Vectors & Matrices

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Contents

1 Complex Numbers

Prop. $|z_1 z_2| = |z_1||z_2|$ (composition property)
$|z_1 + z_2| \leq |z_1| + |z_2|$ (triangle property)

Proof (of composition property)**.** Compute square of each side and compare.

Proof (of triangle inequality)**.** $\text{LHS}^2 = (z_1 + z_2)(\overline{z_1} + \overline{z_2}) = |z_1|^2 + z_1\overline{z_2} + \overline{z_1}z_2 + |z_2|^2$
$\text{RHS}^2 = |z_1|^2 + |z_2|^2 + 2|z_1||z_2|$
Note:

$$z_1\overline{z_2} + \overline{z_1}z_2 \leq 2|z_1||z_2|$$
$$\iff \frac{1}{2}(z_1 z_2 + \overline{(z_1\overline{z_2})}) \leq |z_1||\overline{z_2}|$$
$$\iff \text{Re}(z_1\overline{z_2}) \leq |z_1\overline{z_2}|$$

which is true. ☐

Alternative form of the Δ inequality:
$|z_2 - z_1| \geq |z_2| - |z_1|$ by replacing z_2 with $z_2 - z_1$ in original form and rearranging
or $|z_2 - z_1| \geq |z_1| - |z_2|$
so $|z_2 - z_1| \geq ||z_2| - |z_1||$

Theorem (De Moivre's Theorem)**.** $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$

Proof. Induction.

Definition. $\exp(z) = e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$
$\cos(z) = \frac{1}{2}(e^{iz} + e^{-iz})$
$\sin(z) = \frac{1}{2i}(e^{iz} - e^{-iz})$

Prop. $e^z e^w = e^{z+w}$

Proof. Multiply series on LHS and find terms of degree n
$\sum_{r=0}^{n} \frac{1}{r!}z^r \frac{1}{(n-r)!}w^{n-r} = \frac{1}{n!}(z+w)^n$ by Binomial Theorem.

Equation of a line through z_0, parallel to w: $\overline{w}z - w\overline{z} = \overline{w}z_0 - w\overline{z_0}$
Equation of a circle center c, radius ρ: $|z|^2 - \overline{c}z - c\overline{z} = \rho^2 - |c|^2$

2 Vectors in 3D

Definition. $\text{span}\{\mathbf{a}\} = \{\lambda\mathbf{a} : \lambda \in \mathbb{R}\}$

Definition. $\mathbf{a}$, $\mathbf{b}$ **parallel** ($\mathbf{a}||\mathbf{b}$) iff $\mathbf{a} = \lambda\mathbf{b}$ or $\mathbf{b} = \lambda\mathbf{a}$ for some $\lambda \in \mathbb{R}$ (λ can be 0)

Definition. $\text{span}\{\mathbf{a,b}\} = \{\alpha\mathbf{a} + \beta\mathbf{b} : \alpha, \beta \in \mathbb{R}\}$

$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}) = -\mathbf{a} \cdot (\mathbf{c} \times \mathbf{b}) = -\mathbf{b} \cdot (\mathbf{a} \times \mathbf{c}) = -\mathbf{c} \cdot (\mathbf{b} \times \mathbf{a})$

$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{b} \cdot \mathbf{c})\mathbf{a}$

$\mathbf{r} = \mathbf{a} + \lambda\mathbf{u}$, $\lambda \in \mathbb{R}$
Has alternative form $\mathbf{u} \times (\mathbf{r} - \mathbf{a}) = 0$

3 Vectors in general: $\mathbb{R}^n$, $\mathbb{C}^n$

Definition. **Inner product** on $\mathbb{R}^n$ defined by:
$\mathbf{x} \cdot \mathbf{y} = \sum_i x_i y_i = x_1 y_1 + \cdots + x_n y_n$

Properties:

(i) Symmetric $\mathbf{x} \cdot \mathbf{y} = \mathbf{y} \cdot \mathbf{x}$

(ii) Bilinear

$$(\lambda\mathbf{x} + \lambda'\mathbf{x}') \cdot \mathbf{y} = \lambda(\mathbf{x} \cdot \mathbf{y}) + \lambda'(\mathbf{x}' + \mathbf{y})$$
$$\mathbf{x} \cdot (\mu\mathbf{y} + \mu'\mathbf{y}') = \mu(\mathbf{x} \cdot \mathbf{y}) + \mu'(\mathbf{x} \cdot \mathbf{y}')$$

(iii) Positive definite $\mathbf{x} \cdot \mathbf{x} = \sum_i x_i^2 \geq 0$
and $\mathbf{x} \cdot \mathbf{x} = 0$ iff $\mathbf{x} = 0$.

Prop (Cauchy-Schwarz)**.** $|\mathbf{x} \cdot \mathbf{y}| \leq |\mathbf{x}||\mathbf{y}|$ with equality iff $\mathbf{x}$ and $\mathbf{y}$ are parallel

Proof. If $\mathbf{y} = \mathbf{0}$, result is immediate.
If $\mathbf{y} \neq \mathbf{0}$ then consider $|\mathbf{x} - \lambda\mathbf{y}|^2 = (\mathbf{x} - \lambda\mathbf{y}) \cdot (\mathbf{x} - \lambda\mathbf{y})$
By bilinearity it's $|\mathbf{x}|^2 - \lambda\mathbf{x} \cdot \mathbf{y} + \lambda^2|\mathbf{y}|^2$
So, $|\mathbf{x} - \lambda\mathbf{y}|^2 = |\mathbf{x}|^2 - \lambda\mathbf{x} \cdot \mathbf{y} + \lambda^2|\mathbf{y}|^2 \geq 0$
This is a real quadratic in λ with at most 1 real root so discriminant ≤ 0 which gives desired inequality.

Note. Setting $\mathbf{x} \cdot \mathbf{y} = |\mathbf{x}||\mathbf{y}| \cos\theta$ allows us to define the angle θ between $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^n$

Prop. Cauchy-Schwarz $\implies$ triangle inequality

Proof.

$$\begin{aligned} |\mathbf{x} + \mathbf{y}|^2 &= |\mathbf{x}|^2 + 2\mathbf{x} \cdot \mathbf{y} + |\mathbf{y}|^2 \\ &\leq |\mathbf{x}|^2 + 2|\mathbf{x}||\mathbf{y}|||\mathbf{y}|^2 \\ &= (|\mathbf{x}| + |\mathbf{y}|)^2 \end{aligned}$$

Definition. $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\} \in V$ (a real vector space) form a **linearly independent set** iff:

$$\lambda_1\mathbf{v}_1 + \lambda_2\mathbf{v}_2 + \cdots + \lambda_r\mathbf{v}_r = 0 \quad (1)$$
$$\implies \lambda_i = 0 \,\forall i$$

i.e. only zero trivially.
If (1) holds with at least one $\lambda_i \neq 0$ then they form a linearly dependent set.

In $\mathbb{R}^3$, $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ is linearly independent iff $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) \neq 0$

Definition. For a vector space V, a **basis** is a set $B = \{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ such that:

(i) B spans V : $\mathbf{v} \in V \implies \mathbf{v} = \sum_{i=1}^{n} \mathbf{v}_i \mathbf{e}_i$

(ii) B is linearly independent

Given (ii), the coefficients $\mathbf{v}_i$ are unique since: $\sum_i \mathbf{v}_i \mathbf{e}_i = \sum_i v_i' \mathbf{e}_i \implies \sum_i (\mathbf{v}_i - v_i')\mathbf{e}_i = \mathbf{0} \implies \mathbf{v}_i = v_i'$ from (i).

Theorem. If $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ and $\{\mathbf{f}_1, \ldots, \mathbf{f}_m\}$ are bases for a vector space V then $n = m$

Proof. $\mathbf{f}_a = \sum_i A_{ai} \mathbf{e}_i$ as $\mathbf{e}$ is a basis
and $\mathbf{e}_i = \sum_a B_{ia} \mathbf{f}_a$ for constants $A_{ai}, B_{ia} \in \mathbb{R}$
$i, j = 1, \ldots, n$
$a, b = 1, \ldots, m$

$$\begin{aligned} \mathbf{f}_a &= \sum_i A_{ai} \left(\sum_b B_{ib} \mathbf{f}_b \right) \\ &= \sum_b \left(\sum_i A_{ai} B_{ib} \right) \mathbf{f}_b \end{aligned}$$

But $\mathbf{f}_a, \mathbf{f}_b$ are linearly independent and coefficients wrt a basis are unique, hence:
$\sum_i A_{ai} B_{ib} = \delta_{ab}$ otherwise relation is nontrivial.
Similarly:

$$\mathbf{e}_i = \sum_j \left(\sum_a B_{ia} A_{aj} \right) \mathbf{e}_j$$

Hence, $\sum_a B_{ia} A_{aj} = \delta_{ij}$

Now:

$$\begin{aligned} \sum_{i,a} A_{ai} B_{ia} &= \sum_a \delta_{aa} = m \\ &= \sum_i \delta_{ii} = n \end{aligned}$$

So $m = n$ $\square$

Note. Steps in proof of basis theorem within scope of course, but proof without prompts non-examinable.

Definition. The number of vectors is any basis is the **dimension** of the space. (Well-defined due to above).

Prop. Let V be a vector space with finite subsets:
$Y = \{\mathbf{w}_1, \ldots, \mathbf{w}_m\}$ that spans V
$X = \{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ that is linearly independent.
Then $k \leq n \leq m$ where n = dimension of V. And:

(i) A basis can be found as a subset of Y by discarding vectors in Y if necessary
(ii) X can be extended to a basis by adding additional vectors from Y as necessary

Proof.

(i) If Y is lin. indep., then Y is a basis and $m = n = \dim V$.
If Y is not lin. indep., then there is $\sum_{i=1}^{m} \lambda_i \mathbf{w}_i = \mathbf{0}$, $\lambda_i \neq 0$ for some i.
WLOG take $\lambda_m \neq 0$, then:
$\mathbf{w}_m = -\frac{1}{\lambda_m} \sum_{i=1}^{m-1} \lambda_i \mathbf{w}_i$
so span Y = span Y' with $Y' = \{\mathbf{w}_1, \ldots, \mathbf{w}_{m-1}\}$.
Repeat until a basis is obtained.

(ii) If X spans V then it is already a basis and $k = n = \dim V$.
If not, then $\exists \mathbf{u}_{k+1} \in V$ (not in span X)
Bu then since $\mathbf{u}_{k+1} \notin$ span X, if: $\sum_{i=1}^{k+1} \mu_i \mathbf{u}_i = \mathbf{0}$ then $\mu_{k+1} = 0$ (otherwise $\mathbf{u}_{k+1}$ in span X)
Then $\mu_i = 0$ for $i = 1, \ldots .k$ (X lin. indep.)
Hence $X' = \{\mathbf{u}_1, \ldots, \mathbf{u}_k, \mathbf{u}_{k+1}\}$ is lin. indep.
Furthermore, we can choose $\mathbf{u}_{k+1}$ from Y
(if $Y \subseteq$ span X then span $Y \subseteq$ span $X \implies$ span $X = V$)
Repeat $X \to X'$ until a basis is obtained. Process stops as Y is finite. □

Note. Steps in proof of theorem within scope of course, but proof without prompts non-examinable.

Definition. **Inner product** or **scalar product** on $\mathbb{C}^n$ is defined by:

$$(\mathbf{z}, \mathbf{w}) = \sum_j \overline{z}_j w_j = \overline{z}_1 w_1 + \cdots + \overline{z}_n w_n$$

Properties:

(i) Hermitian: $(\mathbf{w}, \mathbf{z}) = (\overline{\mathbf{z}, \mathbf{w}})$
(ii) Linear/ anti-linear
$(\mathbf{z}, \lambda\mathbf{w} + \lambda'\mathbf{w}') = \lambda(\mathbf{z}, \mathbf{w}) + \lambda'(\mathbf{z}, \mathbf{w}')$
$(\lambda\mathbf{z} + \lambda'\mathbf{z}', \mathbf{w}) = \overline{\lambda}(\mathbf{z}, \mathbf{w}) + \lambda'(\mathbf{z}', \mathbf{w}')$
(iii) Positive definite

$$(\mathbf{z}, \mathbf{z}) = \sum_j |z_j|^2, \text{ real and } \geq 0$$
$$= 0 \text{ iff } \mathbf{z} = 0$$

4 Matrices and Linear Maps

Definition (Linear Map)**.** A **linear map** or **linear transformation** is a function $T : V \to W$ between vector spaces V (dim n) and W (dim m) such that:

$$\begin{aligned} T(\lambda \mathbf{x} + \mu \mathbf{y}) &= T(\lambda \mathbf{x}) + T(\mu \mathbf{y}) \\ &= \lambda T(\mathbf{x}) + \mu T(\mathbf{y}) \end{aligned}$$

for $\mathbf{x}, \mathbf{y} \in V$, $\lambda, \mu \in \mathbb{R}$ or $\mathbb{C}$

Note. A linear map is completely determined by its action on a basis (exercise)

$\mathbf{x}' = T(\mathbf{x}) \in W$ is the image of $\mathbf{x} \in V$ under linear map T.
$\mathrm{Im}(T) = \{\mathbf{x}' \in W : \mathbf{x}' = T(\mathbf{x})$ for $\mathbf{x} \in V\}$
$\ker(T) = \{\mathbf{x} \in V : T(\mathbf{x}) = 0\}$

Lemma. $\ker(T)$ is a subspace of V and $\mathrm{Im}(T)$ is a subspace of W

Proof. Exercise.

Definition. $\dim \mathrm{Im}(T)$ or $\mathrm{rank}(T)$ is the **rank** of T $(\leq m)$
$\dim \ker(T)$ or $\mathrm{null}(T)$ is the **nullity** of T $(\leq n)$

Theorem. For $T : V \to W$ a linear map as above,
$\mathrm{rank}(T) +\ \mathrm{null}(T) = n = \dim V$

Proof. non-examinable.

Rotation angle θ about axis $\mathbf{n}$:

$$\mathbf{x} \mapsto (\cos\theta)\mathbf{x} + (1 - \cos\theta)(\mathbf{n} \cdot \mathbf{x})\mathbf{n} + (\sin\theta)\mathbf{n} \times \mathbf{x}$$

Derive by writing $\mathbf{x}$ in parallel and perp. to $\mathbf{n}$ components

Projection to plane with unit normal $\mathbf{n}$:

$$\mathbf{x} \mapsto \mathbf{x} - (\mathbf{x} \cdot \mathbf{n})\mathbf{n}$$

Reflection in plane with unit normal $\mathbf{n}$:

$$\mathbf{x} \mapsto \mathbf{x} - 2(\mathbf{x} \cdot \mathbf{n})\mathbf{n}$$

Dilations along axes defined as one would expect.

Note. Can have different scale factors for different axes.

Let $\mathbf{a}, \mathbf{b}$ be orthogonal unit vectors in $\mathbb{R}^3$, λ a real parameter.
Define a shear parallel to $\mathbf{a}$ scale factor λ:

$$\mathbf{x} \mapsto \mathbf{x} + \lambda \mathbf{a}(\mathbf{x} \cdot \mathbf{b}).$$

Claim. Matrix acting on vector acts as expected ($T(\mathbf{x})$):

Proof. Write $\mathbf{x} = \sum_i x_i \mathbf{e}_i$ and apply linearity. (Have $\mathbf{e}_i \mapsto \mathbf{C}_i$)

Claim. Image of matrix M is span of columns.

Proof. Write any $\mathbf{x}$ in components, apply linearity.

Note. $\mathbf{x}'_a = M_{ai}\mathbf{x}_i = (\mathbf{R}_a)_i \mathbf{x}_i = \mathbf{R} \cdot \mathbf{x}$
$\ker(T) = \ker(M) = \{\mathbf{x} : \mathbf{R}_a \cdot \mathbf{x} = 0 \, \forall a\}$
Kernel of M is the subspace $\perp$ to all rows. (Can use cross product to quickly work out kernel span)

Claim. $(\alpha M + \beta N)\mathbf{x} = \alpha M(\mathbf{x}) + \beta N(\mathbf{x})$

Proof. Consider ith component of $\mathbf{x}$.

Claim. $(AB)^T = B^T A^T$

Proof. consider components.

Definition. M **symmetric** iff $M^T = M$
M **anti-symmetric** iff $M^T = -M$

Any M can be written $M = S + A$
$S = \frac{1}{2}(M + M^T)$
$A = \frac{1}{2}(M - M^T)$

Note. If A is 3×3, antisymmetric, then $A_{ij} = \varepsilon_{ijk} a_k$

$$A = \begin{bmatrix} 0 & a_3 & -a_2 \\ -a_3 & 0 & a_1 \\ a_2 & -a_1 & 0 \end{bmatrix}$$

$(A\mathbf{x})_i = \varepsilon_{ijk} a_k x_j = (\mathbf{x} \times \mathbf{a})_i$

Definition. The **Hermitian conjugate** of a matrix M is: $M^\dagger = (\overline{M^T})$

Definition. M is **Hermitian** if $M = M^\dagger$
M is **anti-Hermitian** if $M = -M^\dagger$

Definition. $\mathrm{tr}(M) = M_{ii}$

$\mathrm{tr}(\alpha M + \beta N) = \alpha \mathrm{tr}(M) + \beta \mathrm{tr}(N)$
$\mathrm{tr}(MN) = \mathrm{tr}(NM)$
$\mathrm{tr}(M^T) = \mathrm{tr}(M)$
$\mathrm{tr}(I) = \delta_{ii} = n$ if I is $n \times n$

We can split a matrix S into traceless and pure trace parts:
$T = S - \frac{1}{n}\mathrm{tr}(S) I$

Definition. A real $n \times n$ matrix U is **orthogonal** iff:

$$U^T U = U U^T = I \text{ i.e. } (U^T = U^{-1})$$

These conditions can be written $U_{ki} U_{kj} = I_{ik} U_{jk} = \delta_i j$. Equivalently: columns of U are orthonormal, or rows of U are orthonormal.
Also equivalently: U is orthogonal iff it preserves the inner product on $\mathbb{R}^n$
i.e. $(U\mathbf{x}) \cdot (U\mathbf{y}) = \mathbf{x} \cdot \mathbf{y} \, \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n$

Definition. A complex $n \times n$ matrix U is **unitary** iff

$$U^\dagger U = U U^\dagger = I \text{ i.e. } (U^\dagger = U^{-1})$$

Equivalently: U is unitary iff it preserves complex inner product on $\mathbb{C}^n$: $(U\mathbf{z}, U\mathbf{w}) = (\mathbf{z}, \mathbf{w})$

Note. Orthogonal matrices from a group. Unitary matrices form a group.

5 Determinants and Inverses

T invertible
$\iff \ker T = \{0\}$
$\iff \operatorname{Im} T = \mathbb{R}^n$ (equivalent by rank-nullity)
If the conditions hold, then $T(\mathbf{e}_1), T(\mathbf{e}_2), \dots, T(\mathbf{e}_n)$ must be a basis and we can define T^{-1} as a linear map by $T^{-1}(T(\mathbf{e}_i)) = \mathbf{e}_i$

Definition. For any $M, (n \times n)$, we define a related matrix $\widetilde{M}$ $(n \times n)$ and a scalar $\det M$ such that $\widetilde{M}M = (\det M)I$ $(*)$
If $\det M \neq 0$ then M is invertible with $M^{-1} = \frac{1}{\det M}\widetilde{M}$.

Definition. The **alternating symbol** or ε symbol in $\mathbb{R}^n$ or $\mathbb{C}^n$ is an n-index object (tensor) defined by:

$$\varepsilon_{ij\dots l} = \begin{cases} +1 & \text{if } i, j, \dots, l \text{ is an even perm of } 1, 2, \dots, n \\ -1 & \text{if } i, j, \dots, l \text{ is an odd perm of } 1, 2, \dots, n \\ 0 & \text{else} \end{cases}$$

Thus if σ is any permulation,
$\varepsilon_{\sigma(1)\sigma(2)\dots\sigma(n)} = \varepsilon(\sigma)$ (i.e. the sign of the permutation)

Definition. Given $\mathbf{v}_1, \dots, \mathbf{v}_n \in \mathbb{R}^n$, the **alternating form** combines them to give a scalar:

$$\begin{aligned}[\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n] &= \varepsilon_{ij\dots l}(\mathbf{v}_1)_i(\mathbf{v}_2)_j \dots (\mathbf{v}_n)_l \\ &= \sum_\sigma \varepsilon(\sigma)(\mathbf{v}_1)_{\sigma(1)}(\mathbf{v}_2)_{\sigma(2)} \dots (\mathbf{v}_n)_{\sigma(n)}\end{aligned}$$

(i) Multilinear:

$$\begin{aligned}&[\mathbf{v}_1, \dots, \mathbf{v}_{p-1}, \alpha\mathbf{u} + \beta\mathbf{w}, \mathbf{v}_{p+1}, \dots, \mathbf{v}_n] \\ &= \alpha[\mathbf{v}_1, \dots, \mathbf{v}_{p-1}, \mathbf{u}, \mathbf{v}_{p+1}, \dots, \mathbf{v}_n] + \beta[\mathbf{v}_1, \dots, \mathbf{v}_{p-1}, \mathbf{w}, \mathbf{v}_{p+1}, \dots, \mathbf{v}_n]\end{aligned}$$

(ii) Totally antisymmetric:

$$[\mathbf{v}_{\sigma(1)}, \mathbf{v}_{\sigma(2)}, \dots, \mathbf{v}_{\sigma(n)}] = \varepsilon(\sigma)[\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n]$$

(justify by checking for a transposition $\tau = (p, q)$, $\sigma' = \sigma\tau$)

(iii)

$$[\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n] = +1$$

These three properties fix alternating form completely and also imply
(iv) if $\mathbf{v}_p = \mathbf{v}_q$ then $[\mathbf{v}_1, \dots, \mathbf{v}_p, \dots, \mathbf{v}_q, \dots, \mathbf{v}_n] = 0$ (from (ii) by exchanging $\mathbf{v}_p, \mathbf{v}_q$)
(v) If $\mathbf{v}_p = \sum_{i \neq p} \lambda_i \mathbf{v}_i$ then

$$[\mathbf{v}_1, \dots, \mathbf{v}_p, \dots, \mathbf{v}_n] = 0$$

(since subbing $\mathbf{v}_p$ in and using multilinearity - use (iv))

Prop.

$$[\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n] \neq 0 \iff \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n \text{ are linearly independent}$$

Proof. To show $\implies$ use property (v)
To show $\impliedby$ use spanning and show multiple of $[\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n]$.

Definition. For an $n \times n$ matrix M with columns $\mathbf{c}_a = M\mathbf{e}_a$, the **determinant** $\det M$ or $|M| \in \mathbb{R}$ or $\mathbb{C}$, is defined by:

$$\begin{aligned}\det M &= [\mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_n] \\ &= [M\mathbf{e}_1, M\mathbf{e}_2, \ldots, M\mathbf{e}_n] \\ &= \varepsilon_{ij\ldots l} M_{i1} M_{j2} \ldots M_{ln} \\ &= \sum_\sigma \varepsilon(\sigma) M_{\sigma(1)1} M_{\sigma(2)2} \ldots M_{\sigma(n)n}\end{aligned}$$

Each of these expressions can be taken as the definition.

Prop. If $\mathbf{R}_a$ are rows of M, then:

$$\begin{aligned}\det M &= [\mathbf{R}_1, \mathbf{R}_2, \ldots, \mathbf{R}_n] \\ &= \varepsilon_{ij\ldots l} M_{1i} M_{2j} \ldots M_{nl} \\ &= \sum_\sigma \varepsilon(\sigma) M_{1\sigma(1)} M_{2\sigma(2)} \ldots M_{n\sigma(n)}\end{aligned}$$

i.e. $\det M = \det M^T$

Proof. Show directly that the $\sum_\sigma$ definitions agree by considering:
$M_{\sigma(1)1} M_{\sigma(2)2} \ldots M_{\sigma(n)n} = M_{1\rho(1)} M_{2\rho(2)} \ldots M_{n\rho(n)}$ for $\rho = \sigma^{-1}$
But $\varepsilon(\sigma) = \varepsilon(\sigma^{-1}) = \varepsilon(\rho)$ so $\sum_\sigma$ is $\sum_\rho$ as required. $\square$

Definition. Define **minor** M^{ia} to be the $(n-1) \times (n-1)$ determinant of matrix obtained by deleting row i and column a from M

Adding multiple of one row to another does not affect the determinant by multilinearity.
Swapping rows changes sign of determinant by alternating property.

Lemma.

$$\varepsilon_{i_1 i_2 \ldots i_n} M_{i_1 a_1} M_{i_2 a_2} \ldots M_{i_n a_n} = (\det M) \varepsilon_{a_1 a_2 \ldots a_n}$$

Proof. LHS and RHS each totally antisymmetric in $a_1, a_2, \ldots, a_n$ and so must be related by a constant factor. To fix this constant, consider $a_1 = 1, a_2 = 2, \ldots, a_n = n$ and check it works, then result follows.

Theorem. For $n \times n$ matrices M, N:

$$\det(MN) = \det M \det N$$

Proof.

$$\begin{aligned}\det(MN) &= \varepsilon_{i_1 \dots i_n}(MN)_{i_1 1} \dots (MN)_{i_n n} \\ &= \varepsilon_{i_1 \dots i_n} M_{i_1 k_1} \dots M_{i_n k_n} N_{k_1 1} \dots N_{k_n n} \\ &= (\det M)\varepsilon_{k_1 \dots k_n} N_{k_1 1} \dots N_{k_n n} \text{ by lemma} \\ &= (\det M)(\det N)\square\end{aligned}$$

Prop.

$$\begin{aligned}\det M &= \sum_i (-1)^{i+a} M_{ia} M^{ia} \; (a \text{ fixed}) \\ &= \sum_a (-1)^{i+a} M_{ia} M^{ia} \; (i \text{ fixed}))\end{aligned}$$

Proof. consider a column of matrix M $(n \times n)$ and write it:
$\mathbf{c}_a = \sum_i M_{ia}\mathbf{e}_i$

$$\begin{aligned}\implies \det M &= [\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_a, \dots, \mathbf{c}_n] \\ &= [\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_{a-1}, \sum_i M_{ia}\mathbf{e}_i, \mathbf{c}_{a+1}, \dots, \mathbf{c}_n] \\ &= \sum_i M_{ia}\Delta_{ia} \text{ where } \Delta_{ia} = [\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_{a-1}, \mathbf{e}_i, \mathbf{c}_{a+1}, \dots, \mathbf{c}_n]\end{aligned}$$

Now consider what Δ_{ia} matrix looks like and use row ops to reach desired result.
Can work similarly considering rows.

Definition. Δ is the **matrix of cofactors**(entries Δ_{ia}).

Definition. The **adjugate** of M:
$\widetilde{M} = \text{adj } M = \Delta^T$

Claim. $M\widetilde{M} = (\det M)I$

Proof. Consider $\mathbf{c}_b = \sum_i M_{ib}\mathbf{e}_i$
Then

$$[\mathbf{c}_1, \dots, \mathbf{c}_{a-1}, \mathbf{c}_b, \mathbf{c}_{a+1}, \dots, \mathbf{c}_n] = \sum_i M_{ib}\Delta_{ia} = \begin{cases} \det M & \text{if } a = b \\ 0 & \text{if } a \neq b \end{cases} = (\det M)\delta_{ab}$$

Similarly, expanding a row:

$$\sum_a M_{ja}\Delta_{ia} = (\det M)\delta_{ij}$$

Remaining steps trivial.

Hence, if $\det M \neq 0$ then $M^{-1} = \frac{1}{\det M}\widetilde{M}$

For linear equations, just methods.
Image is span of columns.
Kernel is perpendicular to all rows (cross product).

6 Eigenvalues and Eigenvectors

Definition. For a linear map $T : V \to V$, a vector $\mathbf{v}$ with $\mathbf{v} \neq \mathbf{0}$ is an **eigenvector** of T with **eigenvalue** λ if $T(\mathbf{v}) = \lambda\mathbf{v}$.
If $V = \mathbb{R}^n$ or $\mathbb{C}^n$ and T is given by an $n \times n$ matrix A then:

$$A\mathbf{v} = \lambda\mathbf{v} \iff (A - \lambda I)\mathbf{v} = \mathbf{0}$$

So given λ, this holds for some $\mathbf{v} \neq \mathbf{0}$ iff $\det(A - \lambda I) = 0$

Definition. For an eigenvalue λ of a matrix A, define the **eigenspace**:

$$E_\lambda = \{\mathbf{v} : A\mathbf{v} = \lambda\mathbf{v}\} = \ker(A - \lambda I)$$

The set of all non-zero $\mathbf{v} \in E_\lambda$ are the eignevectors.

Definition. The **geometric multiplicity** is $m_\lambda = \dim E_\lambda =$ no. of linearly indep. evecs with eval λ.
$m_\lambda = \text{null } (A - \lambda I)$

Definition. The **algebraic multiplicity** is M_λ, multiplicity of λ as a root of $\chi_A(t)$ i.e. $\chi_A(t) = (t - \lambda)^{M_\lambda} f(t)$ with $f(\lambda) \neq 0$

Prop. $M_\lambda \geq m_\lambda$ (and $m_\lambda \geq 1$ as λ a root of $\chi_A(t)$)

Proof. Later

Prop. Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ be eigenvectors of a matrix $A\,(n \times n)$ with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_r$. If eigenvalues are distinct, then $\mathbf{v}_i$ are linearly independent.

Proof. Note that $\mathbf{w} = \sum_{j=1}^{r} \alpha_j \mathbf{v}_j \implies$

$(A - \lambda I)\mathbf{w} = \sum_{j=1}^{r} \alpha_j (\lambda_j - \lambda)\mathbf{v}_j$

Way 1. Suppose evecs linearly dependent. So $\exists$ a linear combination $\mathbf{w} = 0$ with no. of non-zero coefficients $p \geq 2$.
Pick such a $\mathbf{w}$ for which p is least. WLOG $\alpha_1 \neq 0$.
Then $(A - \lambda I)\mathbf{w} = \sum_{i=1}^{j>1} \alpha_j(\lambda_j - \lambda_1)\mathbf{v}_j = 0$, a linear relation with $p-1$ nonzero coefficients.
※

Way 2. Given a linear relation $\mathbf{w} = 0 \implies \prod_{j \neq k} (A - \lambda_j I)\mathbf{w} = \alpha_k (\prod_{j \neq k} (\lambda_k - \lambda_j))\mathbf{v}_k = 0$ for k fixed.
Eigenvalues distinct so $\alpha_k = 0$.
I.e. eigenvectors are linearly independent. ☐

Corollary. With conditions as in prop, let B_{λ_i} be a basis for the eigenspace λ_i $i = 1, 2, \ldots, r$ Then $B_{\lambda_1} \cup B_{\lambda_2} \cup \cdots \cup B_{\lambda_r}$ is linearly independent.

Proof. Consider general linear combination of all these vecs:
has form $\mathbf{w} = \mathbf{w}_1 + \mathbf{w}_2 + \cdots + \mathbf{w}_r$, where $\mathbf{w}_i \in E_{\lambda_i}$.
Apply same argument as in prop to deduce if $\mathbf{w} = 0$ then $\mathbf{w}_i = 0$ for each i.
Each $\mathbf{w}_i$ is trivial linear combination of elements of B_{λ_i} and the result follows. ☐

Prop. For an $n \times n$ matrix A, acting on $V = \mathbb{R}^n$ or $\mathbb{C}^n$, the following condition are equivalent:

(i) There exists a basis of eigenvectors for V, $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ with $A\mathbf{v}_i = \lambda_i \mathbf{v}_i$ (not summation)

(ii) There exists an $n \times n$ invertible matrix P with $P^{-1}AP = D = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$

Proof. Note that for any matrix P, AP has columns $Ac_i(P)$ and PD has columns $\lambda_i c_i(P)$
Then (i) and (ii) are related by choosing $\mathbf{v}_i = c_i(P)$:

$$P^{-1}AP = B \iff AP = PD \iff A\mathbf{v}_i = \lambda_i \mathbf{v}_i \square$$

i.e. given an eigenvector basis as in (i), this relation defines P; conversel, given a matrix P as in (ii), its columns are a basis of eigenvectors.

Prop. Consider $n \times n$ matrix A.

(i) A is diagonalisable if it has n distinct eigenvalues (sufficient)

(ii) A is diagonalisable iff for every eigenvalue A, the multiplicity coincides: $M_\lambda = m_\lambda$ (necessary and sufficient)

Proof. Use prop and corr above.

(i) If n distinct evals, then n lin indep evecs so they form a basis.

(ii) If λ_i with $i = 1, 2, \ldots, r$ are all the distinct evals then $B_{\lambda_1} \cup \cdots \cup B_{\lambda_r}$ is lin indep but no. of elements is $\sum_i m_{\lambda_i}$ (dim of each E_{λ_i}) $= \sum_i M_{\lambda I} = n$ (degree of char. poly) where B_{λ_i} is a basis for E_{λ_i}. So we have a basis. □

Definition. Matrices A and $B\,(n \times n)$ are **similar** if $B = P^{-1}AP$ for some invertible $P\,(n \times n)$, an equivalence relation.

Prop. If A and B are similar, then

(i) tr $B =$ tr A

(ii) $\det B = \det A$

(iii) $\chi_B(t) = \chi_A(t)$

Proof. (i) trivial using cyclic property

(ii) trivial using multiplicative properrt of det

(iii) consider $\det(B - tI) = \det\big(P^{-1}AP - tP^{-1}P\big)$ and factor

Obervation: if A is hermitian, then $(A\mathbf{v})^\dagger \mathbf{w} = \mathbf{v}^\dagger (A\mathbf{w})$ for all $\mathbf{v}, \mathbf{w} \in \mathbb{C}^n$ since $(\mathbf{v}^\dagger A^\dagger)\mathbf{w} = \mathbf{v}^\dagger A^\dagger \mathbf{w} = \mathbf{v}^\dagger A \mathbf{w} = \mathbf{v}^\dagger (A\mathbf{w})$

Theorem. For a matrix A that is hermitian $(n \times n)$

(i) Every eigenvalue λ is real

(ii) Eigenvectors $\mathbf{v}, \mathbf{w}$ with distinct eigenvalues $\lambda \neq \mu$ are orthogonal. $(\mathbf{v}^\dagger \mathbf{w} = 0)$

(iii) If A is real and symmetric, then for each λ in (i), we can choose a real eigenvector $\mathbf{v}$ and (ii) becomes

$$\mathbf{v}^T w = \mathbf{v} \cdot \mathbf{w} = 0$$

Proof. (i) $\mathbf{v}^\dagger (A\mathbf{v}) = (A\mathbf{v})^\dagger \mathbf{v}$

$\implies \mathbf{v}^\dagger (\lambda \mathbf{v}) = (\lambda \mathbf{v})^\dagger \mathbf{v}$

$\mathbf{v} \neq 0$ so $\mathbf{v}^\dagger \mathbf{v} \neq 0$, so $\mathbf{v} = \overline{\lambda}$ so real.

(ii) $\mathbf{v}^\dagger (A\mathbf{w}) = (A\mathbf{v})^\dagger \mathbf{w}$

$\implies \mathbf{v}^\dagger (\mu \mathbf{w}) = (\lambda \mathbf{v})^\dagger \mathbf{w}$

$\implies \mu \mathbf{v}^\dagger \mathbf{w} = \lambda \mathbf{v}^\dagger \mathbf{w}$

$\lambda \neq \mu$, so $\mathbf{v}^\dagger \mathbf{w} = 0$

(iii) Given $A\mathbf{v} = \lambda \mathbf{v}$ with $\mathbf{v} \in \mathbb{C}^n$ but A and λ real,

Let $\mathbf{v} = \mathbf{u} + i\mathbf{u}'$ with $\mathbf{u}, \mathbf{u}' \in \mathbb{R}^n$

then $A\mathbf{u} = \lambda \mathbf{u}, A\mathbf{u}' = \lambda \mathbf{u}'$ (Re and Im parts)

but $\mathbf{v} \neq 0 \implies$ one of $\mathbf{u}, \mathbf{u}' \neq 0$, so $\exists \geq 1$ real eigenvector. □

Note. Shows sets of evecs with distinct evals lin indep, but for Hermitian matrices, we have they are orthogonal $\implies$ linear independence.
Furthermore, previously considered bases B_λ for each eigenspace E_λ, now natural to choose bases B_λ to be orthonormal

Theorem. Any $n \times n$ hermitian matrix is diagonalisable if:

(i) $\exists$ a basis of eigenvectors $\mathbf{u}_1, \dots, \mathbf{u}_n \in \mathbb{C}^n$ with $A\mathbf{u}_i = \lambda \mathbf{u}_i$. Equivalently,

(ii) $\exists\, n \times n$ invertible matrix P with $P^{-1}AP = D = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$

Definition. A **quadric form** is a function $\mathbb{R}^2 \to \mathbb{R}$ given by $F(x) = \mathbf{x}^T A \mathbf{x} = x_i A_{ij} x_j$ where A is a real symmetric $n \times n$ matrix. (anti anti-symmetric part of A would not contribute).

A can be diagonalised. The principal axes of F are the evecs of A.

Definition. $\exp(A) = I + A + \frac{1}{2}A^2 + \dots + \frac{1}{r!}A^r + \dots$ (always converges)

Theorem (Cayley-Hamilton)**.**

$$\chi_A(A) = c_0 I + c_1 A + \dots + c_n A^n = 0$$

'a matrix satisfies its own characteristic equation.'
General case not examinable.

Proof. (i) General 2×2: trivial by substitution.
(ii) Diagonalisable $n \times n$ matrix: write $\chi_A(A) = \chi_A(PDP^{-1}) = P\chi_A(D)P^{-1} = 0$.

Note.

$$-c_0 I = A(c_1 I + \dots + c_n A^{n-1}$$

If $c_0 = \det A \neq 0$ then A invertible and:

$$A^{-1} = -\frac{1}{c_0}(c_1 A + \dots + c_n A^{n-1})$$

7 Changing Bases, Canonical Forms and Symmetries

Change of basis from $\{\mathbf{e}_i\}$ to $\{\mathbf{e}'_i\}$ and $\{\mathbf{f}_a\}$ to $\{\mathbf{f}'_a\}$is given by:

$$\mathbf{e}'_i = \sum_j \mathbf{e}_j P_{ji}$$

$$\mathbf{f}'_a = \sum_b \mathbf{f}_b Q_{ba}$$

Entries in column i of P are components of a new basis vector $\mathbf{e}'_i$ wrt old basis vectors $\mathbf{e}_j$, similar for Q.

Prop. With definitions above: $A' = Q^{-1}AP$, change of basis formula for a linear map

Proof.

$$\begin{aligned} T(\mathbf{e}'_i) &= T\left(\sum_j \mathbf{e}_j P_{ji}\right) \\ &= \sum_j T(\mathbf{e}_j) P_{ji} \\ &= \sum_{j,a} \mathbf{f}_a A_{aj} P_{ji} \end{aligned}$$

But also

$$\begin{aligned} T(\mathbf{e}'_i) &= \sum_b \mathbf{f}'_b A'_{bi} \\ &= \sum_{a,b} \mathbf{f}_a Q_{ab} A'_{bi} \end{aligned}$$

Equating coefficients of $\mathbf{f}_a$ gives:

$$\sum_{j,a} A_{aj} P_{ji} = \sum_{a,b} Q_{ab} A'_{bi}$$

Hence $AP = QA'$ or $A' = Q^{-1}AP$ as required. □

Consider changes in vector components

$$\mathbf{x} = \sum_i x_i \mathbf{e}_i = \sum_j x'_j \mathbf{e}'_j = \sum_i \left(\sum_j P_{ij} x'_j \right) \mathbf{e}_i$$

$$\implies x_i = P_{ij} x_j$$

(using Σ convention).

Write X for $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$, X' for $\begin{bmatrix} x'_1 \\ \vdots \\ x'_n \end{bmatrix}$

Then $X = PX'$ or $X' = P^{-1}X$

Similarly, $Y = QY'$ or $Y' = Q^{-1}Y$

Matrices representing the same linear map wrt to different basis are similar (and conversely holds).
For hermitian matrices, change of basis matrix to diagonalise is unitary.

Prop. Any 2×2 complex matrix A is similar to one of:

(i) $A' = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$

(ii) $A' = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$

(iii) $A' = \begin{bmatrix} \lambda & 0 \\ 1 & \lambda \end{bmatrix}$

Proof. $\chi_A(t)$ has 2 roots over $\mathbb{C}$

(i) For distinct roots/ evals, (λ_1, λ_2) we have $M_{\lambda_i} = m_{\lambda_i} = 1$ so matrix of evecs is change of basis matrix

(ii) Repeated root: if $m_\lambda = 2$ then same root applies

(iii) Repeated root: if $m_\lambda = 1$ then let v be evec and w any other lin indep vector (note w component in Aw is λw as repeated)

Theorem. Any $n \times N$ complex mtrix A is similar yo a matrix of the following form:

$$\begin{bmatrix} [J_{n_1}(\lambda_1)] & & & \\ & [J_{n_2}(\lambda_2)] & & \\ & & \ddots & \\ & & & [J_{n_r}(\lambda_r)] \end{bmatrix}$$

$$J_{n_i}(\lambda_i) = \begin{bmatrix} \lambda_i & 1 & & \\ & \lambda_i & 1 & \\ & & \ddots & 1 \\ & & & \lambda_i \end{bmatrix}$$

Definition. A **quadric** in $\mathbb{R}^n$ is a hypersurface defined by

$$Q(x) = \mathbf{x}^T A\mathbf{x} + \mathbf{b}^T\mathbf{x} + c = 0$$

for some non-zero symmetric real matrix A. $\mathbf{b} \in \mathbb{R}^n$, $c \in R$

$$Q(x) = A_{ij}x_i x_j + c = 0$$

Note. A invertible iff it has no zero eigenvalues. In this case, we can complete the square in eqn by setting $\mathbf{y} = \mathbf{x} + \frac{1}{2}A^{-1}\mathbf{b}$ and considering $\mathbf{y}^T A\mathbf{y}$
Ellipsoid or hyperboloid.

Quadrics in $\mathbb{R}^2$ called conics.
Possible solutions: Ellipse, point, no soln
Hyperbola, pair of lines

Polar case sets one focus at center.

Name	Cartesian Form	Cartesian info [foci]	Polar	Polar info
Ellipse	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	$e < 1$ $b^2 = a^2(1 - e^2)$ $[x = \pm ae]$	$r = \frac{l}{1+ecos\theta}$	$e < 1$, $l = a(1 - e^2)$
Parabola	$y^2 = 4ax$	$e = 1$ $[x = +a]$	$r = \frac{l}{1+ecos\theta}$	$e = 1$, $l = 2a$
Hyperbola	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	$e > 1$ $b^2 = a^2(e^2 - 1)$ $[x = \pm ae]$	$r = \frac{l}{1+ecos\theta}$	$e > 1$, $l = a(e^2 - 1)$

Equation for a cone in $\mathbb{R}^3$: let $\mathbf{c}$ be apex, $\mathbf{n}$ axis, (unit vec), $\alpha(< \frac{\pi}{2})$ angle

$$(\mathbf{x} - \mathbf{c}) \cdot \mathbf{n} = |\mathbf{x} - \mathbf{c}| \cos\alpha$$

Squaring gives double cone:

$$((\mathbf{x} - \mathbf{c}) \cdot \mathbf{n})^2 = |\mathbf{x} - \mathbf{c}|^2 \cos^2\alpha$$

R is orthogonal $\iff R^T R = I \iff (R\mathbf{x}) \cdot (R\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}\, \forall \mathbf{x}, \mathbf{y} \iff$ rows or cols of R orthonormal

Consider a new "inner product" on $\mathbb{R}^2$ given by: $(\mathbf{x}, \mathbf{y}) = \mathbf{x}^T J\mathbf{y}$ where $J = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = x_0 y_0 - x_1 y_1$

and label componenets in $\mathbb{R}^2$ by $\mathbf{x} = \begin{bmatrix} x_0 \\ x_1 \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}$
This is not positive definite but still bilinear and symmetric.

Definition. New inner product called the **Minkowski metric** on $\mathbb{R}^2$. $\mathbb{R}^2$ with this metric is called Minkowski space.

M pereserves Minkowski metric iff:
$(M\mathbf{x}, M\mathbf{y}) = (\mathbf{x}, \mathbf{y})\, \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^2$
$\iff (M\mathbf{x})^T J(M\mathbf{y}) = \mathbf{x}^T M^T J M \mathbf{y} = \mathbf{x}^T J \mathbf{y}$
$\iff M^T J M = J$

Definition. The set of such matrices form a group. (note again $\det M = \pm 1$)
Furthermore, $|M_{00}|^2 \geq 1$ so $M_{00} \geq 1$ or $M_{00} \leq -1$
The subgroup with $\det M = +1$ and $M_{00} \geq 1$ is the **Lorentz group**

General form for M: find this by using cols $M\mathbf{e}_0, M\mathbf{e}_1$ are orthonormal in the same sense as $\mathbf{e}_0$ and $\mathbf{e}_1$.

$$(M\mathbf{e}_0, M\mathbf{e}_0) = M_{00}^2 - M_{10}^2 = (\mathbf{e}_0, \mathbf{e}_0) = 1$$

So we can write $M\mathbf{e}_0 = \begin{bmatrix} \cosh\theta \\ \sinh\theta \end{bmatrix}$ for some real θ as $M_{00}^2 \geq 1$

Considering $(M\mathbf{e}_0, M\mathbf{e}_1)$ and $(M\mathbf{e}_1, M\mathbf{e}_1)$ we deduce $M\mathbf{e}_1 = \pm \begin{bmatrix} \sinh\theta \\ \cosh\theta \end{bmatrix}$

Imposing $\det M = +1$, we have:

$$M = \begin{bmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{bmatrix}$$

Note. Matrices found obey:

$$M(\theta_1)M(\theta_2) = M(\theta_1 + \theta_2)$$

using hyperbolic addition formulas.

Physical interpretation/ application:
Set $M(\theta) = \gamma(v) \begin{bmatrix} 1 & v \\ v & 1 \end{bmatrix}$ where $v = \tanh\theta$, $\gamma(v) = (1 - v^2)^{-\frac{1}{2}}$, new parameter $-1 < v < 1$
Rename $x_0 \to t$ time coordinate
$x_1 \to x$ space coordinate
Then $\mathbf{x}' = M\mathbf{x} \iff t' = \gamma(t + vx)$ and $x' = \gamma(x + vt)$
Lorentz transformation or boost relating time and coordinates for observers moving with relative velocity v in Special Relativity, in units with speed of light $c = 1$.
γ factor in Lorentz transformation gives rise to time dilation and length contraction effects.

Group property $M(\theta_3) = M(\theta_1)M(\theta_2)$ with $\theta_3 = \theta_1 + \theta_2$
$\implies$ related composition of velocities $v_i = \tanh\theta_i$, $i = 1, 2, 3$
$v_3 = \frac{v_1 + v_2}{1 + v_1 v_2}$ (addition formula for tanh) consistent with $|v_i| < 1$