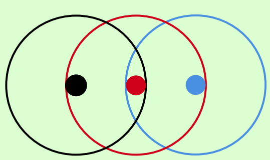
Consider that $\varphi(\mathbf{x}) = \varphi(\mathbf{a})$. Apply same argument to

$$|\mathbf{x} - \mathbf{a}| = \varepsilon' < \varepsilon$$

Deduce $\varphi(\mathbf{x}) = \varphi(\mathbf{a})$ on $|\mathbf{x} - \mathbf{a}| \leq \varepsilon$



Introduce bunch of overlapping balls such that the centre of the $(n+1)$th ball is contained inside the nth.



Everywhere inside 1^{st} ball, have $\varphi(\mathbf{x}) = \varphi(\mathbf{a})$.
In particular, on center of second ball have $\varphi(\mathbf{x}) = \varphi(\mathbf{a})$.
Using previous argument get $\varphi(\mathbf{x}) = \varphi(\mathbf{a})$ throughout second ball. Carry on until you get to $\mathbf{y}$. Find $\varphi(\mathbf{y}) = \varphi(\mathbf{a})$ i.e. φ constant. $\square$

Corollary. If φ is harmonic on Ω then

$$\varphi(\mathbf{x}) \leq \max_{\mathbf{y}\in\partial\Omega} \varphi(\mathbf{y}) \ (\mathbf{x} \in \Omega)$$

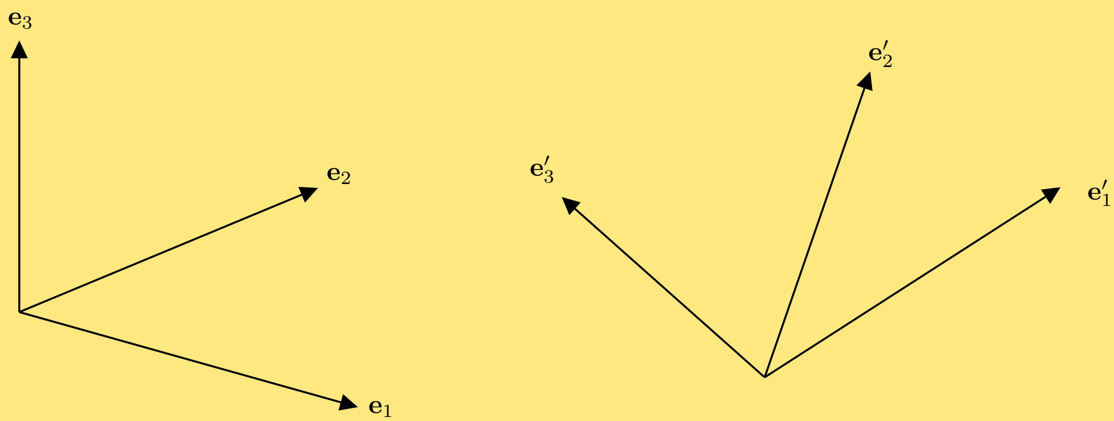
(Maximum principle)

8 Cartesian Tensors

Remark. Throughout this section we deal solely with Cartesian coordinate systems

8.1 A Closer Look at Vectors

Method. Let $\{\mathbf{e}_i\}$ be a right-handed, orthonormal basis with respect to a fixed set of Cartesian axes



Write vector as

$$\mathbf{x} = x_i \mathbf{e}_i$$

We shouldn't identify $\mathbf{x}$ with the components $\{x_i\}$ since these will change if we use a different basis. If we instead used $\{\mathbf{e}'_i\}$ (also right-handed and orthonormal), then same vector is

$$\mathbf{x} = x'_i \mathbf{e}'_i$$

We have

$$x_j \mathbf{e}_j = x'_j \mathbf{e}'_j \tag{*}$$

Since $\{\mathbf{e}_j\}$ and $\{\mathbf{e}'_j\}$ orthonormal

$$\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$$

$$\mathbf{e}'_i \cdot \mathbf{e}'_j = \delta_{ij}$$

Method (cont.)**.** From (*)

$$x'_i = \delta_{ij}x'_j = (\mathbf{e}'_i \cdot \mathbf{e}'_j)x_j = \mathbf{e}'_i \cdot (\mathbf{e}'_j x'_j) = (\mathbf{e}'_i \cdot \mathbf{e}_j)x_j$$

Set $R_{ij} = \mathbf{e}'_i \cdot \mathbf{e}_j$, then

$$x'_i = R_{ij}x_j$$

Alternatively

$$x'_i = \delta_{ij}x'_j = (\mathbf{e}_i \cdot \mathbf{e}_j)x_j = \mathbf{e}_i \cdot (\mathbf{e}'_j x'_j) = (\mathbf{e}'_j \cdot \mathbf{e}_i)x_j$$

i.e.

$$x_i = R_{ji}x'_j = R_{ki}x'_k$$

$$x_j = R_{kj}x'_k$$

$$x'_i = R_{ij}x_j = R_{ij}R_{kj}x'_k$$

So we find

$$(\delta_{ik} - R_{ij}R_{jk})s'_k = 0$$

Since this true for ALL choices $\{x'_k\}$ get

$$R_{ij}R_{kj} = \delta_{ik}$$

If R is matrix with entries $\{R_{ij}\}$, this reads

$$RR^T = I$$

So $\{R_{ij}\}$ are components of an orthogonal matrix.
Since:

$$x_j\mathbf{e}_j = x'_i\mathbf{e}'_i = R_{ij}x_j\mathbf{e}'_i$$

holds for ALL $\{x_j\}$, also have

$$\mathbf{e}_j = R_{ij}\mathbf{e}'_i$$

and since both $\{\mathbf{e}_i\}$ and $\{\mathbf{e}'_i\}$ right-handed

$$\begin{aligned} 1 = \mathbf{e}_1 \cdot (\mathbf{e}_2 \times \mathbf{e}_3) &= R_{i1}R_{j2}R_{k3}\mathbf{e}'_i \cdot (\mathbf{e}'_j \times \mathbf{e}'_k) \\ &= R_{i1}R_{j2}R_{k3}\varepsilon_{ijk} = \det(R) \end{aligned}$$

Remark. So matrix R s=is orthogonal and $\det R = 1$. So $\{R_{ij}\}$ are components of a rotation matrix

Moral. If we transform fom $\{\mathbf{e}_i\}$ to $\{\mathbf{e}'_i\}$ then the components of a vector $\mathbf{v}$ transform as

$$v'_i = R_{ij}v_j$$

where $R_{ij} = \mathbf{e}'_i \cdot \mathbf{e}_j$ are components of a rotation matrix. Call objects whose components transform in this way rank 1 tensors, or vectors.

8.2 A Closer Look at Scalars

Method. Consider

$$\sigma = \mathbf{a} \cdot \mathbf{b}$$

Using $\{\mathbf{e}_i\}$ with $\mathbf{a} = a_i \mathbf{e}_i$ etc.

$$\begin{aligned} \sigma &= a_i b_j (\mathbf{e}_i \cdot \mathbf{e}_j) \\ &= a_i b_j \delta_{ij} \\ &= a_i b_j \delta_{ij} \\ &= a_i b_i \end{aligned}$$

Instead use $\{\mathbf{e}'_i\}$ would find

$$\sigma' = a'_i b'_i$$

Using $a'_i = R_{ip} a_p,\ b'_i = R_{iq} b_q$

$$\sigma' = R_{ip} R_{iq} a_p b_q = \delta_{pq} a_p b_q = a_p b_p = \sigma$$

We call objects that transform in this way scalars.

Moral. objects that transform as

$$\sigma' = \sigma$$

when we change from $\{\mathbf{e}_i\}$ to $\{\mathbf{e}'_i\}$ are called scalars, or rank 0 tensors.

8.3 A Closer Look at Linear Maps

Method. Let $\mathbf{n} \in \mathbb{R}^3$ be a fixed unit vector and define linear map

$$T : \mathbf{x} \mapsto \mathbf{y} = T(\mathbf{x}) = \mathbf{x} - (\mathbf{x} \cdot \mathbf{a})\mathbf{n}$$

Using $\{\mathbf{e}_i\}$ with $\mathbf{x} = x_i\mathbf{e}_i$, $\mathbf{y} = y_i\mathbf{e}_i$ etc.

$$\begin{aligned} y_i\mathbf{e}_i &= T(x_j\mathbf{e}_j) \\ &= x_jT(\mathbf{e}_j) \\ &= x_j(\mathbf{e}_j - n_in_j\mathbf{e}_i \\ &= (\delta_{ij} - n_in_j)x_j\mathbf{e}_i \end{aligned}$$

Set $T_{ij} = \delta_{ij} - n_in_j$. Then

$$y_i\delta_{ij} - n_in_j)x_j = T_{ij}x_j$$

Call $\{T_{ij}\}$ components of linear map $T : \mathbb{R}^3 \to \mathbb{R}^3$ wrt $\{\mathbf{e}_i\}$
If we had instead used $\{\mathbf{e}'_i\}$ would have found

$$y'_i = T'_{ij}x'_j$$

where $T'_{ij} = \delta_{ij} - n'_in'_j$. Using $n'_i = R_{ij}n_j$ give

$$\begin{aligned} T'_{ij} &= \delta_{ij} - R_{ip}R_{jq}n_pn_q \\ &= R_{ip}R_{jp}(\delta_{pq} - n_pn_q) \\ &= R_{ip}R_{jq}T_{pq} \end{aligned}$$

Components of T transform according to

$$T'_{ij} = R_{ip}R_{jq}T_{pq}$$

Objects that transform in this way are called rank 2 tensors.

8.4 Cartesian Tensors of Rank n

Definition. An object whose components $T_{\underbrace{ij\ldots k}_{n \text{ indices}}}$ transform (when we go from $\{\mathbf{e}_i\}$ to $\{\mathbf{e}'_i\}$) according to

$$T'_{ij\ldots k} = \overbrace{R_{ip}R_{jq}\ldots R_{kr}}^{n\ R\text{s}}T_{pq\ldots r}$$

is called a (Cartesian) **tensor of rank n**.
Here

$$R_{ij} = \mathbf{e}'_i \cdot \mathbf{e}_j$$

are components of rotation matrix, so

$$R_{ip}R_{jp} = \delta_{ij}$$

Example. If $u_i, v_k, \dots, w_k$ are components of n vectors, then

$$T_{ij\dots k} = u_i v_j \dots w_k$$

define components of a tensor of rank n.
Can check:

$$\begin{aligned} T'_{ij\dots k} &= u'_i v'_j \dots w'_k \\ &= R_{ip} u_p R_{jq} v_q \dots R_{kr} w_r \\ &= R_{ip} R_{jq} \dots R_{kr} T_{pq\dots r} \end{aligned}$$

Example. Kronecker delta is defined without reference to any vasis via

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

So $\delta'_{ij} = \delta_{ij}$ by definition. But note

$$R_{ip} R_{jq} \delta_{pq} = R_{ip} R_{jp} = \delta_{ij}$$

So we have

$$\delta'_{ij} = R_{ip} R_{jq} \delta_{pq}$$

i.e. δ_{ij} is a rank 2 tensor.

Example. The Levi Civita symbol is defined without reference to any basis

$$\varepsilon_{ijk} = \begin{cases} 1 & \text{if } (i\ j\ k) \text{ is an even perm of } (1\ 2\ 3) \\ -1 & \text{if } (i\ j\ k) \text{ is an odd perm of } (1\ 2\ 3) \\ 0 & \text{otherwise} \end{cases}$$

By definition, $\varepsilon'_{ijk} = \varepsilon_{ijk}$. But

$$\begin{aligned} R_{ip} R_{jq} R_{kr} \varepsilon_{pqr} &= \det(R) \varepsilon_{ijk} \\ &= \varepsilon_{ijk} \end{aligned}$$

So we have

$$\varepsilon'_{ijk} = R_{ip} R_{jq} R_{kr} \varepsilon_{pqr}$$

So ε_{ijk} is a tensor of rank 3.

Example. Experimental evidence suggests a linear relationship between current $\mathbf{J}$ produced in conductive medium exposed to electric field $\mathbf{E}$, so

$$\mathbf{J} = \sigma \mathbf{E}$$

or using suffix notation

$$J_i = \sigma_{ij} \varepsilon_j$$

σ_{ij} is called the electrical conductivity tensor, it really is a rank 2 tensor. Under change of basis

$$\sigma'_{ij} E'_j = J'_i = R_{ip} J_p = R_{qp} \sigma_{pq} E_q$$

Using

$$E'_j = R_{jq} E_q \iff E_q = R_{jq} E'_j$$

we get

$$\sigma_{ij} E'_j = R_{ip} R_{jq} \sigma_{pq} E'_j$$

This holds for ANY $\{E'_j\}$, so

$$\sigma'_{ij} = R_{ip} R_{jq} \sigma_{pq}$$

i.e. σ_{ij} is a rank 2 tensor.
See Quotient Theorem later in course.

Example. Not all things are tensors. For given Cartesian right handed basis $\{\mathbf{e}_i\}$ we define array

$$(A_{ij}) = \begin{bmatrix} \pi & 7 & 0 \\ \sqrt{2} & e & -3 \\ \gamma & 1 & 12 \end{bmatrix}$$

and set $A'_{ij} = 0$ in all other bases $\{\mathbf{e}_i\}$. Then A_{ij} are NOT the components of a rank 2 tensor.

Definition. If $A_{ij\dots k}$ and $B_{ij\dots k}$ are n-th rank tensors, define

$$(A+B)_{ij\dots k} = A_{ij\dots k} + B_{ij\dots k}$$

This is also n-th rank tensor, If α is a scalar then

$$(\alpha A)_{ij\dots k} = \alpha A_{ij\dots k}$$

is an n-th rank tensor.
We define the **tensor product** of an m-th rank tensor $U_{ij\dots k}$ and a an n-th rank tensor $V_{pq\dots r}$ by

$$(U \otimes V)_{ij\dots kpq\dots r} = U_{ij\dots k} V_{pq\dots r}$$

where

$$\underbrace{ij\dots k}_{m \text{ indices}} \underbrace{pq\dots r}_{n \text{ indices}}$$

Claim. This is a tensor of rank $n+m$.

Proof.

$$\begin{aligned} U'_{i\dots j}V'_{p\dots q} &= R_{ia}\dots R_{jb}U)a\dots bR_{pc}\dots R_{qd}V_{c\dots d} \\ &= \underbrace{R_{ia}\dots R_{jb}R_{pc}\dots R_{qd}}_{n+m \text{ terms}} \underbrace{U_{a\dots b}V_{c\dots d}}_{(U\otimes V)_{a\dots bc\dots d}} \end{aligned}$$

Method. Given n-th rank tensor $T_{ijk\dots d}$ $n \geq 2$, we can define tensor of rank $n-2$ by contracting on pair of indices. For instance, contracting on i and j is defined by

$$\delta_{ij}T_{ijk\dots d} = T_{iik\dots d}$$

Note.

$$\begin{aligned} T'_{ijk\dots d} &= \underbrace{R_{ip}R_iq}_{\delta_{pq}} R_{kr}\dots R_{ls}T_{pqr\dots s} \\ &= R_{kq}\dots R_{ls}T_{ppr\dots s} \end{aligned}$$

So $T_{iik\dots d}$ transforms as tensor of rank $n-2$

Definition. Say $T_{ij\dots k}$ is **symmetric** in (i,j) if

$$T_{ih\dots k} = T_{ji\dots k}$$

This really is well-defined property of the tensor

$$\begin{aligned} T'_{ij\dots k} &= R_{ip}R_{jq}\dots R_{kr}T_{pq\dots r} \\ &= R_{ip}R_{jq}\dots R_{kr}T_{qp\dots r} \\ &= R_{iq}R_{jp}\dots R_{kr}T_{pq\dots r} \\ &= T'_{ji\dots k} \end{aligned}$$

Similarly, we say $A_{ij\dots k}$ is **anti-symmetric** in (i,j) if

$$A_{ij\dots k} = -A_{ji\dots k}$$

Say a tensor is **totally (anti-)symmetric** if it is (anti-)symmetric in every pair of indices.

Example. Tensors δ_{ij} and $a_i a_j a_k$ are both totally symmetric.
ε_{ijk} is a totally anti-symmetric tensor.
In fact, the only totally anti-symmetric tensor on $\mathbb{R}^3$ of rank $n = 3$ is proportional to ε_{ijk}, and there are no non-zero high rank ones. Indeed, if $T_{ij\dots k}$ totally anti-symmetric of rank n, then $T_{ij\dots k} = 0$ if any two indices are the same

$$T_{22\dots k} = -T_{22\dots k} \implies T_{22\dots k} = 0$$

So by pigeonhole principle, there will always be two or more matching indices if $n > 3$. If $n = 3$, there are only $3! = 6$ non-zero components. If

$$T_{123} = T_{231} = T_{312} = \lambda$$

$$T_{213} = T_{321} = T_{132} = -\lambda$$

Thus $T_{ijk} = \lambda\varepsilon_{ijk}$

8.5 Tensor Calculus

Remark. "vector field" gives vector $\mathbf{v}(\mathbf{x})$ for $\mathbf{x} \in \mathbb{R}^3$
"scalar field" gives vector $\varphi(\mathbf{x})$ for $\mathbf{x} \in \mathbb{R}^3$
A tensor field of rank n, $T_{ij\dots k}(\mathbf{x})$, gives an n-th rank tensor at each $\mathbf{x} \in \mathbb{R}^3$.

Equation. Recall

$$x_i' = R_{ij}x_j \iff x_j = R_{ij}x_i'$$

Differentiating RHS wrt x_k'

$$\frac{\partial x_j}{\partial x_k'} = R_{ij}\frac{\partial x_i'}{\partial x_k'} = R + oj\delta_{ik} = R_{kj}$$

So by chain rule

$$\frac{\partial}{\partial x_i'} = \frac{\partial x_j}{\partial x_i'}\frac{\partial}{\partial x_j} = R_{ij}\frac{\partial}{\partial x_j}$$

"$\frac{\partial}{\partial x_i}$ transforms like a rank 1 tensor"

Prop. If $T_{i\dots j}(\mathbf{x})$ is tensor field of rank n then

$$\underbrace{\left(\frac{\partial}{\partial x_p}\right)\dots\left(\frac{\partial}{\partial x_q}\right)}_{m \text{ terms}} T_{i\dots j}(\mathbf{x}) = \text{ tensor field of rank } n+m$$

Proof. Label LHS by $A_{p\dots qi\dots j}$

$$\begin{aligned} A_{p\dots qi\dots j} = \left(\frac{\partial}{\partial x_p'}\right)\dots\left(\frac{\partial}{\partial x_q'}\right) T_{i\dots j}'(\mathbf{x}) \\ &= \left(R_{pa}\frac{\partial}{\partial x_a}\right)\dots\left(R_{qb}\frac{\partial}{\partial x_b}\right) R_{ic}\dots R_{jd}T_{c\dots d} \\ &= R_{pa}\dots R_{qb}R_{ic}\dots R_{jd}A_{a\dots bc\dots d} \end{aligned}$$

So have tensor field of rank $n + m$. $\square$

Example. If $\varphi = \varphi(\mathbf{x})$ scalar field then

$$[\nabla\varphi]_i = \frac{\mathrm{d}\varphi}{\mathrm{d}x_i}$$

So $\nabla\varphi$ is rank $0 + 1 = 1$ tensor field, i.e. a vector field.

Example. For vector field $\mathbf{v}$ have divergence

$$\nabla \cdot \mathbf{v} = \frac{\partial v_i}{\partial x_i}$$

Note:

$$\begin{aligned}
\frac{\partial v_i'}{\partial x_i'} &= R_{ip}\frac{\partial}{\partial x_p}R_{iq}v_q \\
&= R_{ip}R_{iq}\frac{\partial v_q}{\partial x_p} \\
&= \delta_{pq}\frac{\partial v_q}{\partial x_p} \\
&= \frac{\partial v_p}{\partial x_p}
\end{aligned}$$

i.e. $\nabla \cdot \mathbf{v}$ is scalar field.

Example. If $\mathbf{v}$ vector field, consider curl $\nabla \times \mathbf{v}$. Then

$$[\nabla \times \mathbf{v}]_i = \varepsilon_{ijk}\frac{\partial v_k}{\partial x_j}$$

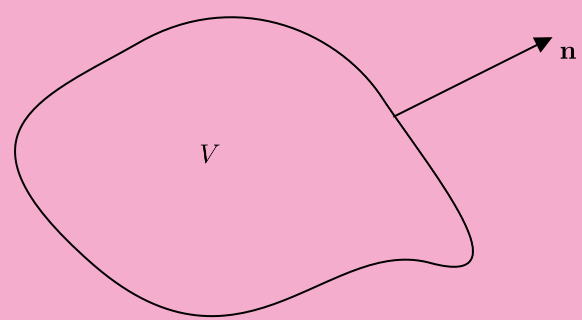
Then:

$$\begin{aligned}
\varepsilon_{ijk}'\frac{\partial v_k'}{\partial x_j'} &= R_{ia}R_{jb}R_{kc}\varepsilon_{abc}R_{jp}\frac{\partial}{\partial x_p}R_{kp}v_q \\
&= R_{ia}\varepsilon_{abc}\underbrace{R_{jb}R_{jp}}_{\delta_{pb}}\underbrace{R_{kc}R_{kq}}_{\delta_{cq}}\frac{\partial v_p}{\partial x_p} \\
&= R_{ia}\varepsilon_{abx}\frac{\partial v_c}{\partial x_b}
\end{aligned}$$

So $\nabla \times \mathbf{v}$ is vector field.

Prop. For tensor field $T_{ij\ldots k\ldots l}(\mathbf{x})$:

$$\int_V \frac{\partial}{\partial x_k} T_{ij\ldots k\ldots l}\,\mathrm{d}V = \int_{\partial V} T_{ij\ldots k\ldots l} n_k\,\mathrm{d}S$$



Proof. Apply divergence theorem to

$$v_k = a_i b_j \ldots c_l T_{ij\ldots k\ldots l} \tag{†}$$

where $a_i, b_j, \ldots, c_l$ are components of constant vector fields. So by div theorem

$$\begin{aligned}\int_V \frac{\partial v_k}{\partial x_k}\,\mathrm{d}V &= a_i b_j \ldots c_l \int_V \frac{\partial}{\partial x_k} T_{ij\ldots k\ldots l}\,\mathrm{d}V \\ &= \int_{\partial V} v_k n_k\,\mathrm{d}S \text{ (div thm on LHS)} \\ &= a_i b_j \ldots c_l \int_{\partial V} T_{ij\ldots k\ldots l} n_k\,\mathrm{d}S\end{aligned}$$

Result now follows because the constant vector fields $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ were arbitrary.
E.g. if we wanted to check (†) when a;; free indices $i, j, \ldots, l$ were $= 1$

$$a_i = \delta_{i1},\ b_j = \delta_{j1},\ \ldots,\ c_l = \delta_{l1}$$

$$LHS = \int_V \frac{\partial}{\partial x_k} T_{11\ldots k\ldots 1}\,\mathrm{d}V$$

$$RHS = \int_{\partial V} T_{11\ldots k\ldots 1} n_k\,\mathrm{d}S$$

Similar idea for other choice of free indices. □

8.6 Rank 2 Tensors

Remark. Observe for rank 2 tensor T_{ij}

$$\begin{aligned} T_{ij} &= \frac{1}{2}(T_{ij} + T_{ji}) + \frac{1}{2}(T_{ij} - T_{ji}) \\ &= S_{ij} + A_{ij} \end{aligned}$$

which is symmetric + anti-symmetric

$$\underbrace{\begin{bmatrix} * & * & * \\ & * & * \\ & & * \end{bmatrix}}_{\text{6 indep components}} \quad \underbrace{\begin{bmatrix} 0 & * & * \\ & 0 & * \\ & & 0 \end{bmatrix}}_{\text{3 indep components}}$$

This is food since $3 + 6 = 9$. Intuitively, seems like info contained in A_{ij} caould be written in terms of some vector (3 indep components).

Prop. Every ran 2 tensor can be written uniquely as

$$T_{ij} = S_{ij} + \varepsilon_{ijk}\omega_k$$

where

$$\omega_i = \frac{1}{2}\varepsilon_{ijk}T_{jk}$$

and

$$S_{ij} \text{ is symmetric}$$

Proof. We can identify (from earlier)

$$S_{ij} = \frac{1}{2}(T_{ij} + T_{ji}$$

Remains to show that

$$e_{ijk}\omega_k = \frac{1}{2}(T_{ij} - T_{ji})$$

$$\begin{aligned}\varepsilon_{ijk}\omega_k &= \frac{1}{2}\varepsilon_{ijk}\varepsilon_{klm}T_{lm} \\ &= \frac{1}{2}(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})T_{lm} \\ &= \frac{1}{2}(T_{ij} - T_{ji}\end{aligned}$$

For uniqueness, suppose

$$(T_{ij} =)S_{ij} + A_{ij} + \tilde{S}_{ij} + \tilde{A}_{ij}(= \tilde{T}_{ij})$$

Take symmetric parts of both side i.e.

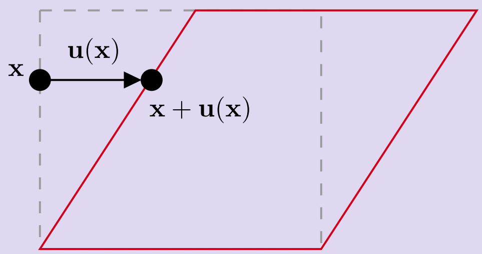
$$\frac{1}{2}(T_{ij} + T_{ji}) = \frac{1}{2}(\tilde{T}_{ij} + \tilde{T}_{ji})$$

Then $S_{ij} = \tilde{S}_{ij}$ and so $A_{ij} = \tilde{A}_{ij}$. i.e. decomposition is unique

$$\varepsilon_{ijk}\omega_k = \varepsilon_{ijk}\tilde{\omega}_k \iff \omega_k = \tilde{\omega}_k \;\square$$

Note. See Truesdell + Noll, Nonlinear Continuum Mechanics

Example. Each point $\mathbf{x}$ in an elastic body undergoes small displacement $\mathbf{u}(\mathbf{x})$



Two nearby points $\mathbf{x}+\delta\mathbf{x}$ and $\mathbf{x}$ that were initially separated by $\delta\mathbf{x}$ become separated by

$$(\mathbf{x}+\delta\mathbf{x}+\mathbf{u}(\mathbf{x}+\delta\mathbf{x}))-(\mathbf{x}+\mathbf{u}(\mathbf{x}))=\delta\mathbf{x}+\underbrace{[\mathbf{u}(\mathbf{x}+\delta\mathbf{x})-\mathbf{u}(\mathbf{x})]}_{\text{change in displacement}}$$

Change in displacement:

$$\mathbf{u}(\mathbf{x}+\delta\mathbf{x})-\mathbf{u}(\mathbf{x})]$$

This tells us how much deformation happens to the body. Using Taylor's theorem:

$$u_i(\mathbf{x}+\delta\mathbf{x})-u_i(\mathbf{x})=\frac{\partial u_i}{\partial x_j}\delta x_j+o(\delta\mathbf{x})$$

We decompose $\frac{\partial u_i}{\partial x_j}$ as follows:

$$\frac{\partial u_i}{\partial x_j}=e_{ij}+\varepsilon_{ijk}\omega_k$$

where

$$e_{ij}=\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)$$

is called LINEAR STRAIN TENSOR and

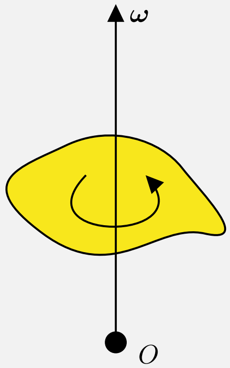
$$\omega_i=\frac{1}{2}\varepsilon_{ijk}\frac{\partial u_j}{\partial x_k}=-\frac{1}{2}(\nabla\times\mathbf{u})_i$$

So:

$$u_i(\mathbf{x}+\delta\mathbf{x})-u_i(\mathbf{x})=\underbrace{e_{ij}\delta x_j}_{\text{measure of deformation}}+\underbrace{[\delta\mathbf{x}\times\boldsymbol{\omega}]_i}_{\text{corresponds to rotation}}+o(\delta\mathbf{x})$$

So e_{ij} gives info about how much body compresses or stretches.

A well known symmetric rank 2 tensor is the inertia tensor. Suppose body with density $\rho(\mathbf{x})$ occupies volume $V \subseteq \mathbb{R}^3$. Each point in the body rotating at constant angular velocity $\boldsymbol{\omega}$



So elocity of point $\mathbf{x} \in V$ is $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{x}$. Total angular velocity about origin is:

$$\begin{aligned}\mathbf{L} &= \int_V \rho(\mathbf{x})(\mathbf{x} \times \mathbf{v})\,\mathrm{d}V \\ &= \int_V \rho(\mathbf{x})[\mathbf{x} \times (\boldsymbol{\omega} \times \mathbf{x})]\,\mathrm{d}V\end{aligned}$$

Using suffix notation

$$\begin{aligned}L_i &= \int_{\mathcal{V}} \rho(\mathbf{x})(x_k x_k \omega_i - x_i x_k \omega_j)\,\mathrm{d}V \\ &= I_{ij}\omega_j\end{aligned}$$

(by writing $\omega_i = \delta_{ij}\omega_j$)
where we have defined inertia tensor

$$I_{ij} = \int_{\mathcal{V}} \rho(\mathbf{x})(x_k x_k \delta_{ij} - x_i x_j)\,\mathrm{d}V$$

where integral is taken over

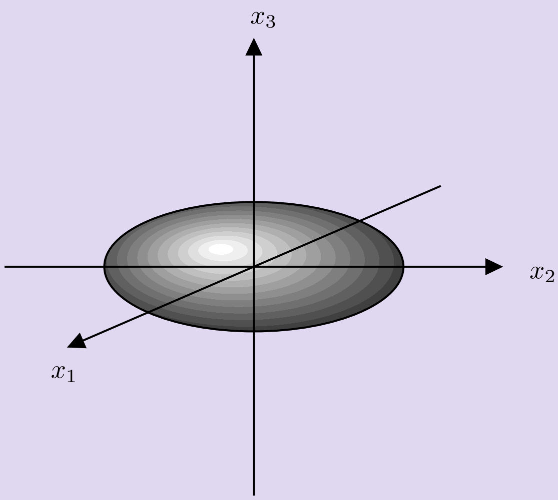
$$\mathcal{V} = \{x_i : x_i \mathbf{e}_i \in V\}$$

Had we used different frame $\{\mathbf{e}_i'\}$ where $\mathbf{x} = x_i' \mathbf{e}_i'$ etc, would have found

$$\begin{aligned}I_{ij}' &= \int_{\mathcal{V}'} \rho(\mathbf{x})(x_k' x_k' \delta_{ij} - x_i' x_j')\,\mathrm{d}V \\ &= R_{ip}R_{jq} \int_{\mathcal{V}} \rho(\mathbf{x})(x_k x_k \delta_{pq} - x_p x_q)\,\mathrm{d}V \\ &= R_{ip}R_{jq}I_{pq}\end{aligned}$$

where $\mathcal{V}' = \{x_i' : x_i \mathbf{e}_i' \in V\}$. So I_{ij} is a rank 2 tensor. It is symettric, $I_{ij} = I_{ji}$.

Example. Consider ellipsoid described by



$$\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} + \frac{x_3^2}{c^2} \leq 1$$

with uniform mass density ρ_0 so mass is

$$M = \rho_0 \frac{4\pi}{3} abc$$

To compute components of inertia in this frame, use scaled spherical polars to compute integrals.

$$x_1 = ar \cos\phi \sin\theta \ 0 \leq \phi < 2\pi$$

$$x_2 = br \sin\phi \sin\theta \ 0 \leq \theta \leq \pi$$

$$x_3 = cr \cos\theta \ 0 \leq r \leq 1$$

Note that if $i \neq j$ then

$$\int_v \rho_0 x_i x_j = 0 \text{ by symmetry}$$

Example (cont.)**.** Also

$$
\begin{aligned}
I_{11} = \rho_0 \int_V (x_2^2 + x_3^2)\,\mathrm{d}V \\
&= \rho_0 abc \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \int_{r=0}^{1} r^2 b^2 \sin^2\phi \sin^2\theta + c^2\cos\theta) r^2 \sin\theta \,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi \\
&= \rho_0 \frac{abc}{5} \int_0^{\pi} (\pi b^2 \sin^2\theta + 2\pi c^2 \cos\theta)\sin\theta\,\mathrm{d}\theta \\
&= \frac{3M}{4}\frac{1}{5} \int_0^{\pi} (b^2\sin^2\theta + (2c^2 - b^2)\cos^2\theta\sin\theta)\,\mathrm{d}\theta \\
&= \frac{3M}{20}\left(2b^2 + \frac{2}{3}(2c^2 - b^2)\right) \\
&= \frac{M}{5}(b^2 + c^2)
\end{aligned}
$$

By symmetry

$$I_{22} = \frac{M}{5}(a^2 + c^2),\ I_{33} = \frac{M}{5}(a^2 + b^2)$$

i.e.

$$(I_{ij}) = \frac{M}{5}\begin{bmatrix} b^2 + c^2 & 0 & 0 \\ 0 & a^2 + c^2 & 0 \\ 0 & 0 & a^2 + b^2 \end{bmatrix}$$

If $a = b = c$:

$$I_{ij} = \frac{2}{5} M \delta_{ij}$$

Prop. If T_{ij} is symmetric then there exist choice of $\{\mathbf{e}_i\}$ for which

$$(T_{ij}) = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0\gamma & \end{bmatrix}$$

The corresponding coordinate aces are called the principal axes of the tensor.

Proof. Direct consequence of the fact that any real symmetric matrix can be diagonalised via orthogonal transformation R for which $\det(R) = 1$ WLOG.

$$[T' = R^T T R] \text{ see IA V+M}$$

Moral. So can always choose set of axes so that I_{ij} is diagonal.

8.7 Invariant and Isotropic Tensors

Definition. We say that a tensor is **isotropic** if it is invariant under changes in Cartesian coords, i.e.

$$T'_{ij\dots k} = R_{ip}R_{jq}\dots R_{kr}T_{pq\dots r} \qquad = T_{ij\dots k}$$

for any choice of rotation R.

Example.

(i) Every scalar (rank 0 tensor) is isotropic

(ii) The Kronecker delta is isotropic

$$\begin{aligned}\delta'_{ij} &= R_{ip}R_{jq}\delta_{pq} \\ &= R_{ip}R_{jp} \\ &= \delta_{ij}\end{aligned}$$

(iii) The Levi-Civita tensor

$$\varepsilon'_{ijk} = R_{ip}R_{jq}R_{kr}\varepsilon_{pqr} = \det(R)\varepsilon_{ijk} = \varepsilon_{ijk}$$

Remark. We can actually classify ALL isotropic tensors on $\mathbb{R}^3$ [General result: Herman Weyls: The Classical Groups]

Prop. Isotropic tensors on $\mathbb{R}^3$ are classified as:

(i) All rank 0 tensors isotropic

(ii) There are no non-zero rank 1 tensors

(iii) The most general isotropic tensor of rank 2 is $\alpha\delta_{ij}$ (α scalar)

(iv) The most general isotropic tensor of rank 3 is $\beta\varepsilon_{ijk}$ (β scalar)

(v) The most general isotropic tensor of rank 4 is

$$\alpha\delta_{ij}\delta_{kl} + \beta\delta_{ik}\delta_{jl} + \gamma\delta_{il}\delta_{jk}$$

(vi) The most general isotropic tensor of rank >4 is a linear combination of products of δ and ε (e.g. $\delta_{ij}\varepsilon_{klm}$

Proof (Sketch)**.**

(i) By definition

(ii) If v_i are components of an isotropic tensor of rank 1 then

$$v_i = R_{ij}v_j = v_i'$$

holds for ANY rotation. Take

$$(R_{ij}) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \ \pi \text{ about } z\text{-axis}$$

then:

$$v_1 = R_{1j}v_j = -v_1$$

$$v_2 = R_{2j}v_j = -v_2$$

i.e. $v_1 = v_2 = 0$. Using

$$(R_{ij}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \ \pi \text{ about } x\text{-axis}$$

then

$$v_3 = R_{3j}v_j = -v_3$$

i.e. $v_3 = 0$ so $v_i = 0$ and this holds in all frames.

Prop.

Proof.

(iii) If T_{ij} isotropic then

$$T_{ij} = R_{ip}R_{jq}T_{pq}$$

holds for ANY R. Take R to be rotation by $\pi/2$ about each axis.

$$(R_{ij}) = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Then

$$T_{13} = R_{1p}R_{3q}T_{pq} = R_{12}R_{33}T_{23} = T_{23}$$

$$T_{23} = R_{2p}R_{3q}T_{pq} = R_{21}R_{33}T_{13} = -T_{13}$$

So

$$T_{13} = T_{23} = 0$$

Also

$$T_{11} = R_{1p}R_{1q}R_{pq} = R_{12}R_{12}T_{22} = T_{22}$$

i.e. $T_{11} = T_{22}$

Now choosing

$$(R_{ij} = \begin{bmatrix} 1 & 10 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix}$$

Then

$$T_{32} = R_{3p}R_{2q}T_{pq} = R_{32}R_{23}T_{23} = -T_{23}$$

So

$$T_{32} = 0$$

$$T_{12} = R_{1p}R_{2q}T_{pq} = R_{11}R_{23}T_{13} = -T_{13} = 0$$

$$T_{12} = 0$$

$$T_{31} = R_{3p}R_{1q}T_{pq} = R_{32}R_{11}T_{21} = -T_{21}$$

$$T_{21} = R_{2p}R_{1q}T_{pq} = R_{23}R_{11}T_{31}$$

i.e.

$$T_{31} = T_{21} = 0$$

.

Finally

$$T_{22} = R_{2p}T_{pq} = R_{2323}T_{33} = T_{33}$$

i.e.

$$T_{22} = T_{33} = T_{11}$$

In conclusion $T_{ij} = 0$ if $i \neq j$ and $T_{11} = T_{22} = T_{33}$. So

$$T_{ij} = \alpha\delta_{ij}$$

for some scalar α

(iv) Same idea, more indices. □

Method. Consider integral of form

$$T_{ij\dots k} = \int_{|\mathbf{x}|<R} = f(r) x_i x_j \dots x_k \,\mathrm{d}V(\mathbf{x})$$

where $x_k x_k = r^2$ and $V(\mathbf{x}) = \mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}x_3$.
Note $f(r)$ and $\{\mathbf{x} : |\mathbf{x}| < R\}$ are invariant under rotations.
We have:

$$\begin{aligned} T_{ij\dots k} &= \int_{|\mathbf{x}|<R} f(r) x'_i x'_j \dots x'_k \underbrace{\mathrm{d}V(\mathbf{x})}_{\mathrm{d}x'_1\,\mathrm{d}x'_2\,\mathrm{d}x'_3} \\ &= \int_{|\mathbf{x}|<R} f(r) R_{ip} x_p R_{jq} x_q \dots R_{kr} x_r \,\mathrm{d}V(\mathbf{x}) \end{aligned}$$

Make substitution $y_i = R_{ij} x_j$, $\mathrm{d}V = \mathrm{d}y_1\,\mathrm{d}y_2\,\mathrm{d}y_3$

$$T'_{ij\dots k} = \int_{|\mathbf{x}|<R} f(r) y_i y_i \dots y_k \,\mathrm{d}V(\mathbf{y})$$

Sine $\{y\}$ is dummy variable

$$T'_{ij\dots k} = \int_{\mathbf{x}|<R} f(r) x_i x_j \dots x_k \,\mathrm{d}V(\mathbf{x}) = T_{ij\dots k}$$

So $T_{ij\dots k}$ is isotropic!
Take $R \to \infty$ corresponds to integrating over all $\mathbb{R}^3$.

Example. Consider

$$T_{ij} = \int_{\mathbb{R}^3} e^{-r^5} x_i x_j \,\mathrm{d}V$$

By previous, $T_{ij} = \alpha \delta_{ij}$. Contracting on (i, j)

$$\begin{aligned} \alpha \delta_{ii} = 3\alpha &= \int_{\mathbb{R}^3} e^{-r^5} r^2 \,\mathrm{d}V \\ &= 4\pi \int_0^\infty r^2 e^{-r^5} r^2 \,\mathrm{d}r \\ &= 4\pi \int_0^\infty \frac{1}{5} \frac{\mathrm{d}}{\mathrm{d}r} \left(e^{-r^5} \right) \mathrm{d}r \\ &= \frac{4\pi}{5} \end{aligned}$$

i.e. $\alpha = \frac{4\pi}{15}$ and

$$T_{ij} = \frac{4\pi}{15} \delta_{ij}$$

Example. The inertia tensor of ball of radius R, constant density ρ_0 [mass $M = \frac{4\pi}{3}R^3\rho_0$]

$$I_{ij} = \int_{|\mathbf{x}|<R} \rho_0(x_k x_k \delta_{ij} - x_i x_j)\,\mathrm{d}V$$

This is sum of two isotropic tensors, hence

$$I_{ij} = \alpha\delta_{ij} \text{ for some } \alpha$$

Contracting on (i, j)

$$\begin{aligned}
3\alpha &= \int_{|\mathbf{x}|<R} \rho_0[3r^2 - r^2]\,\mathrm{d}V \\
&= 4\pi\rho_0 \cdot 2\int_0^R r^4\,\mathrm{d}r \\
&= \left[\frac{4\pi}{3}\rho_0 R^4\right]\frac{3}{R^3} \cdot 2 \cdot \frac{R^5}{5} \\
&= \frac{6MR^2}{5}
\end{aligned}$$

So $\alpha = \frac{2MR^2}{5}$ and

$$I_{ij} = \frac{2M}{5}R^2\delta_{ij}$$

8.8 Tensors as Multi-Linear Maps and the Quotient Rule

Method. For a tensor T_{ij} consider bilinear map $t : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ defined by

$$t(\mathbf{a}, \mathbf{b}) := T_{ij}a_i b_j$$

LHS well defined since RHS does not depend on which basis we use (it's a scalar).
So rank two tensor gives rise to bilinear map.
Conversely, suppose $t : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ is bilinear, then for a given basis $\{\mathbf{e}_i\}$ it defines an array T_{ij} via

$$\begin{aligned}
t(\mathbf{a}, \mathbf{b}) &= t(a_i\mathbf{e}_i, b_j\mathbf{e}_j) \\
&= a_i b_j t(\mathbf{e}_i, \mathbf{e}_j) \\
&:= a_i b_j T_{ij}
\end{aligned}$$

If we use different basis $\{\mathbf{e}'_i\}$ with $\mathbf{e}'_i = R_{ip}\mathbf{e}_p$ then by linearity

$$\begin{aligned}
T'_{ij} &= t(\mathbf{e}'_i, \mathbf{e}'_j) \\
&= t(R_{ip}\mathbf{e}_p, R_{jq}\mathbf{e}_q) \\
&= R_{ip}R_{jq}t(\mathbf{e}_p, \mathbf{e}_q) \\
&= R_{ip}R_{jq}T_{pq}
\end{aligned}$$

So T_{ij} is rank 2 tensor I.e. bilinear map t gives rise to rank 2 tensor.

Moral. Have a one-to-one correspondence between bilinear maps and rank 2 tensors. In particular if the map

$$(\mathbf{a}, \mathbf{b}) \mapsto T_{ij} a_i b_j$$

is genuinely bilinear, independent of basis, then T_{ij} are components of rank 2 tensor.

Remark. Same idea works for higher rank tensors: if the map

$$(\mathbf{a}, \mathbf{b}, \dots, \mathbf{c}) \mapsto T_{ij\dots k} a_i b_j \dots c_k$$

genuinely defines a n-multilinear map (indep of basis) then $T_{ij\dots k}$ are components of rank n tensor.

Note. Recall from earlier that we showed σ_{ij} (conductivity tensor) was tensor from definition

$$J_i = \sigma_{ij} E_j$$

Could have used quotient theorem.

Prop. Let $T_{i\dots jp\dots q}$ be an array of numbers defined in each Cartesian coord system such that

$$\underbrace{v_{i\dots j}}_{A} := \underbrace{T_{i\dots jp\dots q}}_{A+B} \underbrace{u_{p\dots q}}_{B}$$

is a tensor for each tensor $u_{p\dots q}$. Then $T_{i\dots jp\dots q}$ is a tensor.

Proof. Take special case $u_{p\dots q} = c_p \dots d_q$ for vectors $\{\mathbf{c}, \dots, \mathbf{d}\}$. Then

$$v_{i\dots j} := T_{i\dots jp\dots q} c_p \dots d_q$$

is a tensor and in particular

$$v_{i\dots j} a_i \dots b_j = T_{i\dots jp\dots q} a_i \dots b_j c_p \dots d_q$$

is a scalar for each $\{\mathbf{a}, \dots \mathbf{b}, \mathbf{c}, \dots, \mathbf{d}\}$. So RHS is scalra (indep of basis) and gives rise to well-defined multilinear map via

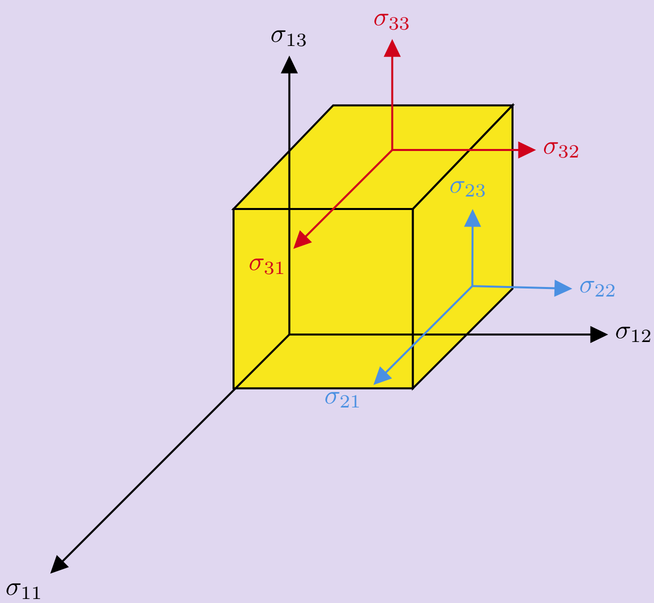
$$t(\mathbf{a}, \dots, \mathbf{b}, \mathbf{c}, \dots, \mathbf{d}) := T_{i\dots jp\dots q} a_i \dots b_j c_p \dots d_q$$

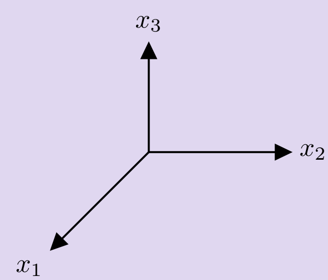
so by previous discussion, $T_{i\dots jp\dots q}$ is a tensor. □

Example. Seen linear strain tensor

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

where $\mathbf{u}(\mathbf{x})$ measures change in displacement at $\mathbf{x}$
Experiment suggests that the internal forces experiences by a body that has undergone deformation depend linearly on strain at each point.
Stresses are described by a stress tensor σ_{ij}





(shows stress in each direction on 3 faces)
So $\exists$ an array of $3^4 = 81$ numbers c_{ijkl} such that

$$\sigma_{ij} = c_{ijkl} e_{kl} \tag{†}$$

Warning. CAN'T APPLY QUOTIENT THEOREM at this point as e_{kl} symmetric

If $c_{ijkl} = c_{ijlk}$ then can apply quotient theorem (ES4) - call this the stiffness tensor (it is a property of the material under deformation). Suppose our material is isotropic, then we should write

$$c_{ijkl} = \lambda \delta_{ij}\delta_{kl} + \beta \delta_{ik}\delta_{jl} + \gamma \delta_{il}\delta_{kl}$$

Use this in (†)

$$\sigma_{ij} = \lambda \delta_{ij} e_{kk} + \beta e_{ij} + \gamma_{ji} \qquad = \lambda \delta_{ij} e_{kk} + 2\mu e_{ij}$$

where $2\mu = \beta + \gamma$, This is higher dimension version of Hooke's law ($F = -kx$).
Can invert - contract on (i, j)

$$\sigma_{ii} = (3\lambda + 2\mu) e_{ii}$$

i.e.

$$e_{kk} = \frac{\sigma_{kk}}{3\lambda + 2\mu} \ (3\lambda + 2\mu \neq 0)$$

So we get:

$$2\mu e_{ij} = \sigma_{ij} - \left(\frac{\lambda}{3\lambda + 2\mu}\right) \sigma_{kk} \delta_{ij}$$