Vector Calculus Summary

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Lent 2021

Contents

1 Differential Geometry of curves

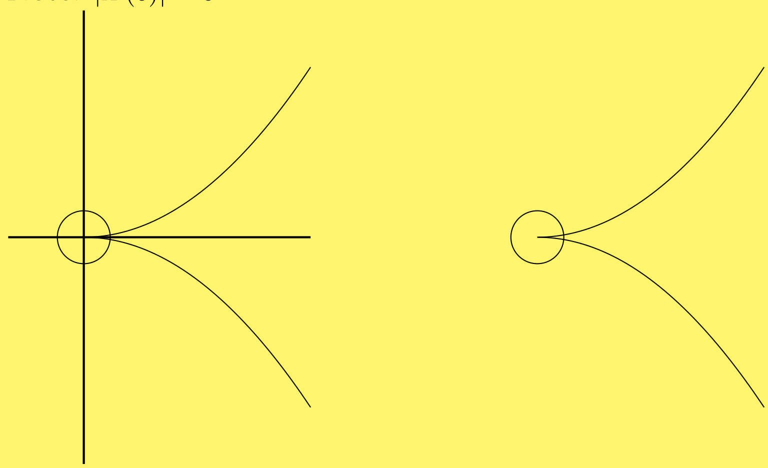
1.1 Parametrised Curves and Arc Length

Definition. We say curve C is **regular** if $|\mathbf{x}'(t)| \neq 0$

Definition. If C is differentiable and regular, say C is **smooth**

Remark. Why "regular" condition?
Consider $\mathbf{x}(t) = (t^2, t^3)$. Clearly differentiable but $\mathbf{x}(t)$ has cusp at $t = 0$.

Note. $|\mathbf{x}'(0)| = 0$



Equation. if $C : t \mapsto x(t),\ t \in [a, b]$

$$l(C) = \int_a^b |\mathbf{x}'(t)|\,\mathrm{d}t$$
$$= \int_C \mathrm{d}s$$

$$\mathrm{d}s = |\mathbf{x}'(t)|\,\mathrm{d}t$$

s is the "arc-length element"
Similarly define

$$\int_C f(\mathbf{x})\,\mathrm{d}s = \int_a^b f(\mathbf{x}(t))|\mathbf{x}'(t)|\,\mathrm{d}t$$

Remark. Suppose C has two different parametrisations:

$$\mathbf{x} = \mathbf{x}_1(t),\ a \le t \le b$$

$$\mathbf{x} = \mathbf{x}_2(\tau),\ \alpha \le t \le \beta$$

Must have $\mathbf{x}_2(\tau) = \mathbf{x}_1(t(\tau))$ for some function $t(\tau)$. Assume $\frac{\mathrm{d}t}{\mathrm{d}\tau} \neq 0$ so map between t and τ invertible and differentiable. Note

$$\mathbf{x}_2'(\tau) = \frac{\mathrm{d}t}{\mathrm{d}\tau}\mathbf{x}_1'(t(\tau))$$

From definitions,

$$\int_C f(\mathbf{x})\,\mathrm{d}s = \int_a^b f(\mathbf{x}(t))|\mathbf{x}'(t)|\,\mathrm{d}t$$

Make substitution $t = t(\tau)$, and assume $\frac{\mathrm{d}t}{\mathrm{d}\tau} > 0$, latter integral becomes

$$\int_\alpha^\beta f(\mathbf{x}_2(\tau))\,\underbrace{|\mathbf{x}_1'(t(\tau)|\frac{\mathrm{d}t}{\mathrm{d}\tau}\,\mathrm{d}\tau}_{|\mathbf{x}_2'(\tau)|\,\mathrm{d}\tau}$$

Which is precisely the same as $\int_C f(\mathbf{x})\mathrm{d}s$ using $\mathbf{x}_2(\tau)$ parametrisation. Similar holds when $\frac{\mathrm{d}t}{\mathrm{d}\tau} < 0$ (exercise). So definition of $\int_C f(\mathbf{x})\,\mathrm{d}s$ does not depend on choice of parametrisation of C.

Definition. The **arc-length function** for a curve $[a, b] \ni t \mapsto \mathbf{x}(t)$ by

$$s(t) = \int_a^t |\mathbf{x}'(\tau)|\,\mathrm{d}\tau$$

So $s(a) = 0$ and $s(b) = l(c)$.
Also:

$$\frac{\mathrm{d}s}{\mathrm{d}t} = |\mathbf{x}'(t)| \ge 0$$

Note. For regular curves have $\frac{\mathrm{d}s}{\mathrm{d}t} > 0$, so can invert relationship between s and t to find

$$t = t(s)$$

So we can parametrise regular curves wrt arc-length, If we write $\mathbf{r}(s) = \mathbf{x}(t(s))$ where $0 \le s \le l(C)$, then by chain rule:

$$\frac{\mathrm{d}t}{\mathrm{d}s} = \frac{1}{|\mathbf{x}'(t(s))|}$$

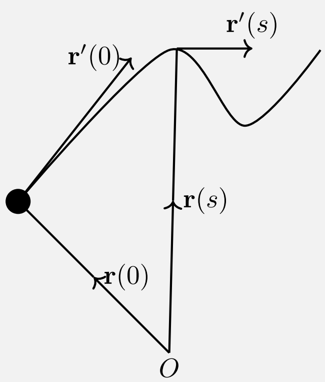
So

Equation.

$$\mathbf{r}'(s) = \frac{\mathbf{x}'(t(s))}{|\mathbf{x}'(t(s))|}$$

i.e. $|\mathbf{r}'(s)| = 1$. This (consistently) gives

$$l(C) = \int_0^{l(C)} |\mathbf{r}'(s)| \, \mathrm{d}s = \int_0^{l(C)} \mathrm{d}s \checkmark$$



1.2 Curvature and Torsion

Note. Throughout this section talk about generic regular curve C parametrised by arc-length, write $s \mapsto \mathbf{r}(s)$

Definition. Tangent vector

$$\mathbf{t}(s) = \mathbf{r}'(s)$$

Already know $|\mathbf{t}(s)| = 1$. Since $|\mathbf{t}(s)|$ doesn't change, the second dervative $\mathbf{r}''(s) = \mathbf{t}'(s)$ only measures change in direction

So intuitively, if $|\mathbf{r}''(s)|$ is large then curve rapidly changes direction, whereas if $|\mathbf{r}''(s)|$ is small, expect curve to be approximately flat.

Definition. The curvature

$$\kappa(s) = |\mathbf{r}''(s)| = |\mathbf{t}'(s)|$$

Equation.

$$\mathbf{t} \cdot \mathbf{t}' = 0$$

Definition. The **principle normal** is defined by the formula

$$\mathbf{t}'(s) = \kappa \mathbf{n}$$

$\mathbf{n}$ is the principle normal

Note. $\mathbf{n}$ is everywhere normal to C since

$$\mathbf{t} \cdot \mathbf{n} = 0$$

Definition. Can extend $\{\mathbf{t}, \mathbf{n}\}$ to orthonormal basis by defining the **binormal**

$$\mathbf{b} = \mathbf{t} \times \mathbf{n}$$

Since $|\mathbf{b}| = 1$ have $\mathbf{b}' \cdot \mathbf{b} = 0$. Also since $\mathbf{t} \cdot {=}0$ and $\mathbf{n} \cdot \mathbf{b} = 0$

$$\begin{aligned} 0 = (\mathbf{t} \cdot \mathbf{b})' &= \mathbf{t}' \cdot \mathbf{b} + \mathbf{t} \cdot \mathbf{b}' \\ &= \underbrace{\kappa \mathbf{n} \cdot \mathbf{b}}_{=0} + \mathbf{t} \cdot \mathbf{b}' \end{aligned}$$

So $\mathbf{b}'$ is orthogonal to both $\mathbf{t}$ and $\mathbf{b}$ i.e. it is parallel to $\mathbf{n}$.

Definition. The **torsion** of a curve is defined by the formula

$$\mathbf{b}' = -\tau \mathbf{n}$$

τ is the torsion

Remark. Have two equations

$$\mathbf{t}' = \kappa \mathbf{n}, \ \mathbf{b}' = -\tau \mathbf{n}$$

Prop. The curvature $\kappa(s)$ and torsion $\tau(s)$ define a curve up to translation/ orientation.

Proof. Since $\mathbf{n} = \mathbf{b} \times \mathbf{t}$, have two coupled equations:

$$\mathbf{t}' = \kappa(\mathbf{b} \times \mathbf{t})$$

$$\mathbf{b}' = -\tau(\mathbf{b} \times \mathbf{t})$$

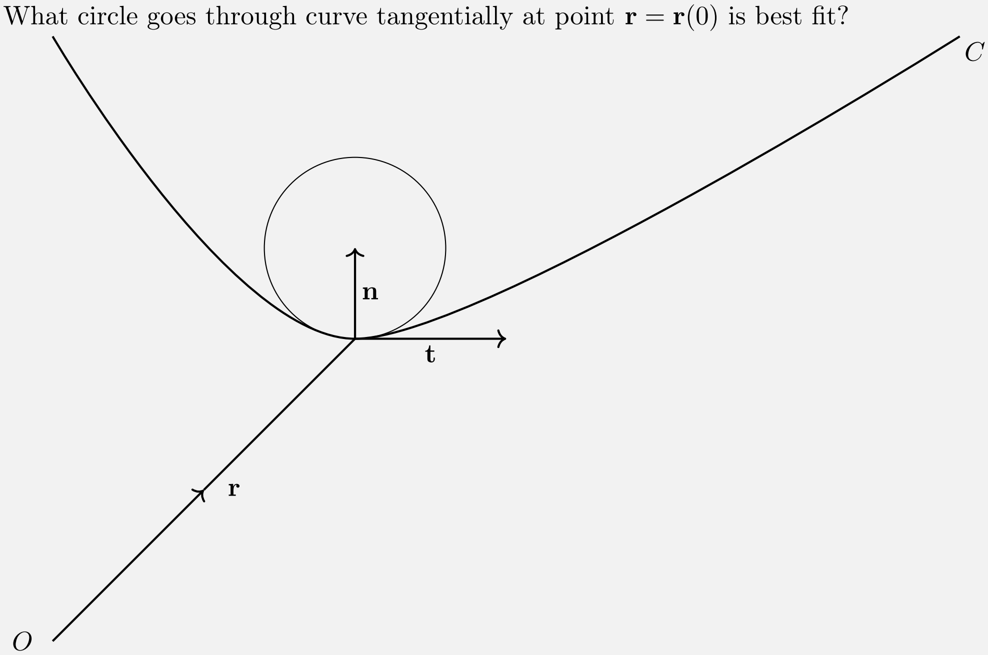
This gives six equations for six unknowns.
Given $\kappa(s)$, $\tau(s)$, $\mathbf{t}(0)$, $\mathbf{b}(0)$, can construct $\mathbf{t}(s)$, $\mathbf{b}(s)$ and hence $\mathbf{n} = \mathbf{b} \times \mathbf{t}$. Hence result $\square$

1.3 Radius of Curvature

Taylor expand a generic curve $s \mapsto \mathbf{r}(s)$ about $s = 0$. Write $\mathbf{t} = \mathbf{t}(0)$, $\mathbf{n} = \mathbf{n}(0)$ etc.

$$\begin{aligned}\mathbf{r}(s) &= \mathbf{r}(0) + s\mathbf{r}'(0) + \frac{1}{2}s^2\mathbf{r}''(0) + o(s^2) \\ &= \mathbf{r} + s\mathbf{t} + \frac{1}{2}s^2\kappa\mathbf{n} + o(s^2)\end{aligned}$$

Suppose, WLOG, that $\mathbf{t}$ is horizontal.
What circle goes through curve tangentially at point $\mathbf{r} = \mathbf{r}(0)$ is best fit?



Equation of circle

$$\mathbf{x}(\theta) = \mathbf{r} + R(1 - \cos\theta)\mathbf{n} + R\sin\theta\mathbf{t}$$

Expand for $|\theta|$ small

$$\mathbf{x}(\theta) = \mathbf{r} + R\theta\mathbf{t} + \frac{1}{2}R\theta^2\mathbf{n} + o(\theta^2)$$

Arc length on circle is $s = R\theta$. So

$$\mathbf{x}(\theta) = \mathbf{r} + s\mathbf{t} + \frac{1}{2}\frac{1}{R}s^2\mathbf{n} + o(s^2)$$

To match equation for curve up to second order, would require

$$R = \frac{1}{\kappa}$$

Definition. We say $R(s) = \frac{1}{\kappa(s)}$ is the **radius of curvature** of curve $s \mapsto \mathbf{r}(s)$

2 Coordinates, Differentials + Gradients

2.1 Differentials + First Order Changes

Definition. The **differential** of f, written $\mathrm{d}f$, by

$$\mathrm{d}f = \frac{\partial f}{\partial u_i}\,\mathrm{d}u_i$$

Call $\{\mathrm{d}u_i\}$ **differential forms**. These are L.I. if $\{u_1, \dots, u_n\}$ are independent.
Similarly, if $\mathbf{x} = \mathbf{x}(u_1, \dots, u_n)$ we define

$$\mathrm{d}\mathbf{x} = \frac{\partial \mathbf{x}}{\partial u_i}\,\mathrm{d}u_i$$

2.2 Coordinates and Line Elements

Definition. The **line element** is:

$$\mathrm{d}\mathbf{x} = \frac{\partial \mathbf{x}}{\partial u_1}\,\mathrm{d}u_1 + \frac{\partial \mathbf{x}}{\partial u_2}\,\mathrm{d}u_2$$

It tells us how small changes in coord produce changes in position vectors.

For polars (r, θ)

$$\mathbf{x}(r,\theta) = \begin{bmatrix} r\cos\theta \\ r\sin\theta \end{bmatrix} \equiv r\mathbf{e}_r$$

where we have used basis vectors $\{\mathbf{e}_2, \mathbf{e}_\theta\}$

$$\mathbf{e}_r = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}, \ \mathbf{e}_\theta = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$

Warning. $\{\mathbf{e}_r, \mathbf{e}_\theta\}$ are orthonormal at each (r, θ), but NOT the same for each (r, θ)

Note. As before,

$$\mathbf{e}_r = \frac{\frac{\partial}{\partial r}\mathbf{x}(r,\theta)}{|\frac{\partial}{\partial r}\mathbf{x}(r,f)|}, \ \mathbf{e}_\theta = \frac{\frac{\partial}{\partial \theta}\mathbf{x}(r,\theta)}{|\frac{\partial}{\partial \theta}\mathbf{x}(r,\theta)|}$$

Since $\{\mathbf{e}_r, \mathbf{e}_\theta\}$ are orthogonal, makes sense to call (r, θ) orthogonal curvilinear coordinates.

For polars, have line element

$$\begin{aligned}\mathrm{d}\mathbf{x} &= \frac{\partial \mathbf{x}}{\partial r}\,\mathrm{d}r + \frac{\partial \mathbf{x}}{\partial \theta}\,\mathrm{d}\theta \\ &= \mathbf{e}_r\,\mathrm{d}r + r\,\mathrm{d}\theta\,\mathbf{e}_\theta\end{aligned}$$

See that a change $\theta \mapsto \theta + \delta\theta$ produces a (first order) change

$$\mathbf{x} \mapsto \mathbf{x} + r\delta\theta\,\mathbf{e}_\theta$$

Warning. NOT $\mathbf{x} \mapsto \mathbf{x} + \delta\theta\,\mathbf{e}_\theta$

2.2.1 Orthogonal Curvilinear Coordinates

Definition. We say that (u, v, w) are a **set of orthogonal curvilinear coords** if the vectors

$$\mathbf{e}_u = \frac{\frac{\partial \mathbf{x}}{\partial u}}{\left|\frac{\partial \mathbf{x}}{\partial u}\right|},\ \mathbf{e}_v = \frac{\frac{\partial \mathbf{x}}{\partial v}}{\left|\frac{\partial \mathbf{x}}{\partial v}\right|},\ \mathbf{e}_w = \frac{\frac{\partial \mathbf{x}}{\partial w}}{\left|\frac{\partial \mathbf{x}}{\partial w}\right|}$$

form a right-handed handed basis for each (u, v, w)

Note. Right handed means $\mathbf{e}_u \times \mathbf{e}_v = \mathbf{e}_w$

Warning. Just as with polar coordinates, $\{\mathbf{e}_u, \mathbf{e}_v, \mathbf{e}_w\}$ form orthonormal basis for $\mathbb{R}^3$ at each (u, v, w), but NOT necessarily the same basis at each point.

Notation. It is standard to write

$$h_u = \left|\frac{\partial \mathbf{x}}{\partial u}\right|, h_v = \left|\frac{\partial \mathbf{x}}{\partial v}\right|, h_w = \left|\frac{\partial \mathbf{x}}{\partial w}\right|$$

Definition. Call $\{h_u, h_v, h_w\}$ **scale factors**

Note. Line element is

$$\begin{aligned}\mathrm{d}\mathbf{x} &= \frac{\partial \mathbf{x}}{\partial u}\,\mathrm{d}u + \frac{\partial \mathbf{x}}{\partial v}\,\mathrm{d}v + \frac{\partial \mathbf{x}}{\partial w}\,\mathrm{d}w \\ &= h_u\mathbf{e}_u\,\mathrm{d}u + h_v\mathbf{e}_v\,\mathrm{d}v + h_w\mathbf{e}_w\,\mathrm{d}w\end{aligned}$$

Tells us how small changes in coordinates “scale-up” to changes in position $\mathbf{x}$

2.2.2 Cylindrical Polar Coords

Definition. **Cyclindrical polars** (ρ, ϕ, z) defined by:

$$\mathbf{x}(\rho, \phi, z) = \begin{bmatrix} \rho \cos\phi \\ \rho \sin\phi \\ z \end{bmatrix}$$

with:

$$0 \leq \rho < \infty$$

$$0 \leq \phi < 2\pi$$

$$-\infty < z < \infty$$

Find

$$\mathbf{e}_\rho = \begin{bmatrix} \cos\phi \\ \sin\phi \\ 0 \end{bmatrix}, \ \mathbf{e}_\phi \begin{bmatrix} -\sin\phi \\ \cos\phi \\ 0 \end{bmatrix}$$

$$\mathbf{e}_z = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$h_\rho = 1, \ h_\phi = \rho, \ h_z = 1$$

$$\mathrm{d}\mathbf{x} = \mathrm{d}\rho \, \mathbf{e}_\rho + \rho \, \mathrm{d}\phi \, \mathbf{e}_\phi + \mathrm{d}z \, \mathbf{e}_z$$

Note.

$$\mathbf{x} = \rho \, \mathbf{e}_\rho + z \, \mathbf{e}_z$$

Warning. STILL DEPENDENT ON ϕ AS $\mathbf{e}_\rho$ DEPENDS ON ϕ

2.2.3 Spherical Polar Coordinates

Definition. **Spherical polars** (r, θ, ϕ) defined by:

$$\mathbf{x}(r, \theta, \phi) = \begin{bmatrix} r \cos\phi \sin\theta \\ r \sin\phi \sin\theta \\ r \cos\theta \end{bmatrix}$$

with:

$$0 \leq r < \infty$$

$$0 \leq \theta \leq \pi$$

$$0 \leq \phi < 2\pi$$

$$\mathbf{e}_r = \begin{bmatrix} \cos\phi\sin\theta \\ \sin\phi\sin\theta \\ \cos\theta \end{bmatrix}, \ \mathbf{e}_\theta = \begin{bmatrix} \cos\phi\cos\theta \\ \sin\phi\cos\theta \\ -\sin\theta \end{bmatrix}$$

$$\mathbf{e}_\phi = \begin{bmatrix} -\sin\phi \\ \cos\phi \\ 0 \end{bmatrix}$$

$$h_r = 1, \ h_\theta = r, \ h_\phi = r\sin\theta$$

i.e.

$$\mathrm{d}\mathbf{x} = \mathrm{d}r\,\mathbf{e}_r + r\,\mathrm{d}\theta\,\mathbf{e}_\theta + r\sin\theta\,\mathrm{d}\phi\,\mathbf{e}_\phi$$

Note.

$$\mathbf{x} = r\begin{bmatrix} \cos\phi\sin\theta \\ \sin\phi\sin\theta \\ \cos\theta \end{bmatrix} = r\,\mathbf{e}_r$$

Warning. STILL DEPENDENT ON ϕ, θ AS $\mathbf{e}_r$ DEPENDS ON ϕ, θ

2.3 Gradient Operator

Definition. For $f : \mathbb{R}^3 \to \mathbb{R}$, define **gradient** of f, written ∇f, by

$$f(\mathbf{x} + \mathbf{h}) = f(\mathbf{x}) + \nabla f(\mathbf{x}) \cdot \mathbf{h} + o(\mathbf{h}) \tag{*}$$

Definition. **Directional derivative** of f in direction $\mathbf{v}$, denoted by $D_\mathbf{v} f$ or $\frac{\partial f}{\partial \mathbf{v}}$, is defined by

$$\frac{\partial f}{\partial \mathbf{v}} = \lim_{t\to 0} \frac{f(\mathbf{x} + t\mathbf{v}) - f(\mathbf{x})}{t}$$

I.e.

$$f(\mathbf{x} + t\mathbf{v}) = f(\mathbf{x}) + tD_\mathbf{v} f(\mathbf{x}) + o(t) \tag{**}$$

Equation. Setting $\mathbf{h} = t\mathbf{v}$ in (*)

$$f(\mathbf{x} + t\mathbf{v}) = f(\mathbf{x}) + t\nabla f(\mathbf{x}) \cdot \mathbf{v} + o(t)$$

Comparing to previous equation (**), we have:

$$\frac{\partial f}{\partial \mathbf{v}} = \mathbf{v} \cdot \nabla f$$

Note. By Cauchy-Schwarz know that $\mathbf{a} \cdot \mathbf{b}$ is maximised when $\mathbf{a}$ points in same direction as $\mathbf{b}$.

So ∇f points in direction of greatest increase of f
Similarly,
$-\nabla f$ points in direction of greatest decrease of f

Equation. Suppose we have a curve $t \mapsto \mathbf{x}(t)$. How does f change as we move along this curve. Write

$$F(t) = f(\mathbf{x}(t))$$

$$\frac{\mathrm{d}F}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t} f(\mathbf{x}(t)) = \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} \cdot \nabla f(\mathbf{x}(t))$$

2.4 Computing the gradient

Equation.

$$\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{bmatrix}$$

Equation.

$$\nabla f \cdot \mathrm{d}\mathbf{x} = \mathrm{d}f$$

Note. Coordinate independent statement!

Prop. If (u, v, w) are O.C.C and $f = f(u, v, w)$,

$$\nabla f = \frac{1}{h_u} \frac{\partial f}{\partial u} \mathbf{e}_u + \frac{1}{h_v} \frac{\partial f}{\partial v} \mathbf{e}_v + \frac{1}{h_w} \frac{\partial f}{\partial u} \mathbf{e}_w$$

Proof. Use above equation and linear independence of $\{\mathrm{d}u, \mathrm{d}v, \mathrm{d}w\}$

Equation. In cyclindrical polars (ρ, ϕ, z), $h_\rho = 1$, $h_\phi = \rho$, $h_z = 1$

$$\nabla f = \frac{\partial f}{\partial \rho} \mathbf{e}_\rho + \frac{1}{\rho} \frac{\partial f}{\partial \phi} \mathbf{e}_\phi + \frac{\partial f}{\partial z} \mathbf{e}_z$$

Equation. In spherical polars (r, θ, ϕ), $h_r = 1$, $h_\theta = r$, $h_\phi = r \sin\theta$,

$$\nabla f = \frac{\partial f}{\partial r} \mathbf{e}_r + \frac{1}{r} \frac{\partial f}{\partial \theta} \mathbf{e}_\theta + \frac{1}{r \sin\theta} \frac{\partial f}{\partial \phi} \mathbf{e}_\phi$$

3 Integration over lines, surfaces and volumes

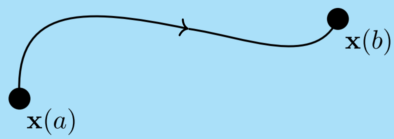
3.1 Line Integrals

Definition. For a vector field $\mathbf{F} = \mathbf{F}(\mathbf{x})$ and piecewise smooth parametrised curve

$$C : [a, b] \ni t \mapsto \mathbf{x}(t)$$

We define **line integral**

$$\int_C \mathbf{F} \cdot \mathrm{d}\mathbf{x} = \int_a^b \mathbf{F}(\mathbf{x}(t)) \cdot \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} \, \mathrm{d}t$$



Definition. We say a curve

$$[a, b] \ni t \mapsto \mathbf{x}(t)$$

is **closed** if $\mathbf{x}(a) = \mathbf{x}(b)$.
In this case, write

$$\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{x}$$

Sometimes call integrals of this form the circulation of $\mathbf{F}$ about C

3.2 Conservative Forces and Exact Differentials

We've seen how to interpret things like $\mathbf{F} \cdot \mathrm{d}\mathbf{x}$ when they're inside an integral. This is another differential form i.e. in coords (u, v, w)

$$\mathbf{F} \cdot \mathrm{d}\mathbf{x} = (\ )\mathrm{d}u + (\ )\mathrm{d}v + (\ )\mathrm{d}w$$

Definition. We say that $\mathbf{F} \cdot \mathrm{d}\mathbf{x}$ is **exact** if

$$\mathbf{F} \cdot \mathrm{d}\mathbf{x} = \mathrm{d}f$$

for some scalar f. Recall that

$$\mathrm{d}f = \nabla f \cdot \mathrm{d}\mathbf{x}$$

So $\mathbf{F} \cdot \mathrm{d}\mathbf{x}$ is exact iff $\mathbf{F} = \nabla f$ for some scalar f. Call such vector fields conservative.

Claim. So we have

$$\mathbf{F} \cdot \mathrm{d}\mathbf{x} \text{ is exact} \iff \mathbf{F} \text{ is conservative.}$$

Remark. Using properties $\mathrm{d}(\alpha f + \beta g) = \alpha \mathrm{d}f + \beta \mathrm{d}g$ (α, β) constant, $\mathrm{d}(fg) = g\mathrm{d}f + f\mathrm{d}g$ etc. usually easy to see if form $\mathbf{F} \cdot \mathrm{d}\mathbf{x}$ is exact

Prop. If θ is exact differential form then

$$\oint_C \theta = 0$$

for any closed curve C

Proof. By previous, if θ exact, then $\theta = \nabla f \cdot \mathrm{d}\mathbf{x}$ for some scalar f. Then substitute in integral, spot derivative and use FTC

Note. Equivalently, if $\mathbf{F}$ is conservative then circulation of $\mathbf{F}$ around any closed loop curve C vanishes

$$\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{x} = 0$$

Prop. If $\mathbf{F}$ conservative ($\mathbf{F} \cdot \mathrm{d}\mathbf{x}$ exact), then line integral between points $A = \mathbf{x}(a)$ and $B = \mathbf{x}(b)$ is independent of path

Proof. If $C = C_1 - C_2$,

$$\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{x} = 0$$

$$\iff \int_{C_1} \mathbf{F} \cdot \mathrm{d}\mathbf{x} = \int_{C_2} \mathbf{F} \cdot \mathrm{d}\mathbf{x}$$

Claim. Let $(u_1, u_2, u_3) \equiv (u, v, w)$ be set of OCC. Let

$$\begin{aligned}\mathbf{F} \cdot \mathrm{d}\mathbf{x} = \theta &= A(u,v,w)\,\mathrm{d}u + B(u,v,w)\,\mathrm{d}v + C(u,v,w)\,\mathrm{d}w \\ &= \theta_i\,\mathrm{d}u_i\end{aligned}$$

A necessary condition for θ to be exact is

$$\frac{\partial \theta_i}{\partial u_j} = \frac{\partial \theta_j}{\partial u_i} \text{ each } i, j \tag{$\dagger$}$$

Proof. Indeed, if θ exact, then $\theta = \mathrm{d}f$ which we can use to show the result

Definition. Call differential forms $\theta = \theta_i$ that obey (†) **closed**. So

$$\theta \text{ exact} \implies \theta \text{ closed}$$

Note. The reverse implication is true if the domain $\Omega \subseteq \mathbb{R}^3$ on which θ is defined is simply-connected.

3.3 Integration in $\mathbb{R}^2$

Method. If $f(x,y) = g(x)h(y)$ and D is a rectangle

$$D = \{(x,y) : a \leq x \leq b,\ c \leq y \leq d\}$$

Then

$$\int_A f(x,y)\,\mathrm{d}A = \left(\int_a^b g(x)\,\mathrm{d}x\right)\left(\int_c^d h(y)\,\mathrm{d}y\right)$$

Method. Often useful to introduce change of variables to compute

$$\int_a^b f(x)\,\mathrm{d}x$$

If we introduce $x = x(u)$ with $x(\alpha) = a$ and $x(\beta) = b$ then:

$$\int_a^b f(x)\,\mathrm{d}x = \begin{cases} +\int_\alpha^\beta f(x(u))\frac{\mathrm{d}x}{\mathrm{d}u}\,\mathrm{d}u\ (\beta > \alpha,\ \frac{\mathrm{d}x}{\mathrm{d}u} > 0) \\ -\int_\beta^\alpha f(x(u))\frac{\mathrm{d}x}{\mathrm{d}u}\,\mathrm{d}u\ (\alpha > \beta,\ \frac{\mathrm{d}x}{\mathrm{d}u} < 0) \end{cases}$$

If $I = [a,b]$ and $I' = x(I)$

$$\int_I f(x)\,\mathrm{d}x = \int_{I'} f(x(u))\left|\frac{\mathrm{d}x}{\mathrm{d}u}\right|\,\mathrm{d}u$$

Note. Similar formula in 2D

Prop. Let $x = x(u,v)$ and $y = y(u,v)$ be a smooth, invertible transformation with smooth inverse that maps the region D' in the (u,v) plane to the region D in the (x,y)-plane. Write $x = x(u,v)$, then

$$\iint_D f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \iint_{D'} f(x(u,v),y(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right|\,\mathrm{d}u\,\mathrm{d}v$$

Where

$$\frac{\partial(x,y)}{\partial(u,v)} = \det\begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} = \det\left[\frac{\partial \mathbf{x}}{\partial u}\middle|\frac{\partial \mathbf{x}}{\partial v}\right]$$

is the Jacobian, often denoted by J. Short version is $\mathrm{d}x\,\mathrm{d}y = |J|\,\mathrm{d}u\,\mathrm{d}v$

Equation.

$$\mathrm{d}x\,\mathrm{d}y = |J|\,\mathrm{d}u\,\mathrm{d}v$$

Example.

$$\mathrm{d}x\,\mathrm{d}y = \rho\,\mathrm{d}\rho\,\mathrm{d}\phi$$

3.4 Integration in $\mathbb{R}^3$

Method. to integrate over regions V in $\mathbb{R}^3$, use similar ideas to those in section 3.3. Let

$$\int_V f(\mathbf{x})\,\mathrm{d}V = \lim_{\varepsilon\to 0}\sum_{i,j,k} f(x_i, y_i, z_i)\,\delta V_{ijk}$$

In this case the volume element satisfies

$$\mathrm{d}V = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

Note. Can do integrals in any order.

Prop. Let $x = x(u,v,w),\ y = y(u,v,w),\ z = z(u,v,w)$ be a continuously differentiable bijection with continuously differentiable inverse that maps the volume V' to the volume V.

$$\iiint_V f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \iiint_V f(x(u,v,w), y(u,v,w), z(u,v,w))|J|\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}w$$

Where

$$J = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \det\left[\frac{\partial\mathbf{x}}{\partial u}\middle|\frac{\partial\mathbf{x}}{\partial v}\middle|\frac{\partial\mathbf{x}}{\partial w}\right]$$

and

$$\mathbf{x} = \begin{bmatrix} x(u,v,w) \\ \vdots \\ z(u,v,w)\end{bmatrix}$$

Short version:

$$\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = |J|\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}w$$

Example. Find in cylindrical polars $(u,v,w) = (\rho,\phi,z)$

$$\mathrm{d}V = \rho\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}z$$

$$|J| = \rho$$

In spherical polars $(u,v,w) = (r,\theta,\phi)$

$$\mathrm{d}V = r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi$$

$$|J| = r^2\sin\theta$$

3.5 Integration over surfaces

Remark. A two dimensional in $\mathbb{R}^3$ can be defined implicitly using a function $f : \mathbb{R}^3 \to \mathbb{R}$

$$S = \{\mathbf{x}\in\mathbb{R}^3 : f(\mathbf{x}) = 0\}$$

Normal to S at $\mathbf{x}$ is parallel to $\nabla f(\mathbf{x})$.
Call surface regular if $\nabla f(\mathbf{x}) \neq 0$ for $\mathbf{x}\in S$

Example. Some surfaces have a boundary, e.g.

$$S = \{(x, y, z) : x^2 + y^2 + z^2 = 1,\ z \geq 0\}$$

Label the boundary by ∂S

$$\partial S = \{(x, y, z) : x^2 + y^2 = 1,\ z = 0\}$$

In this course, a surface S will either have no boundary ($\partial S = \varnothing$), or it will have boundary made of piecewise smooth curves. If S has no boundary, say S is a closed surface.

Moral. It is often useful to parametrise a surface using some coordinates (u, v)

$$S = \{\mathbf{x} = \mathbf{x}(u, v),\ (u, v) \in D\}$$

D some region in (u, v)-plane

Definition. Call parametrisation of S **regular** if

$$\frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v} \neq 0 \text{ on } S$$

In this case, we can define normal

$$\mathbf{n} = \frac{\frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v}}{\left|\frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v}\right|}$$

Note. This normal will vary smoothly wrt (u, v).
Choosing a normal consistently over S gives us a way of orientating the boundary ∂S: make the convention that normal vectors in your immediate vicinity should be on your left as you traverse ∂S

Definition. This leads us to define the **scalar area element** and **vector area element**

$$\mathrm{d}S = \left|\frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v}\right| \mathrm{d}u\, \mathrm{d}v$$

$$\mathrm{d}\mathbf{S} = \frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v} \mathrm{d}u\, \mathrm{d}v = \mathbf{n}\, \mathrm{d}S$$

Equation. Gives area of S:

$$\text{area}(S) = \int_S \mathrm{d}S = \iint_D \left|\frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v}\right| \mathrm{d}u\, \mathrm{d}v$$

and

$$\int_S f\, \mathrm{d}S = \iint_D f(\mathbf{x}(u, v)) \left|\frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v}\right| \mathrm{d}u\, \mathrm{d}v$$

Example. Suppose velocity of fluid is written $\mathbf{u} = \mathbf{u}(\mathbf{x})$. Given S, how to calculate how much fluid passes through it per unit time? On small patch δS on S, fluid passing through would be $(\mathbf{u} \cdot \delta \mathbf{S})\delta t$ in time δt. So amount of fluid that passes over S in ∂t is

$$\delta t \int_S \mathbf{u} \cdot \, \mathrm{d}\mathbf{S}$$

This is the rate at which fluid passes through surface S times δt.
Called "flux" integrals.

Are these surface integrals dependant on choice of parametrisation of S?
Let $\mathbf{x} = \mathbf{x}(u, v)$ and $\mathbf{x} = \tilde{\mathbf{x}}(\tilde{u}, \tilde{v})$ be two different parametrisations of S with $(u, v) \in D$ and $(\tilde{u}, \tilde{v}) \in \tilde{D}$.
Must have relationship

$$\mathbf{x}(u, v) = \tilde{\mathbf{x}}((\tilde{u}(u, v), \tilde{v}(u, v))$$

$$\begin{aligned}
\frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v} &= \left(\frac{\partial \tilde{\mathbf{x}}}{\partial \tilde{u}} \frac{\partial \tilde{u}}{\partial u} + \frac{\partial \tilde{\mathbf{x}}}{\partial \tilde{v}} \frac{\partial \tilde{v}}{\partial u} \right) \times \left(\frac{\partial \tilde{\mathbf{x}}}{\partial \tilde{u}} \frac{\partial \tilde{u}}{\partial v} + \frac{\partial \tilde{\mathbf{x}}}{\partial \tilde{v}} \frac{\partial \tilde{v}}{\partial v} \right) \\
&= \left(\frac{\partial \tilde{u}}{\partial u} \frac{\partial \tilde{v}}{\partial v} - \frac{\partial \tilde{u}}{\partial v} \frac{\partial \tilde{v}}{\partial u} \right) \frac{\partial \tilde{\mathbf{x}}}{\partial \tilde{u}} \times \frac{\partial \tilde{\mathbf{x}}}{\partial \tilde{v}} \\
&= \frac{\partial(\tilde{u}, \tilde{v})}{\partial(u, v)} \frac{\partial \tilde{\mathbf{x}}}{\partial \tilde{u}} \times \frac{\partial \tilde{\mathbf{x}}}{\partial \tilde{v}}
\end{aligned}$$

4 Divergence, Curl and Laplacians

4.1 Definitions

Seen gradient operator ∇, acts on functions $f : \mathbb{R}^3 \to \mathbb{R}$. In Cartesians,

$$\nabla = \mathbf{e}_i \frac{\partial}{\partial x_i}$$

Definition. For a vector field $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$, define **divergence** of $\mathbf{F}$ by

$$\operatorname{div}(\mathbf{F}) = \nabla \cdot \mathbf{F}$$

Equation. So in Cartesians,

$$\nabla \cdot \mathbf{F} = \frac{\partial F_i}{\partial x_i}$$

(can show)

Note. Divergence of a vector field is a scalar field.

Definition. For a vector field $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$, define **curl** of $\mathbf{F}$ by

$$\operatorname{curl}(\mathbf{F}) = \nabla \times \mathbf{F}$$

Equation. So in Cartesians

$$\nabla \times \mathbf{F} = \left(\varepsilon_{ijk} \frac{\partial F_k}{\partial x_j} \right) \mathbf{e}_i$$

So in Cartesians,

$$[\nabla \times \mathbf{F}]_i = \varepsilon_{ijk} \frac{\partial}{\partial x_j} F_k$$

Note. Curl of vector field is another vector field. In terms of a "formal" determinant

$$\nabla \times \mathbf{F} = \det \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ F_1 & F_2 & F_3 \end{bmatrix}$$

Definition. For scalar field $f : \mathbb{R}^3 \to \mathbb{R}$, define **Laplacian** of f

$$\nabla^2 f = \nabla \cdot \nabla f \ (= \text{div}(\text{grad } f))$$

In Cartesians, $[\nabla f] = \frac{\partial f}{\partial x_i}$, so

$$\nabla^2 f = \frac{\partial^2 f}{\partial x_i \partial x_i}$$

Prop. For f, g scalar fields, $\mathbf{F}, \mathbf{G}$ vector fields

$$\begin{aligned}
\nabla(fg) &= (\nabla f)g + (\nabla g)f \\
\nabla \cdot (f\mathbf{F}) &= (\nabla f) \cdot \mathbf{F} + f(\nabla \cdot \mathbf{F}) \\
\nabla \times (f\mathbf{F}) &= (\nabla f) \times \mathbf{F} + f(\nabla \times \mathbf{F}) \\
\nabla(\mathbf{F} \cdot \mathbf{G}) &= \mathbf{F} \times (\nabla \times \mathbf{G}) + \mathbf{G} \times (\nabla \times \mathbf{F}) + (\mathbf{F} \cdot \nabla)\mathbf{G} + (\mathbf{G} \cdot \nabla)\mathbf{F} \\
\nabla \times (\mathbf{F} \times \mathbf{G}) &= \mathbf{F}(\nabla \cdot \mathbf{G}) - \mathbf{G}(\nabla \cdot \mathbf{F}) + (\mathbf{G} \cdot \nabla)\mathbf{F} - (\mathbf{F} \cdot \nabla)\mathbf{G} \\
\nabla \cdot (\mathbf{F} \times \mathbf{G}) &= (\nabla \times \mathbf{F}) \cdot \mathbf{G} - \mathbf{F} \cdot (\nabla \times \mathbf{G})
\end{aligned}$$

Proof.

Note.

$$\begin{aligned}
[(\mathbf{F} \cdot \nabla)\mathbf{G}]_i &= \left(F_j \frac{\partial}{\partial x_j}\right) G_i \\
&= F_j \frac{\partial G_i}{\partial x_j}
\end{aligned}$$

Proofs are just algebra

Remark. These identities hold in ANY OCC, but are most easily established using Cartesians

Equation. For general OCC, divergence defined by same formula $\nabla \cdot \mathbf{F}$, i.e.

$$\left(\mathbf{e}_u \frac{1}{h_u} \frac{\partial}{\partial u} + \mathbf{e}_v \frac{1}{h_v} \frac{\partial}{\partial v} + \mathbf{e}_w \frac{1}{h_w} \frac{\partial}{\partial w}\right) \cdot (F_u \mathbf{e}_u + \cdots + F_w \mathbf{e}_w)$$

Remark. Gets quite messy as $\{\mathbf{e}_u, \mathbf{e}_v, \mathbf{e}_w\}$ will depend on (u, v, w). Just state results:

$$\nabla \cdot \mathbf{F} = \frac{1}{h_u h_v h_w}\left[\frac{\partial}{\partial u}(h_v h_w F_u) + \frac{\partial}{\partial v}(h_u h_w F_v) + \frac{\partial}{\partial w}(h_u h_v F_w)\right]$$

$$\nabla \times \mathbf{F} = \frac{1}{h_u h_v h_w} \det \begin{bmatrix} h_u \mathbf{e}_u & h_v \mathbf{e}_v & h_w \mathbf{e}_w \\ \frac{\partial}{\partial u} & \frac{\partial}{\partial v} & \frac{\partial}{\partial w} \\ h_u F_u & h_v F_v & h_w F_w \end{bmatrix}$$

AND

$$\nabla^2 f = \frac{1}{h_u h_v h_w}\left[\frac{\partial}{\partial u}\left(\frac{h_v h_w}{h_u}\frac{\partial f}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{h_u h_w}{h_v}\frac{\partial f}{\partial v}\right) + \frac{\partial}{\partial w}\left(\frac{h_u h_v}{h_w}\frac{\partial f}{\partial w}\right)\right]$$

Since

$$[\nabla f]_u = \frac{1}{h_u}\frac{\partial f}{\partial u} \text{ etc.}$$

Example. In cylindrical polars (ρ, ϕ, z),

$$(h_\rho, h_\phi, h_z) = (1, \rho, 1)$$

So

$$\nabla^2 f = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial f}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2}$$

Definition.

$$\nabla^2 \mathbf{F} = \nabla(\nabla \cdot \mathbf{F}) - \nabla \times (\nabla \times \mathbf{F})$$

4.2 Relations between div, grad and curl

Prop. For a scalar field f and a vector field $\mathbf{F}$

$$\nabla \times \nabla f = 0$$

$$\nabla \cdot (\nabla \times \mathbf{F}) = 0$$

i.e. curl $\cdot$ grad $= 0$, div $\cdot$ curl $= 0$

Proof. Algebra

Note. Recall $\mathbf{F}$ was conservative if $\mathbf{F} = \nabla f$.

Definition. Say $\mathbf{F}$ is **irrotational** if

$$\nabla \times \mathbf{F} = 0$$

Remark. So from proposition

$$\mathbf{F} \text{ conservative} \implies \mathbf{F} \text{ irrotational}$$

Reverse implication is true if domain of $\mathbf{F}$ is simply connected (or "1-connected")
e.g. $\mathbb{R}^3$ is 1-connected byt $\mathbb{R}^3 \setminus \{z\text{-axis}\}$ is not 1-connected

Remark. Similarly, if there exists a vector potential for $\mathbf{F}$ i.e.

$$\mathbf{F} = \nabla \times \mathbf{A}$$

then

$$\nabla \cdot \mathbf{F} = 0$$

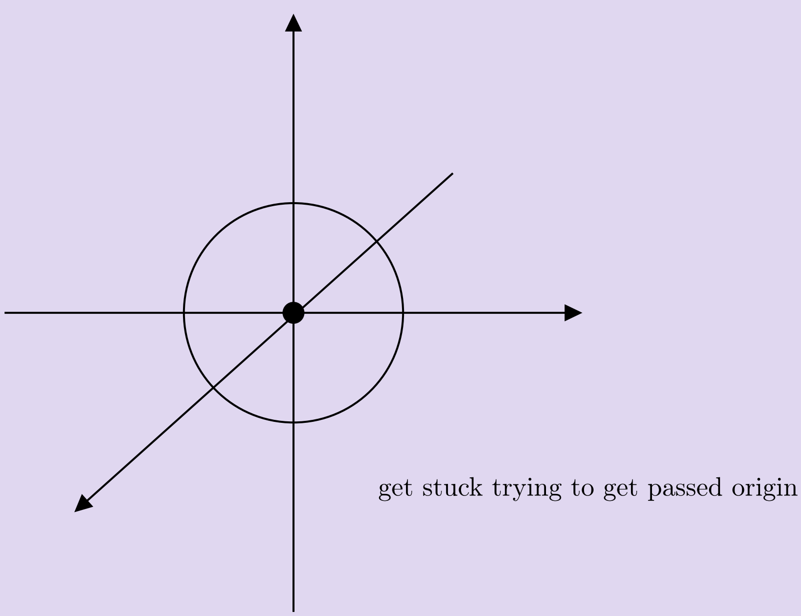
Here $\mathbf{A}$ is called the vector potential for $\mathbf{F}$

Definition. When $\nabla \cdot \mathbf{F} = 0$, say that $\mathbf{F}$ is **solenoidal**

Remark. So existence of vector potential for $\mathbf{F} \implies \mathbf{F}$ solenoidal
Reverse implication is true if domain of $\mathbf{F}$ is 2-connected.

Definition. Say $\Omega \subseteq \mathbb{R}^3$ is **2-connected** if it is 1-connected and every sphere in Ω can be continuously shrunk to any point in Ω

Example. $\mathbb{R}^3$ is 2-connected. $\mathbb{R}^3 \setminus \{0\}$ is 1-connected, but not 2-connected



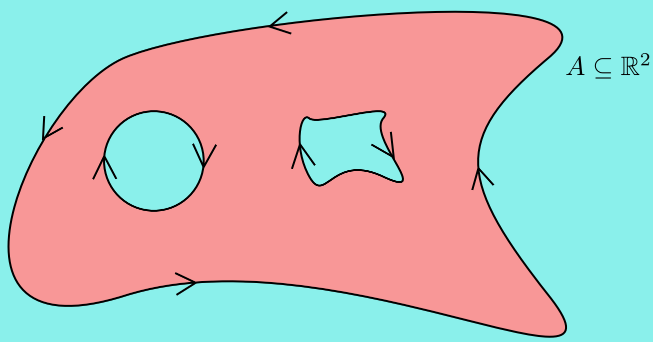
5 Integral Theorems

5.1 Greens Theorem: Statement and Examples

Theorem. If $P = P(x, y)$, $Q = Q(x, y)$ are continuously differentiable functions on $A \cup \partial A$ and ∂A is piecewise smooth, then

$$\oint_{\partial A} P\,\mathrm{d}x + Q\,\mathrm{d}y = \iint_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y$$

Orientation of ∂A is such that A lies to your left as you traverse it.



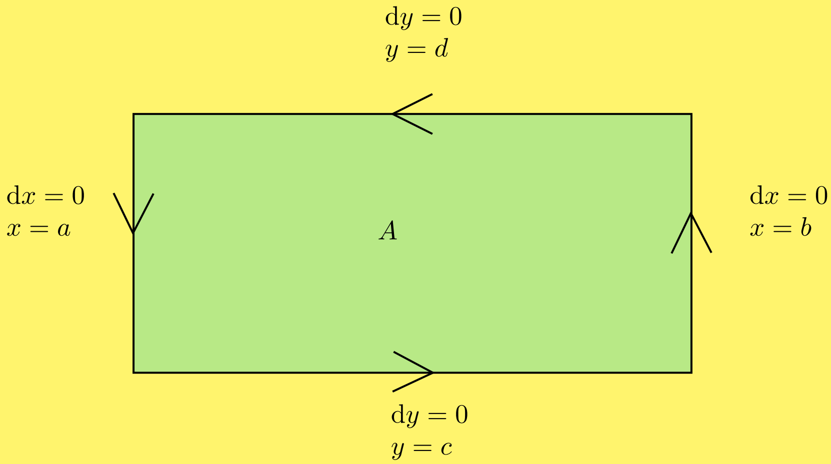
Proof. Proved later through other integral theorems

Note. It is easy to establish this result for

$$A = \{(x,y) : a \leq x \leq b, c \leq y \leq d\}$$

In this case, RHS is

$$\begin{aligned} &\int_c^d \left(\int_a^b \frac{\partial Q}{\partial x} \, \mathrm{d}x \right) \mathrm{d}y - \int_a^b \left(\int_c^d \frac{\partial P}{\partial y} \, \mathrm{d}y \right) \mathrm{d}x \\ &= \int_c^d [Q(b,y) - Q(a,y)] \, \mathrm{d}y + \int_a^b [P(x,c) - P(x,d)] \, \mathrm{d}x \\ &\equiv \oint_{\partial A} P \, \mathrm{d}x + Q \, \mathrm{d}y \end{aligned}$$



5.2 Stoke's Theorem: Statement and Examples

Theorem. If $\mathbf{F} = \mathbf{F}(\mathbf{x})$ is a continuously differentiable vector field and S is an orientable, piece-wise regular surface with piecewise smooth boundary ∂S then

$$\int_S (\nabla \times \mathbf{F}) \cdot \mathrm{d}\mathbf{S} = \oint_{\partial D} \mathbf{F} \cdot \mathrm{d}\mathbf{x}$$

Note. Generalisation of FTC

Remark. The "orientable" bit means there's a consistent choice of normal vector at each point of S. I.e. S has "two sides".

Example. If S is an orientable, closed surface and $\mathbf{F}$ is continuously differentiable then

$$\int_S \nabla \times \mathbf{F} \cdot \mathrm{d}\mathbf{S} = 0$$

Prop. If $\mathbf{F}$ is continuously differentiable and for every loop C

$$\oint_C \mathbf{F} \cdot \mathrm{d}\mathbf{x} = 0$$

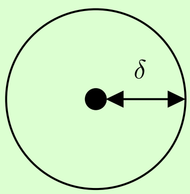
then $\nabla \times \mathbf{F} = 0$. So $\mathbf{F}$ irrotational $\Longleftarrow$ $\mathbf{F}$ has zero circulation any closed loop.

Proof. Assume result is false i.e. $\exists$ unit vector is such that

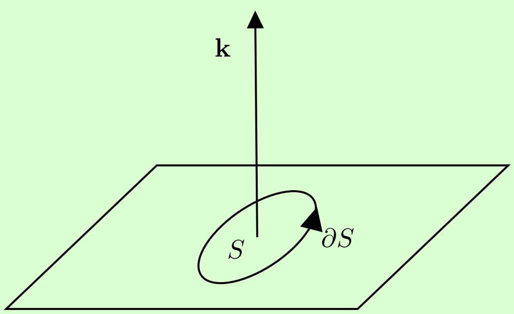
$$\mathbf{k} \cdot \underbrace{\nabla \times \mathbf{F}(\mathbf{x}_0)}_{\varepsilon} > 0$$

for some $\mathbf{x} = \mathbf{x}_0$.
By continuity, for $\delta > 0$, sufficiently small so that, by continuity

$$\mathbf{k} \cdot \nabla \times \mathbf{F}(\mathbf{x}) > \frac{1}{2}\varepsilon \text{ for } |\mathbf{x} - \mathbf{x}_0| < \delta$$



Take loop in this ball $\{\mathbf{x} : |\mathbf{x} - \mathbf{x}_0| < \delta\}$ that lies entirely in a plane with normal $\mathbf{k}$



Then:

$$\begin{aligned} 0 &= \oint_{\partial S} \mathbf{F} \cdot \mathrm{d}\mathbf{x} \\ &= \int_S \nabla \times \mathbf{F} \cdot \mathbf{k} \,\mathrm{d}S \\ &> \frac{1}{2}\varepsilon \int \mathrm{d}S \\ &> 0 ※ \end{aligned}$$

5.3 Divergence Theorem: Statement and Examples (Gauss' Theorem)

Theorem. If $\mathbf{F} = \mathbf{F}(\mathbf{x})$ is continuously differentiable vector field and V is a volume with piecewise regular boundary ∂V then

$$\int_V \nabla \cdot \mathbf{F} \, \mathrm{d}V = \int_{\partial V} \mathbf{F} \cdot \, \mathrm{d}\mathbf{S}$$

where normal to ∂V points OUT of V

Prop. If $\mathbf{F} = \mathbf{F}(\mathbf{x})$ is continuously differentiable and $D \subseteq \mathbb{R}^2$ is a planar region with piecewise sooth boundary ∂D then

$$\int_D \nabla \cdot \mathbf{F} \, \mathrm{d}A = \oint_{\partial D} \mathbf{F} \cdot \mathbf{n} \, \mathrm{d}x$$

(s arc-length)
again $\mathbf{n}$ points OUT of D.

Prop. If $\mathbf{F} = \mathbf{F}(\mathbf{x})$ is continuously differentiable and for every closed surface S

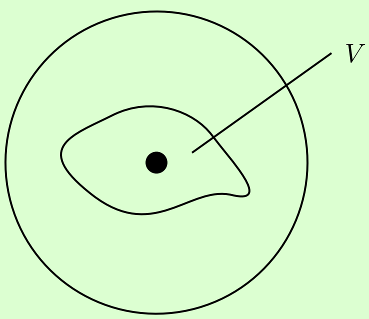
$$\int_S \mathbf{F} \cdot \, \mathrm{d}\mathbf{S} = 0$$

then $\nabla \cdot \mathbf{F} = 0$

Proof. Suppose result is false. So $\nabla \cdot \mathbf{F} = \varepsilon > 0$. By continuity, for $\delta > 0$ sufficiently small

$$\nabla \cdot \mathbf{F}(\mathbf{x}) > \frac{1}{2}\varepsilon$$

$$|\mathbf{x} - \mathbf{x}_0| < \delta$$



Choose a volume V inside ball $|\mathbf{x} - \mathbf{x}_0| < \delta$. Then by assumption

$$0 = \int_{\partial V} \mathbf{F} \cdot \, \mathrm{d}\mathbf{S} = \int_V \nabla \cdot \mathbf{F} \, \mathrm{d}V > \frac{1}{2}\varepsilon \int_V \, \mathrm{d}V > 0 \text{ ※}$$

Conclude that if vector field E has zero net flux through any closed surface then it is solenoidal ($\nabla \cdot \mathbf{F} = 0$) $\square$

Example. Many equations in mathematical physics can be written in the form

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0 \tag{†}$$

Call these CONSERVATION LAWS.
Suppose both ρ and $|\mathbf{J}|$ decrease rapidly as $|\mathbf{x}| \to \infty$. $(\rho = (\rho(\mathbf{x}, t),\ \mathbf{J} = \mathbf{J}(\mathbf{x}, t)$. Define charge:

$$Q = \int_{\mathbb{R}^3} \rho(\mathbf{x}, t) \, \mathrm{d}V$$

We have conservation of charge:

$$\begin{aligned} \frac{\mathrm{d}Q}{\mathrm{d}t} &= -\int_{\mathbb{R}^3} \frac{\partial \rho}{\partial t} \, \mathrm{d}V \\ &= -\int_{\mathbb{R}^3} \nabla \cdot \mathbf{J} \, \mathrm{d}V \\ &= -\lim_{R \to \infty} \int_{|\mathbf{x}| \le R} \nabla \cdot \mathbf{J} \, \mathrm{d}V \\ &= -\lim_{R \to \infty} \int_{|\mathbf{x}| = R} \mathbf{J} \cdot \mathrm{d}\mathbf{S} \\ &= 0 \end{aligned}$$

as $|\mathbf{J}| \to 0$ rapidly as $|\mathbf{x}| \to \infty$
So (†) gives "conservation of charge"

5.4 Sketch Proofs

Prop. The divergence theorem is true

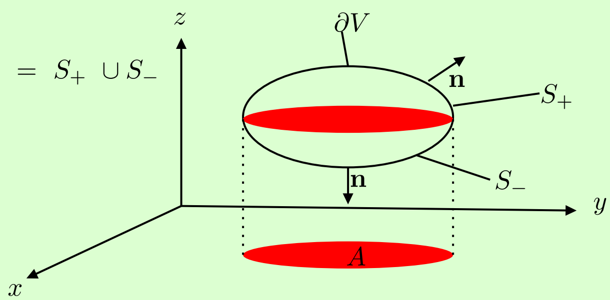
Proof. Suppose first that

$$\mathbf{F} = F_z(x, y, z)\mathbf{e}_z$$

Then divergence thm says

$$\int_V \frac{\partial F_z}{\partial z}\,\mathrm{d}V = \int_{\partial V} F_z\mathbf{e}_z \cdot \mathrm{d}\mathbf{S} \tag{$\dagger$}$$

$\partial V = S_+ \cup S_-$



We write:

$$S_\pm = \{\mathbf{x}(x, y) = \begin{bmatrix} x \\ y \\ g_\pm(x, y) \end{bmatrix},\ (x, y) \in A\}$$

Then

$$\begin{aligned}\int_V \frac{\partial F_z}{\partial z}\,\mathrm{d}V &= \iint_A \left[\int_{g_-(x,y)}^{g_+(x,y)} \frac{\partial F_z}{\partial z}\,\mathrm{d}z\right]\mathrm{d}x\,\mathrm{d}y \\ &= \iint_A \left[F_z(x, y, g_+)x, y) - F_z(x, y, g_-(x, y))\right]\mathrm{d}x\,\mathrm{d}y\end{aligned}$$

To calculate RHS of ($\dagger$) over $S_\pm$

$$\mathrm{d}\mathbf{S} = \frac{\partial \mathbf{x}}{\partial x} \times \frac{\partial \mathbf{x}}{\partial y}\,\mathrm{d}x\,\mathrm{d}y = \begin{bmatrix} -\frac{\partial g_\pm}{\partial x} \\ -\frac{\partial g_\pm}{\partial y} \\ 1 \end{bmatrix}\mathrm{d}x\,\mathrm{d}y$$

Since we want $\mathbf{n}$ to point OUT of V, on $S_\pm$, we have

$$\mathrm{d}\mathbf{S}|_{S_\pm} = \pm\begin{bmatrix} -\frac{\partial g_\pm}{\partial x} \\ -\frac{\partial g_\pm}{\partial y} \\ 1 \end{bmatrix}\mathrm{d}x\,\mathrm{d}y$$

$$\begin{aligned}\implies \int_{\partial V} \mathbf{F}\cdot\mathrm{d}\mathbf{S} &= \left[\int_{S_+} + \int_{S_-}\right] F_z\mathbf{e}_z \cdot \mathrm{d}\mathbf{S} \\ &= \iint_A F_z(x, y, g_+(x, y))\,\mathrm{d}x\,\mathrm{d}y - \iint_A F_z(x, y, g_-(x, y))\,\mathrm{d}x\,\mathrm{d}y \\ &= \int_V \frac{\partial F_z}{\partial z}\,\mathrm{d}V\end{aligned}$$

Prop (cont.).

Proof (cont.). So (†) holds. In exactly the same way

$$\int_V \frac{\partial F_x}{\partial x} \,\mathrm{d}V = \int_{\partial V} F_x \mathbf{e}_x \cdot \mathrm{d}\mathbf{S}$$

$$\int_V \frac{\partial F_y}{\partial y} \,\mathrm{d}V = \int_{\partial V} F_y \mathbf{e}_y \cdot \mathrm{d}\mathbf{S}$$

Adding these three together

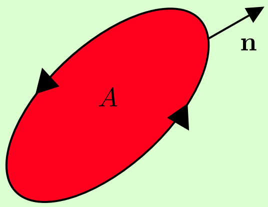
$$\int_V \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \,\mathrm{d}V = \int_{\partial V} F_x \mathbf{e}_x + F_y \mathbf{e}_y + F_z \mathbf{e}_z \cdot \mathrm{d}\mathbf{S}$$

which is the divergence thm □

Prop. Div thm $\implies$ Green's thm

Proof. Use 2D div thm with $\mathbf{F} = \begin{bmatrix} Q \\ -P \end{bmatrix}$. Then

$$\iint_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathrm{d}x \, \mathrm{d}y = \int_A \nabla \cdot \mathbf{F} \, \mathrm{d}A = \oint_{\partial A} \mathbf{F} \cdot \mathbf{x} \, \mathrm{d}s$$



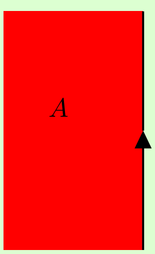
If ∂A is parametrised wrt arc length, so unit tangent vector is

$$\mathbf{t} = \begin{bmatrix} x'(s) \\ y'(s) \end{bmatrix}$$

Then the normal vector must be

$$\mathbf{n} = \begin{bmatrix} y'(s) \\ -x'(s) \end{bmatrix}$$

Check: if $\mathbf{t}$ points vertically upwards then A would be to our left:



$$\mathbf{t} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

And so

$$\begin{aligned} \mathbf{F} \cdot \mathbf{n} \, \mathrm{d}s &= \begin{bmatrix} Q \\ -P \end{bmatrix} \cdot \begin{bmatrix} y'(s) \\ -x'(s) \end{bmatrix} \mathrm{d}s \\ &= P \frac{\mathrm{d}x}{\mathrm{d}s} \, \mathrm{d}s + Q \frac{\mathrm{d}y}{\mathrm{d}s} \, \mathrm{d}s \\ &= P \, \mathrm{d}x + Q \, \mathrm{d}y \end{aligned}$$

i.e.

$$\iint_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathrm{d}x \, \mathrm{d}y = \oint_{\partial A} \mathbf{F} \cdot \mathbf{x} \, \mathrm{d}s$$

Prop. Green's thm $\implies$ Stoke's thm

Proof. Consider regular surface

$$S = \{\mathbf{x} = \mathbf{x}(u, v) : (u, v) \in A\}$$

Then the boundary is

$$\partial S = \{\mathbf{x} = \mathbf{x}(u, v) : (u, v) \in \partial A\}$$

Green's thm gives

$$\oint_{\partial A} P\,\mathrm{d}u + Q\,\mathrm{d}v = \iint_A \left(\frac{\partial Q}{\partial u} - \frac{\partial P}{\partial v}\right) \mathrm{d}u\,\mathrm{d}v$$

Make choices

$$P(x, y) = \mathbf{F}(\mathbf{x}(u, v)) \cdot \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}u}$$

$$Q(x, y) = \mathbf{F}(\mathbf{x}(u, v)) \cdot \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}v}$$

Then

$$\begin{aligned} P\,\mathrm{d}u + Q\,\mathrm{d}v &= \mathbf{F}(\mathbf{x}(u, v)) \cdot \left(\frac{\partial \mathbf{x}}{\partial u}\,\mathrm{d}u + \frac{\partial \mathbf{x}}{\partial v}\,\mathrm{d}v\right) \\ &= \mathbf{F}(\mathbf{x}(u, v)) \cdot \mathrm{d}\mathbf{x}(u, v) \end{aligned}$$

And so

$$\oint_{\partial A} P\,\mathrm{d}u + Q\,\mathrm{d}v = \oint_{\partial S} \mathbf{F} \cdot \mathrm{d}\mathbf{x}$$

Prop (cont.).

Proof (cont.). For the other side of Stokes'

$$\frac{\partial Q}{\partial u} = \frac{\partial x_j}{\partial u}\frac{\partial F_i}{\partial x_j}\frac{\partial x_i}{\partial v} + F_i\frac{\partial^2 x_i}{\partial v\partial u}$$

$$\frac{\partial P}{\partial v} = \frac{\partial x_j}{\partial v}\frac{\partial F_i}{\partial x_j}\frac{\partial x_i}{\partial u} + F_i\frac{\partial^2 x_i}{\partial u\partial v}$$

So:

$$\begin{aligned}
\frac{\partial Q}{\partial u} - \frac{\partial P}{\partial v} &= \left(\frac{\partial x_i}{\partial v}\frac{\partial x_j}{\partial u} - \frac{\partial x_i}{\partial u}\frac{\partial x_j}{\partial v}\right)\frac{\partial F_i}{\partial x_j} \\
&= (\delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp})\frac{\partial F_i}{\partial x_j}\frac{\partial x_p}{\partial v}\frac{\partial x_q}{\partial u} \\
&= \varepsilon_{ijk}\varepsilon_{pqk}\frac{\partial F_i}{\partial x_j}\frac{\partial x_p}{\partial u}\frac{\partial x_q}{\partial u} \\
&= [-\nabla \times \mathbf{F}]_k\left(-\frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v}\right)_k \\
&= (\nabla \times \mathbf{F}) \cdot \left(\frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v}\right)
\end{aligned}$$

So

$$\begin{aligned}
\iint_A \left(\frac{\partial Q}{\partial u} - \frac{\partial P}{\partial v}\right) \mathrm{d}u\,\mathrm{d}v &= \iint_A (\nabla \times \mathbf{F}) \cdot \left(\frac{\partial \mathbf{x}}{\partial u} \times \frac{\partial \mathbf{x}}{\partial v}\right) \mathrm{d}u\,\mathrm{d}v \\
&= \int_S \nabla \times \mathbf{F} \cdot \mathrm{d}\mathbf{S}
\end{aligned}$$

This is Stokes' theorem. □

6 Maxwell's Equations

6.1 Brief Introduction to Electromagnetism

Notation. Denote by

$$\mathbf{B} = \mathbf{B}(\mathbf{x}, t)$$

the magnetic field and

$$\mathbf{E} = \mathbf{E}(\mathbf{x}, t)$$

electric field. These fields will depend on charge density

$$\rho = \rho(\mathbf{x}, t)$$

(electric charge per unit volume) and on current density

$$\mathbf{J} = \mathbf{J}(x, t)$$

(electric current per unit area)

Equation.

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} \tag{1}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{2}$$

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = \mathbf{0} \tag{3}$$

$$\nabla \times \mathbf{B} - \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} = \mu_0 \mathbf{J} \tag{4}$$

The constants ε_0 and μ_0 are the permittivity and permeability of free space, which obey

$$\frac{1}{\mu_0 \varepsilon_0} = c^2$$

where $c = 299,792,458 \,\mathrm{ms}^{-1}$ is the speed of light.

Method. If we take div of (4), using $\nabla \cdot \nabla \times \mathbf{B} = 0$ and then using (1):

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0$$

conservation law.
This gives rise to conservation of charge

6.2 Integral Formulations

Method. Integrating (1) over volume V and using divergence theorem,

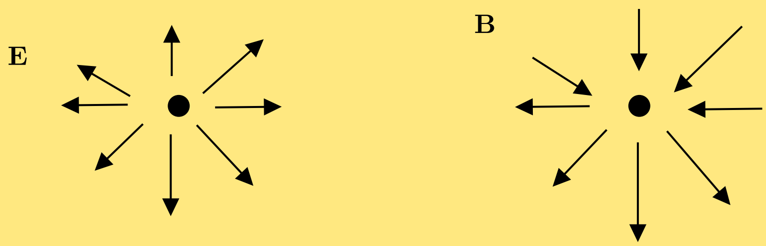
$$\int_{\partial V} \mathbf{E} \cdot \mathrm{d}\mathbf{S} = \frac{1}{\varepsilon_0} \int_V \rho \, \mathrm{d}V \equiv \frac{Q}{\varepsilon_0}$$

where Q is the "total charge in V"
This is called Gauss' Law.

Method. For magnetic fields, (2) gives

$$\int_{\partial V} \mathbf{B} \cdot \mathrm{d}\mathbf{S} = 0$$

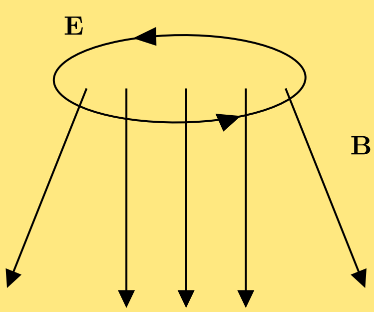
There is no net magnetic flux over any closed surface ∂V.



i.e. there are no magnetic monopoles

Method. Integrating (3) over surface S and use Stoke's theorem

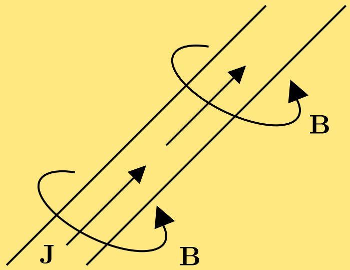
$$\oint_{\partial S} \mathbf{E} \cdot \mathrm{d}\mathbf{x} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot \mathrm{d}\mathbf{S} = -\frac{\mathrm{d}}{\mathrm{d}t} \int_S \mathbf{B} \cdot \mathrm{d}\mathbf{S}$$



The CHANGE in magnetic flux through S induces circulation in $\mathbf{E}$ about ∂S

Method. Integrate (4) over S and use Stokes

$$\oint_{\partial S} \mathbf{B} \cdot \mathrm{d}\mathbf{x} = \mu_0 \int_S \mathbf{J} \cdot \mathrm{d}\mathbf{S} + \mu_0 \varepsilon_0 \frac{\mathrm{d}}{\mathrm{d}t} \int_S \mathbf{E} \cdot \mathrm{d}\mathbf{S}$$



6.3 Electromagnetic Waves

Equation. In Empty space, $\rho = 0, \mathbf{J} = 0$, so (1) to (4) become

$$\nabla \cdot \mathbf{E} = 0 \tag{1}$$
$$\nabla \cdot \mathbf{B} = 0 \tag{2}$$
$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = \mathbf{0} \tag{3}$$
$$\nabla \times \mathbf{B} - \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} = \mathbf{0} \tag{4}$$

Equation. Using (1), (3), (4) and

$$\mu_0 \varepsilon_0 = \frac{1}{c^2}$$

we get

$$\nabla^2 \mathbf{E} - \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0$$

(this is the wave equation in 3-D) So in vacuum, electric field travel at speed c.

Equation. Similarly, using (2), (3), (4)

$$\nabla^2 \mathbf{B} = -\frac{1}{c^2} \frac{\partial^2 \mathbf{B}}{\partial t^2} = 0$$

So electromagnetic waves always travel at speed c in a vacuum

6.4 Electrostatics + Magnetostatics

Equation. Suppose all fields and source terms are independent of t. Then Maxwell's equations decouple

$$(A)\begin{cases}\nabla\cdot\mathbf{E}=\rho/\varepsilon_0\\ \nabla\times\mathbf{E}=0\end{cases}$$

$$(B)\begin{cases}\nabla\cdot\mathbf{B}=0\\ \nabla\times\mathbf{B}=\mu_0\mathbf{J}\end{cases}$$

If we are working on all of $\mathbb{R}^3$ (which is 2-connected), then $\nabla\times\mathbf{E}=0$ and $\nabla\cdot\mathbf{B}=0$ implies

$$\mathbf{E}=-\nabla\phi,\ \mathbf{B}=\nabla\times\mathbf{A}$$

Call ϕ the electric potential and $\mathbf{A}$ the magnetic potential.
Maxwell's equations (A) and (B) become

$$-\nabla^2\phi=\frac{\rho}{\varepsilon_0}$$

$$\nabla\times(\nabla\times\mathbf{A})=\mu_0\mathbf{J}$$

The first is called Poisson's equation, see section 7

7 Poisson's and Laplace Equations

7.1 The Boundary Value Problem

Remark. Many problems in mathematical physics can be reduced to the form

$$\nabla^2 \varphi = F$$

Called Poisson's Equation, or if $F \equiv 0$, call it Laplace's equation. We solve this equation on $\Omega = \mathbb{R}^n$ or $\Omega \subset \mathbb{R}^n,\ n = 2, 3$.
Physical problems involve boundary conditions,
i.e. φ will have prescribed behaviour on $\partial\Omega$ (or as $|x| \to \infty$ if $\Omega = \mathbb{R}^n$).

Example. The Dirichlet problem is

$$\begin{cases} \nabla^2 \varphi = F \text{ in } \Omega \\ \varphi = f \text{ on } \partial\Omega \end{cases}$$

Example. The Neumann problem is

$$\begin{cases} \nabla^2 \varphi = F \text{ in } \Omega \\ \frac{\partial \varphi}{\partial \mathbf{n}} = g \text{ on } \partial\Omega \end{cases}$$

where we have the normal derivative

$$\frac{\partial \varphi}{\partial \mathbf{n}} = \mathbf{n} \cdot \nabla \varphi$$

Must interpret boundary conditions in an appropriate manner: we assume that φ (or $\frac{\partial \varphi}{\partial \mathbf{n}}$) approaches the boundary data f (or g) continuously as $\mathbf{x} \to \partial\Omega$. That is to say, we assume φ and $\nabla\varphi$ are continuous on $\Omega \cup \partial\Omega$.

Warning. If $\nabla^2 \varphi = 0$ in Ω then φ needs to be well-defined on all of Ω. Don't fall into trap of assuming things like

$$\nabla^2 \left(\frac{1}{|\mathbf{x}|} \right) = 0$$

for all $\mathbf{x} \in \mathbb{R}^3$. It is only true for $\mathbf{x} \neq 0$

Example. General spherically symmetric solution for Dirichlet problem:

$$\varphi(r) = A + \frac{B}{r}$$

MUST have $B \equiv 0$ or else φ not well-defined throughout $\Omega = \{r < a\}$

Remark. Want solutions to be unique (or very almost unique)

Method. Consider generic linear problem

$$\begin{cases} L\varphi = F \text{ in } \Omega \\ B\varphi = f \text{ on } \partial\Omega \end{cases} \tag{††}$$

where L, B linear differential operators.
If φ_1 and φ_2 both solve (††), consider $\psi = \phi_1 - \phi_2$. By linearity

$$\begin{cases} L\psi = 0 \text{ in } \Omega \\ B\psi = 0 \text{ on } \partial\Omega \end{cases} \tag{†††}$$

If we can show that the ONLY solution to (†††) is $\psi = 0$, must conclude that $\varphi_1 = \varphi_2$, i.e. solution to (††) is unique.

Moral. The solution to a linear problem is unique iff the only solution to the homogenous problem is the zero solution

Prop. The solution of the Dirichlet problem is unique.
The solution to the Neumann problem is unique up ot the addition of a constant.

Proof. Let $\psi = \varphi_1 - \varphi_2$ be the difference of two solutions to Dirichlet or Neumann problem. so

$$\nabla^2\psi = 0 \text{ in } \Omega$$

$$B\psi = 0 \text{ on } \partial\Omega$$

where $B\psi \equiv \psi$ (Dirichlet) or $B\psi = \frac{\partial\psi}{\partial\mathbf{n}}$ (Neumann)
Consider the non-negative functional:

$$I[\psi] = \int_\Omega |\nabla\psi|^2 \,\mathrm{d}V \geq 0$$

Clearly $I[\psi] = 0$ if and only if $\nabla\psi = 0$ in Ω.
Note:

$$\begin{aligned} I[\psi] &= \int_{\partial\Omega} \psi \frac{\partial\psi}{\partial\mathbf{n}} \,\mathrm{d}S \\ &= 0 \end{aligned}$$

using

$$\mathrm{d}\mathbf{S} = \mathbf{n}\,\mathrm{d}S,\ \mathbf{n}\cdot\nabla\psi = \frac{\partial\psi}{\partial\mathbf{n}}$$

Since $\psi = 0$ on $\partial\Omega$ (Dirichlet) or $\frac{\partial\psi}{\partial\mathbf{n}} = 0$ on $\partial\Omega$ (Neumann). Conclude that $\nabla\psi = 0$ throughout $\Omega \implies \psi = \text{const.}$ throughout Ω.

(i) For Dirichlet, $\psi = 0$ on $\partial\Omega$, so by continuity of ψ on $\Omega\cup\partial\Omega$, must have $\psi = 0$ everywhere. So solution to Dirichlet problem is unique.

(ii) From Neumann, only know $\frac{\mathrm{d}\psi}{\mathrm{d}\mathbf{n}} = 0$ on boundary so can't say any more, so since $\psi =$ const. deduce that

$$\varphi_1 = \varphi_2 + \text{ const.}$$

Any two solutions differ only by a constant. □

7.2 Gauss' Flux Method

Method. Suppose source term F is spherically symmetric, ie. $F = F(r)$, where $r = |\mathbf{x}|$. Write our problem as:

$$\nabla \cdot \nabla \varphi = F(r) \tag{*}$$

and assume $\Omega = \mathbb{R}^3$. Since RHS only depends on r, same is true of LHS. So assume that $\varphi = \varphi(r)$, in which case

$$\nabla \varphi = \varphi'(r)\mathbf{e}_r$$

Integrating (*) over region $|\mathbf{x}| < R$, and use divergence theorem

$$\int_{|\mathbf{x}|<R} \nabla \cdot \nabla \varphi \, \mathrm{d}V = \int_{|\mathbf{x}|=R} \nabla \varphi \cdot \, \mathrm{d}\mathbf{S} = \int_{|\mathbf{x}|<R} F(r) \, \mathrm{d}V$$

The RHS represents the amount of, e.g. mass, inside ball of radius $R > 0$. Set

$$\int_{|\mathbf{x}|<R} F \, \mathrm{d}V = Q(R)$$

where $Q(R)$ is "the amount of stuff inside ball $|\mathbf{x}| < R$"
So our equation is

$$\int_{|\mathbf{x}|=R} \nabla \varphi \cdot \, \mathrm{d}\mathbf{S} = Q(R)$$

Recall that on sphere of radius R

$$\mathrm{d}\mathbf{S} = \mathbf{e}_r R^2 \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi$$

So on $|\mathbf{x}| = R$:

$$\nabla \varphi \cdot \, \mathrm{d}\mathbf{S} = \varphi'(r)\mathbf{e}_r \cdot (\, \mathbf{e}_r \underbrace{R^2 \sin\theta \, \mathrm{d}\theta \mathrm{d}\phi}_{\mathrm{d}S}) \Big|_{|\mathbf{x}|=R} = \varphi'(R) \, \mathrm{d}S$$

So

$$Q(R) = \int_{|\mathbf{x}|<R} \varphi'(R) \, \mathrm{d}S = \varphi'(R) \underbrace{\int_{|\mathbf{x}|<R} \mathrm{d}S}_{4\pi R^2}$$

In summary

$$\varphi'(R) = \frac{Q(R)}{4\pi R^2} \; \forall R > 0$$

$$\implies \nabla \varphi = \frac{Q(R)}{4\pi r^2} \mathbf{e}_r$$

Method. What if our problem is symmetric about the z-axis i.e.

$$\nabla^2\varphi = F(\rho)\ \rho^2 = x^2 + y^2$$

Have "cylindrical symmetry". Integrate

$$\nabla \cdot \nabla\varphi = F(\rho)$$

over cylinder of radius R, height a.
Assuming $\varphi = \varphi(\rho)$, have

$$\nabla\varphi = \varphi'(\rho)\,\mathbf{e}_\rho \text{ (cylindrical polars)}$$

$$\int_V \nabla \cdot \nabla\varphi\,\mathrm{d}V = \int_V F(\rho)\,\mathrm{d}V$$

where V is cylinder

$$LHS = 2\pi aR\varphi'(R)$$

using Div Thm. So

$$\varphi'(R) = \frac{1}{R} \cdot \frac{1}{2\pi a} \int_V F(\rho)\,\mathrm{d}V$$

By evaluating the integral and rearranging, we get

Equation.

$$\varphi'(\rho) = \frac{1}{\rho} \int_0^\rho sF(s)\,\mathrm{d}s$$

7.3 Superposition Principle

Remark. Linear problems are relatively easy because of the following: if we have

$$L\psi_n = F_n\ n = 1, 2, 3, \dots$$

then

$$L\left(\sum_n \psi_n\right) = \sum_n F(n)$$

We can superimpose solutions. Can often break up forcing term $F = \sum_n F_n$, solve each problem

$$L\psi_n = F_n$$

To get solution to $L\psi = F$, write $\psi = \sum_n \psi_n$

Example. Consider electric potential due to pair of point charges $Q_{\mathbf{a}}$ at $x = \mathbf{a}$, $Q_{\mathbf{b}}$ at $x = \mathbf{b}$. Charge density would be

$$\rho(\mathbf{x}) = Q_{\mathbf{a}}\delta(\mathbf{x}-\mathbf{a}) + Q_{\mathbf{b}}\delta(\mathbf{x}-\mathbf{b})$$

For one point charge, electric potential obeys

$$-\nabla^2\phi = \frac{Q_{\mathbf{a}}}{\varepsilon_0}\delta(\mathbf{x}-\mathbf{a})$$

Solution would be

$$\phi(\mathbf{x}) = \frac{Q_{\mathbf{a}}}{4\pi\varepsilon_0}\frac{1}{|\mathbf{x}-\mathbf{a}|}$$

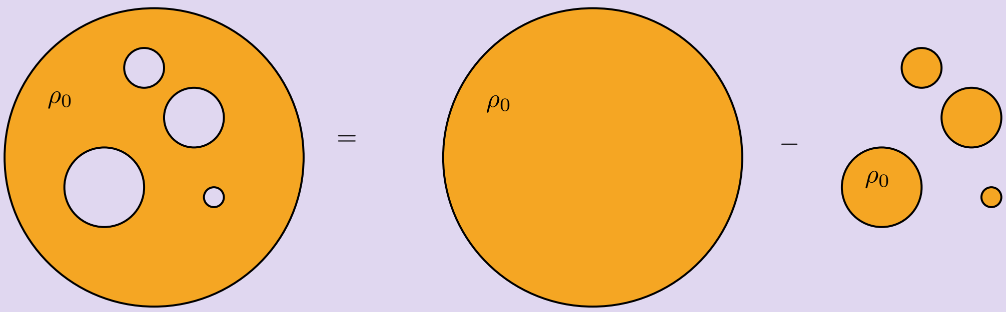
So by superposition principle, electric potential due to point charges at $\mathbf{x} = \mathbf{a}$ and $\mathbf{x} = \mathbf{b}$ is

$$\phi(\mathbf{x}) = \frac{Q_{\mathbf{a}}}{4\pi\varepsilon_0}\frac{1}{|\mathbf{x}-\mathbf{a}|} + \frac{Q_{\mathbf{a}}}{4\pi\varepsilon_0}\frac{1}{|\mathbf{x}-\mathbf{b}|}$$

Example. Consider electric potential outside ball of radius $|\mathbf{x}| < R$ of uniform charge density ρ_0, that has several balls removed from its interior

$$|\mathbf{x}-\mathbf{a}_i| < R_i \ i = 1, \dots, N$$

$$|\mathbf{a}_i| + R_i < R, \ |\mathbf{a}_i - \mathbf{a}_j| > R_i + R_j \text{ for each } i, j$$



Use superposition principle: represent each hole to be a ball of uniform charge density $-\rho_0$. Effective potential in $|\mathbf{x}| > R$ from each hole is

$$\phi(\mathbf{x}) = -\frac{1}{4\pi\varepsilon_0}\frac{Q_i}{|\mathbf{x}-\mathbf{a}_i|}$$

using

$$Q = \left(\frac{4\pi R_i^3}{3}\right)\rho_0$$

by superposition principle

$$\phi(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0}\left[\frac{Q}{|\mathbf{x}|} - \sum_{i=1}^{N}\frac{Q_i}{|\mathbf{x}-\mathbf{a}_i|}\right]$$

7.4 Integral Solutions

Prop. Assume $F \to 0$ rapidly as $|\mathbf{x}| \to \infty$. The unique solution to the Dirichlet problem

$$\begin{cases} \nabla^2 \varphi = F(\mathbf{x}) \text{ for } \mathbf{x} \in \mathbb{R}^3 \\ |\varphi| \to 0 \text{ as } |\mathbf{x}| \to \infty \end{cases}$$

is given by

$$\varphi(\mathbf{x}) = -\frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{F(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|} \, \mathrm{d}V(\mathbf{y})$$

Proof. Certainly have

$$\nabla^2 \left(-\frac{1}{4\pi} \frac{1}{|\mathbf{x}|} \right) = \delta(\mathbf{x}) \ \mathbf{x} \neq 0$$

as $1/|\mathbf{x}|$ a solution to Laplace's equation. If we assume divergence thm works with delta function, on any ball $|\mathbf{x}| < R$

$$\begin{aligned} \int_{|\mathbf{x}|<R} \nabla^2 \left(\frac{1}{|\mathbf{x}|} \right) \mathrm{d}V &= \int_{|\mathbf{x}|=R} \nabla \left(\frac{1}{|\mathbf{x}|} \right) \cdot \mathrm{d}\mathbf{S} \\ &= -4\pi \end{aligned}$$

By evaluating integral. So for any $R > 0$

$$\int_{|\mathbf{x}|<R} \nabla^2 \left(-\frac{1}{4\pi} \frac{1}{|\mathbf{x}|} \right) \mathrm{d}V = 1 = \int_{|\mathbf{x}|<R} \delta(\mathbf{x}) \, \mathrm{d}V$$

We conclude

$$\nabla^2 \left(-\frac{1}{4\pi} \frac{1}{|\mathbf{x}|} \right) = \delta(\mathbf{x})$$

so proposition follows.

Remark. This result is another way of saying

$$\nabla^2 \left(-\frac{1}{4\pi} \frac{1}{|\mathbf{x}|} \right) = \delta(\mathbf{x})$$

Since by differentiating under integral sign

$$\begin{aligned} \nabla^2 \left(-\frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{F(\mathbf{y})}{|\mathbf{x} - \mathbf{y}|} \, \mathrm{d}V(\mathbf{y}) \right) &= -\frac{1}{4\pi} \int_{\mathbb{R}^3} F(\mathbf{y}) \nabla^2 \left(\frac{1}{|\mathbf{x} - \mathbf{y}|} \right) \mathrm{d}V(\mathbf{y}) \\ &= \int_{\mathbb{R}^3} F(\mathbf{y}) \delta(\mathbf{x} - \mathbf{y}) \, \mathrm{d}V(\mathbf{y}) \\ &= F(\mathbf{x}) \end{aligned}$$

7.5 Harmonic Functions

Definition. When the forcing term in Poisson's equation is identically zero, we call it **Laplace's equation**:

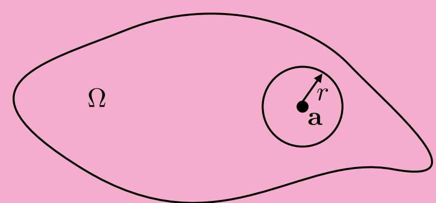
$$\nabla^2\varphi = 0 \tag{†}$$

Solutions to Laplace's equation are called harmonic functions

Prop. If φ harmonic on $\Omega \subseteq \mathbb{R}^3$, then

$$\varphi(\mathbf{a}) = \frac{1}{4\pi r^2}\int_{|\mathbf{x}-\mathbf{a}|=r} \varphi(\mathbf{x})\,\mathrm{d}S \tag{*}$$

for $\mathbf{a} \in \Omega$ and r sufficiently small.



Proof. Let $F(r)$ denote RHS of (*). Then

$$\begin{aligned}
F(r) &= \frac{1}{4\pi r^2}\int_{|\mathbf{x}|=r} \varphi(\mathbf{a}+\mathbf{x})\,\mathrm{d}S \\
&= \frac{1}{4\pi r^2}\int_{\phi=0}^{2\pi}\left[\int_{\theta=0}^{\pi} \varphi(\mathbf{a}+r\mathbf{e}_r)r^2\sin\theta\,\mathrm{d}\theta\right]\mathrm{d}\phi \\
&= \frac{1}{4\pi}\int_{\phi=0}^{2\pi}\left[\int_{\theta=0}^{\pi} \varphi(\mathbf{a}+r\mathbf{e}_r)\sin\theta\,\mathrm{d}\theta\right]\mathrm{d}\phi
\end{aligned}$$

Computing $F'(r)$, using

$$\frac{\mathrm{d}}{\mathrm{d}r}\varphi(\mathbf{a}+r\mathbf{e}_r) = \mathbf{e}_r \cdot \nabla\varphi(\mathbf{a}+r\mathbf{e}_r)$$

$$\begin{aligned}
F'(r) &= \frac{1}{4\pi r^2}\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} \mathbf{e}_r \cdot \nabla\varphi(\mathbf{a}+r\mathbf{e}_r)r^2\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi \\
&= \frac{1}{4\pi r^2}\int_{|\mathbf{x}|=r} \nabla\varphi(\mathbf{a}+\mathbf{x})\cdot\,\mathrm{d}\mathbf{S} \\
&= \frac{1}{4\pi r^2}\int_{|\mathbf{x}-\mathbf{a}|=r} \nabla\varphi\cdot\,\mathrm{d}\mathbf{S} \\
&= \frac{1}{4\pi r^2}\int_{|\mathbf{x}-\mathbf{a}|<r} \nabla^2\varphi\,\mathrm{d}V \\
&= 0
\end{aligned}$$

So $F(r)$ is constant and result follows by taking $r \to 0$. □

Moral. Can use central idea in this proof to examine what the Laplacian helps us measure

Prop. For any smooth $\varphi : \mathbb{R}^3 \to \mathbb{R}$

$$\nabla^2 \varphi(\mathbf{a}) = \lim_{r \to 0} \frac{6}{r^2} \left[\frac{1}{4\pi r^2} \int_{|\mathbf{x}-\mathbf{a}|=r} \varphi(\mathbf{x}) \, \mathrm{d}S - \varphi(\mathbf{a}) \right]$$

In particular, if φ satisfies the MVP then it is harmonic.

Proof. Consider function $G(r)$ defined by

$$G(r) = \frac{1}{4\pi r^2} \int_{|\mathbf{x}-\mathbf{a}|=r} \varphi(\mathbf{x}) \, \mathrm{d}S - \varphi(\mathbf{a})$$

So G measures extent to which φ differs from its average. we have from previous proof

$$G'(r) = F'(r) = \frac{1}{4\pi r^2} \int_{|\mathbf{x}-\mathbf{a}|<r} \nabla^2 \varphi \, \mathrm{d}V$$

Obviously, this vanishes if φ harmonic. Note

$$\begin{aligned} \int_{|\mathbf{x}-\mathbf{a}|<r} \nabla^2 \varphi(\mathbf{x}) \, \mathrm{d}V &= \nabla^2 \varphi(\mathbf{a}) \int_{|\mathbf{x}-\mathbf{a}|<r} \mathrm{d}V + \int_{|\mathbf{x}-\mathbf{a}|<r} (\nabla^2 \varphi(\mathbf{x}) - \nabla^2 \varphi(\mathbf{a}) \, \mathrm{d}V \\ &= \frac{4\pi}{3} r^3 \nabla^2 \varphi(\mathbf{a}) + o(r^3) \ (r \to 0) \end{aligned}$$

So

$$G'(r) = \frac{r}{3} \nabla^2 \varphi(\mathbf{a}) + o(r) \ (r \to 0)$$

Compare this with Taylor expansion

$$G'(r) = G'(0) + rG''(0) + o(r) \ (r \to 0)$$

we deduce:

$$G'(0) = 0, \ G''(0) = \frac{1}{3} \nabla^2 \varphi(\mathbf{a})$$

So

$$\begin{aligned} G(r) &= \underbrace{G(0)}_{=0} + r \underbrace{G'(0)}_{=0} + \frac{r^2}{2} G''(0) + o(r^2) \\ &= \frac{1}{6} \nabla^2 \varphi(\mathbf{a}) r^2 + o(r^2) \ (r \to 0) \end{aligned}$$

$$\implies \nabla^2 \varphi(\mathbf{a}) = \lim_{r \to 0} \left[\frac{6}{r^2} G(r) \right] \implies \text{result} \ \square$$

Prop. If φ is harmonic on $\Omega \subseteq \mathbb{R}^3$ then it cannot have a maximum at any interior point of Ω unless φ is constant.

Proof. Suppose $\mathbf{a} \in \Omega$ is such that

$$\varphi(\mathbf{a}) \geq \varphi(\mathbf{x})$$

for all $\mathbf{x} \in \Omega$. So certainly

$$\varphi(\mathbf{a}) \geq \varphi(\mathbf{x}) \text{ on } 0 < |\mathbf{x} - \mathbf{a}| \leq \varepsilon$$

for some $\varepsilon > 0$. But by mean value thm

$$\varphi(\mathbf{a}) = \frac{1}{4\pi\varepsilon^2} \int_{|\mathbf{x}-\mathbf{a}|=\varepsilon} \varphi(\mathbf{x}) \, \mathrm{d}S$$

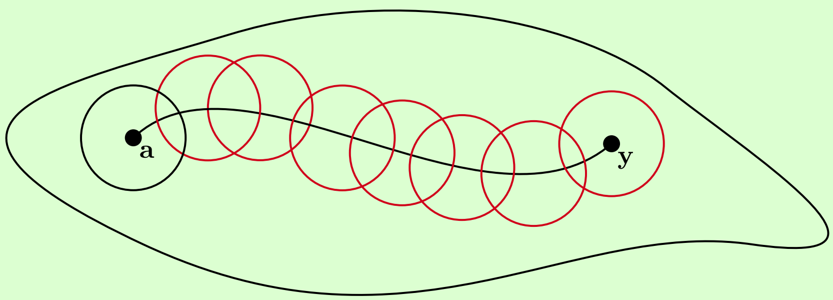
i.e.

$$0 = \frac{1}{4\pi\varepsilon^2} \int_{|\mathbf{x}-\mathbf{a}|=\varepsilon} \underbrace{\varphi(\mathbf{a}) - \varphi(\mathbf{x})}_{\geq 0} \, \mathrm{d}S$$

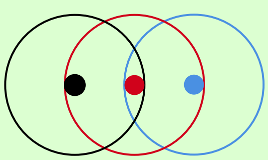
Consider that $\varphi(\mathbf{x}) = \varphi(\mathbf{a})$. Apply same argument to

$$|\mathbf{x} - \mathbf{a}| = \varepsilon' < \varepsilon$$

Deduce $\varphi(\mathbf{x}) = \varphi(\mathbf{a})$ on $|\mathbf{x} - \mathbf{a}| \leq \varepsilon$



Introduce bunch of overlapping balls such that the centre of the $(n+1)$th ball is contained inside the nth.



Everywhere inside 1$^{\text{st}}$ ball, have $\varphi(\mathbf{x}) = \varphi(\mathbf{a})$.
In particular, on center of second ball have $\varphi(\mathbf{x}) = \varphi(\mathbf{a})$.
Using previous argument get $\varphi(\mathbf{x}) = \varphi(\mathbf{a})$ throughout second ball. Carry on until you get to $\mathbf{y}$. Find $\varphi(\mathbf{y}) = \varphi(\mathbf{a})$ i.e. φ constant. $\square$

Corollary. If φ is harmonic on Ω then

$$\varphi(\mathbf{x}) \leq \max_{\mathbf{y}\in\partial\Omega} \varphi(\mathbf{y}) \ (\mathbf{x} \in \Omega)$$

(Maximum principle)

Note. Comes from considering maximum of φ on $\Omega \cup \partial\Omega$

8 Cartesian Tensors

Remark. Throughout this section we deal solely with Cartesian coordinate systems

8.1 A Closer Look at Vectors

Moral. If we transform fom $\{\mathbf{e}_i\}$ to $\{\mathbf{e}'_i\}$ then the components of a vector $\mathbf{v}$ transform as

$$v'_i = R_{ij} v_j$$

where $R_{ij} = \mathbf{e}'_i \cdot \mathbf{e}_j$ are components of a rotation matrix. Call objects whose components transform in this way rank 1 tensors, or vectors.

8.2 A Closer Look at Scalars

Moral. objects that transform as

$$\sigma' = \sigma$$

when we change from $\{\mathbf{e}_i\}$ to $\{\mathbf{e}'_i\}$ are called scalars, or rank 0 tensors.

8.3 Cartesian Tensors of Rank n

Definition. An object whose components $T_{\underbrace{ij\ldots k}_{n \text{ indices}}}$ transform (when we go from $\{\mathbf{e}_i\}$ to $\{\mathbf{e}'_i\}$) according to

$$T'_{ij\ldots k} = \overbrace{R_{ip} R_{jq} \ldots R_{kr}}^{n\ Rs} T_{pq\ldots r}$$

is called a (Cartesian) **tensor of rank n**.
Here

$$R_{ij} = \mathbf{e}'_i \cdot \mathbf{e}_j$$

are components of rotation matrix, so

$$R_{ip} R_{jp} = \delta_{ij}$$

Example. If $u_i, v_k, \ldots, w_k$ are components of n vectors, then

$$T_{ij\ldots k} = u_i v_j \ldots w_k$$

define components of a tensor of rank n
(can check)

Example. Kronecker delta is defined without reference to any basis via

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

So $\delta'_{ij} = \delta_{ij}$ by definition. But note

$$R_{ip}R_{jq}\delta_{pq} = R_{ip}R_{jp} = \delta_{ij}$$

So we have

$$\delta'_{ij} = R_{ip}R_{jq}\delta_{pq}$$

i.e. δ_{ij} is a rank 2 tensor.

Example. The Levi Civita symbol is defined without reference to any basis

$$\varepsilon_{ijk} = \begin{cases} 1 & \text{if } (i\ j\ k) \text{ is an even perm of } (1\ 2\ 3) \\ -1 & \text{if } (i\ j\ k) \text{ is an odd perm of } (1\ 2\ 3) \\ 0 & \text{otherwise} \end{cases}$$

By definition, $\varepsilon'_{ijk} = \varepsilon_{ijk}$. But

$$\begin{aligned} R_{ip}R_{jq}R_{kr}\varepsilon_{pqr} &= \det(R)\varepsilon_{ijk} \\ &= \varepsilon_{ijk} \end{aligned}$$

So we have

$$\varepsilon'_{ijk} = R_{ip}R_{jq}R_{kr}\varepsilon_{pqr}$$

So ε_{ijk} is a tensor of rank 3.

Definition. If $A_{ij\dots k}$ and $B_{ij\dots k}$ are n-th rank tensors, define

$$(A + B)_{ij\dots k} = A_{ij\dots k} + B_{ij\dots k}$$

This is also n-th rank tensor, If α is a scalar then

$$(\alpha A)_{ij\dots k} = \alpha A_{ij\dots k}$$

is an n-th rank tensor.
We define the **tensor product** of an m-th rank tensor $U_{ij\dots k}$ and a an n-th rank tensor $V_{pq\dots r}$ by

$$(U \otimes V)_{ij\dots kpq\dots r} = U_{ij\dots k}V_{pq\dots r}$$

where

$$\underbrace{ij\dots k}_{m \text{ indices}}\ \underbrace{pq\dots r}_{n \text{ indices}}$$

Claim. This is a tensor of rank $n+m$.

Proof.

$$\begin{aligned} U'_{i\dots j}V'_{p\dots q} &= R_{ia}\dots R_{jb}U_{a\dots b}R_{pc}\dots R_{qd}V_{c\dots d} \\ &= \underbrace{R_{ia}\dots R_{jb}R_{pc}\dots R_{qd}}_{n+m \text{ terms}} \underbrace{U_{a\dots b}V_{c\dots d}}_{(U\otimes V)_{a\dots bc\dots d}} \end{aligned}$$

Method. Given n-th rank tensor $T_{ijk\dots d}$ $n \geq 2$, we can define tensor of rank $n-2$ by contracting on pair of indices. For instance, contracting on i and j is defined by

$$\delta_{ij}T_{ijk\dots d} = T_{iik\dots d}$$

Note.

$$\begin{aligned} T'_{iik\dots d} &= \underbrace{R_{ip}R_{iq}}_{\delta_{pq}} R_{kr}\dots R_{ls}T_{pqr\dots s} \\ &= R_{kq}\dots R_{ls}T_{ppr\dots s} \end{aligned}$$

So $T_{iik\dots d}$ transforms as tensor of rank $n-2$

Definition. Say $T_{ij\dots k}$ is **symmetric** in (i,j) if

$$T_{ij\dots k} = T_{ji\dots k}$$

(can check this is well-defined i.e. regardless of basis)
Similarly, we say $A_{ij\dots k}$ is **anti-symmetric** in (i,j) if

$$A_{ij\dots k} = -A_{ji\dots k}$$

Say a tensor is **totally (anti-)symmetric** if it is (anti-)symmetric in every pair of indices.

Example. Tensors δ_{ij} and $a_i a_j a_k$ are both totally symmetric.
ε_{ijk} is a totally anti-symmetric tensor.
In fact, the only totally anti-symmetric tensor on $\mathbb{R}^3$ of rank $n=3$ is proportional to ε_{ijk}, and there are no non-zero high rank ones. Indeed, if $T_{ij\dots k}$ totally anti-symmetric of rank n, then $T_{ij\dots k}=0$ if any two indices are the same

$$T_{22\dots k} = -T_{22\dots k} \implies T_{22\dots k} = 0$$

So by pigeonhole principle, there will always be two or more matching indices if $n>3$. If $n=3$, there are only $3! = 6$ non-zero components. If

$$T_{123} = T_{231} = T_{312} = \lambda$$

$$T_{213} = T_{321} = T_{132} = -\lambda$$

Thus $T_{ijk} = \lambda\varepsilon_{ijk}$

8.4 Tensor Calculus

Remark. "vector field" gives vector $\mathbf{v}(\mathbf{x})$ for $\mathbf{x} \in \mathbb{R}^3$
"scalar field" gives scalar $\varphi(\mathbf{x})$ for $\mathbf{x} \in \mathbb{R}^3$
A tensor field of rank n, $T_{ij\ldots k}(\mathbf{x})$, gives an n-th rank tensor at each $\mathbf{x} \in \mathbb{R}^3$.

Equation. Recall

$$x'_i = R_{ij}x_j \iff x_j = R_{ij}x'_i$$

Differentiating RHS wrt x'_k

$$\frac{\partial x_j}{\partial x'_k} = R_{ij}\frac{\partial x'_i}{\partial x'_k} = R_{ij}\delta_{ik} = R_{kj}$$

So by chain rule

$$\frac{\partial}{\partial x'_i} = \frac{\partial x_j}{\partial x'_i}\frac{\partial}{\partial x_j} = R_{ij}\frac{\partial}{\partial x_j}$$

"$\frac{\partial}{\partial x_i}$ transforms like a rank 1 tensor"

Prop. If $T_{i\ldots j}(\mathbf{x})$ is tensor field of rank n then

$$\underbrace{\left(\frac{\partial}{\partial x_p}\right)\ldots\left(\frac{\partial}{\partial x_q}\right)}_{m \text{ terms}} T_{i\ldots j}(\mathbf{x}) = \text{ tensor field of rank } n+m$$

Proof. Label LHS by $A_{p\ldots qi\ldots j}$ and do the algebra

Example. If $\varphi = \varphi(\mathbf{x})$ scalar field then

$$[\nabla\varphi]_i = \frac{\partial\varphi}{\partial x_i}$$

So $\nabla\varphi$ is rank $0+1=1$ tensor field, i.e. a vector field.

Example. For vector field $\mathbf{v}$ have divergence

$$\nabla \cdot \mathbf{v} = \frac{\partial v_i}{\partial x_i}$$

can check that $\nabla \cdot \mathbf{v}$ is scalar field.

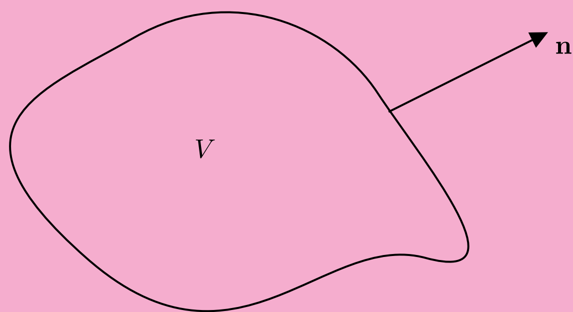
Example. If $\mathbf{v}$ vector field, consider curl $\nabla \times \mathbf{v}$. Then

$$[\nabla \times \mathbf{v}]_i = \varepsilon_{ijk}\frac{\partial v_k}{\partial x_j}$$

can check that $\nabla \times \mathbf{v}$ is vector field.

Prop. For tensor field $T_{ij\dots k\dots l}(\mathbf{x})$:

$$\int_V \frac{\partial}{\partial x_k} T_{ij\dots k\dots l}\,\mathrm{d}V = \int_{\partial V} T_{ij\dots k\dots l} n_k\,\mathrm{d}S$$



Proof. Apply divergence theorem to

$$v_k = a_i b_j \dots c_l T_{ij\dots k\dots l} \qquad (\dagger)$$

where $a_i, b_j, \dots, c_l$ are components of constant vector fields. So by div theorem

$$\int_V \frac{\partial v_k}{\partial x_k}\,\mathrm{d}V = a_i b_j \dots c_l \int_{\partial V} T_{ij\dots k\dots l} n_k\,\mathrm{d}S$$

Result now follows because the constant vector fields $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ were arbitrary.
E.g. if we wanted to check ($\dagger$) when all free indices $i, j, \dots, l$ were $= 1$

$$a_i = \delta_{i1},\ b_j = \delta_{j1},\ \dots,\ c_l = \delta_{l1}$$

Similar idea for other choice of free indices. □

8.5 Rank 2 Tensors

Remark. Observe for rank 2 tensor T_{ij}

$$\begin{aligned} T_{ij} &= \frac{1}{2}(T_{ij} + T_{ji}) + \frac{1}{2}(T_{ij} - T_{ji}) \\ &= S_{ij} + A_{ij} \end{aligned}$$

which is symmetric + anti-symmetric

$$\underbrace{\begin{bmatrix} * & * & * \\ & * & * \\ & & * \end{bmatrix}}_{\text{6 indep components}} \quad \underbrace{\begin{bmatrix} 0 & * & * \\ & 0 & * \\ & & 0 \end{bmatrix}}_{\text{3 indep components}}$$

This is good since $3 + 6 = 9$. Intuitively, seems like info contained in A_{ij} caould be written in terms of some vector (3 indep components).

Prop. Every rank 2 tensor can be written uniquely as

$$T_{ij} = S_{ij} + \varepsilon_{ijk}\omega_k$$

where

$$\omega_i = \frac{1}{2}\varepsilon_{ijk}T_{jk}$$

and

$$S_{ij} \text{ is symmetric}$$

Proof. We can identify (from earlier)

$$S_{ij} = \frac{1}{2}(T_{ij} + T_{ji})$$

Remains to show that

$$\varepsilon_{ijk}\omega_k = \frac{1}{2}(T_{ij} - T_{ji})$$

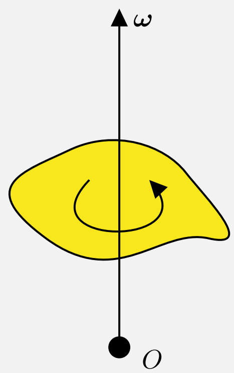
For uniqueness, suppose

$$(T_{ij} =) S_{ij} + A_{ij} = \tilde{S}_{ij} + \tilde{A}_{ij} (= \tilde{T}_{ij})$$

Then consider

$$\frac{1}{2}(T_{ij} + T_{ji})$$

A well known symmetric rank 2 tensor is the inertia tensor. Suppose body with density $\rho(\mathbf{x})$ occupies volume $V \subseteq \mathbb{R}^3$. Each point in the body rotating at constant angular velocity $\boldsymbol{\omega}$



So velocity of point $\mathbf{x} \in V$ is $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{x}$. Total angular velocity about origin is:

$$\begin{aligned} \mathbf{L} &= \int_V \rho(\mathbf{x})(\mathbf{x} \times \mathbf{v}) \, \mathrm{d}V \\ &= \int_V \rho(\mathbf{x})[\mathbf{x} \times (\boldsymbol{\omega} \times \mathbf{x})] \, \mathrm{d}V \end{aligned}$$

Using suffix notation

$$L_i = I_{ij}\omega_j$$

(by writing $\omega_i = \delta_{ij}\omega_j$)
where we have defined inertia tensor

$$I_{ij} = \int_{\mathcal{V}} \rho(\mathbf{x})(x_k x_k \delta_{ij} - x_i x_j) \, \mathrm{d}V$$

where integral is taken over

$$\mathcal{V} = \{(x_1, x_2, x_3) : \mathbf{x} = x_i \mathbf{e}_i \in V\}$$

Can show I_{ij} is a rank 2 tensor

Prop. If T_{ij} is symmetric then there exist choice of $\{\mathbf{e}_i\}$ for which

$$(T_{ij}) = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \gamma \end{bmatrix}$$

The corresponding coordinate axes are called the principal axes of the tensor.

Proof. $T' = R^T T R$. See IA V+M

Moral. So can always choose set of axes so that I_{ij} is diagonal.

8.6 Invariant and Isotropic Tensors

Definition. We say that a tensor is **isotropic** if it is invariant under changes in Cartesian coords, i.e.

$$\begin{aligned} T'_{ij\ldots k} &= R_{ip}R_{jq}\ldots R_{kr}T_{pq\ldots r} \\ &= T_{ij\ldots k} \end{aligned}$$

for any choice of rotation R.

Prop. Isotropic tensors on $\mathbb{R}^3$ are classified as:

(i) All rank 0 tensors isotropic

(ii) There are no non-zero rank 1 tensors

(iii) The most general isotropic tensor of rank 2 is $\alpha\delta_{ij}$ (α scalar)

(iv) The most general isotropic tensor of rank 3 is $\beta\varepsilon_{ijk}$ (β scalar)

(v) The most general isotropic tensor of rank 4 is

$$\alpha\delta_{ij}\delta_{kl} + \beta\delta_{ik}\delta_{jl} + \gamma\delta_{il}\delta_{jk}$$

(vi) The most general isotropic tensor of rank >4 is a linear combination of products of δ and ε (e.g. $\delta_{ij}\varepsilon_{klm}$

Proof. First is by definition. Consider different R to gather more information in the other cases.

Method. Consider integral of form

$$T_{ij\ldots k} = \int_{|\mathbf{x}|<R} f(r)x_ix_j\ldots x_k \,\mathrm{d}V(\mathbf{x})$$

We can show that $T_{ij\ldots k}$ is isotropic so then can use above result
Take $R \to \infty$ corresponds to integrating over all $\mathbb{R}^3$.

8.7 Tensors as Multi-Linear Maps and the Quotient Rule

Method. For a tensor T_{ij} consider bilinear map $t : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ defined by

$$t(\mathbf{a}, \mathbf{b}) := T_{ij} a_i b_j$$

LHS well defined since RHS does not depend on which basis we use (it's a scalar).
So rank two tensor gives rise to bilinear map.
Conversely, suppose $t : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ is bilinear, then for a given basis $\{\mathbf{e}_i\}$ it defines an array T_{ij} via

$$\begin{aligned} t(\mathbf{a}, \mathbf{b}) &= t(a_i \mathbf{e}_i, b_j \mathbf{e}_j) \\ &= a_i b_j t(\mathbf{e}_i, \mathbf{e}_j) \\ &:= a_i b_j T_{ij} \end{aligned}$$

Can show if we use different basis $\{\mathbf{e}'_i\}$ with $\mathbf{e}'_i = R_{ip} \mathbf{e}_p$ then by linearity

$$T'_{ij} = R_{ip} R_{jq} T_{pq}$$

So T_{ij} is rank 2 tensor i.e. bilinear map t gives rise to rank 2 tensor.

Moral. Have a one-to-one correspondence between bilinear maps and rank 2 tensors. In particular if the map

$$(\mathbf{a}, \mathbf{b}) \mapsto T_{ij} a_i b_j$$

is genuinely bilinear, independent of basis, then T_{ij} are components of rank 2 tensor.

Remark. Same idea works for higher rank tensors: if the map

$$(\mathbf{a}, \mathbf{b}, \ldots, \mathbf{c}) \mapsto T_{ij\ldots k} a_i b_j \ldots c_k$$

genuinely defines a n-multilinear map (indep of basis) then $T_{ij\ldots k}$ are components of rank n tensor.

Prop. Let $T_{i\dots jp\dots q}$ be an array of numbers defined in each Cartesian coord system such that

$$\underbrace{v_{i\dots j}}_{A} := \underbrace{T_{i\dots jp\dots q}}_{A+B}\underbrace{u_{p\dots q}}_{B}$$

is a tensor for each tensor $u_{p\dots q}$. Then $T_{i\dots jp\dots q}$ is a tensor.

Proof. Take special case $u_{p\dots q} = c_p \dots d_q$ for vectors $\{\mathbf{c}, \dots, \mathbf{d}\}$. Then

$$v_{i\dots j} := T_{i\dots jp\dots q} c_p \dots d_q$$

is a tensor and in particular

$$v_{i\dots j} a_i \dots b_j = T_{i\dots jp\dots q} a_i \dots b_j c_p \dots d_q$$

is a scalar for each $\{\mathbf{a}, \dots \mathbf{b}, \mathbf{c}, \dots, \mathbf{d}\}$. So RHS is scalar (indep of basis) and gives rise to well-defined multilinear map via

$$t(\mathbf{a}, \dots, \mathbf{b}, \mathbf{c}, \dots, \mathbf{d}) := T_{i\dots jp\dots q} a_i \dots b_j c_p \dots d_q$$

so by previous discussion, $T_{i\dots jp\dots q}$ is a tensor. □

Warning. Need to check holds for all tensors $u_{p\dots q}$